## 11.6: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let $\mathbf{r}(t)$ be a vector function whose range is a set of 3-dimensional vectors:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where $x(t), y(t), z(t)$ are functions of one variable and they are called the **component functions**.

A vector function $\mathbf{r}(t)$ is *continuous* if and only if its component functions $x(t), y(t), z(t)$ are continuous.

EXAMPLE 1. *Given*

$$\mathbf{r}(t) = \left\langle t\ln(t+1), t^2\sin t, e^t\right\rangle.$$

(a) *Find the domain of* $\mathbf{r}(t)$.

$D(t \ln(t+1)) = (-1, \infty)$

$D(t^2 \sin t) = \mathbb{R}$

$D(e^t) = \mathbb{R}$

$D(\vec{r}(t)) = (-1, \infty)$

(b) *Find all* $t$ *where* $\mathbf{r}(t)$ *is continuous.* $(-1, \infty)$

*Space curve is given by* parametric equations*:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), \quad t \text{ in } I\},$$

*where $I$ is an interval and $t$ is a* parameter.

FACT*: Any continuous vector-function* $\mathbf{r}(t)$ *defines a space curve $C$ that is traced out by the tip of the moving vector* $\mathbf{r}(t)$.

*Any parametric curve has a* direction of motion *given by increasing of parameter.*

In Ex. 2 we will eliminate a parameter in order to represent the given curve as an intersection of two surfaces.

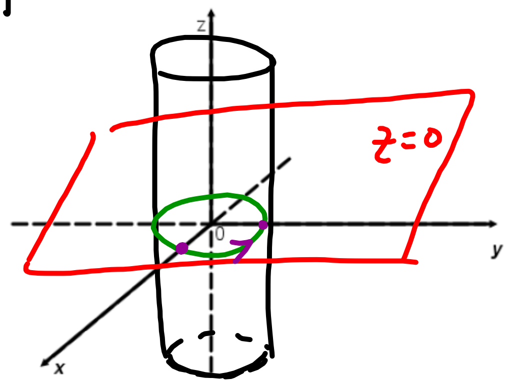
EXAMPLE 2. *Describe the curve defined by the vector function (indicate direction of motion):*

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$

$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\}$ $\underline{x^2 + y^2 = 1}$, cylinder because $\cos^2 t + \sin^2 t = 1$

$\underbrace{z = 0}$ plane

$\vec{r}(t)$ is unit circle in the xy-plane




For direction of motion:

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, 0 \rangle$

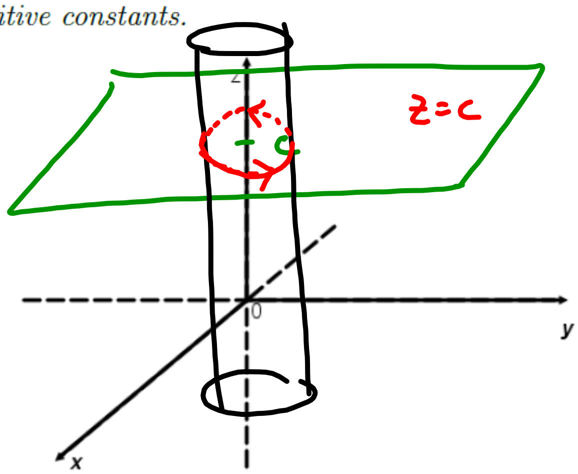
(b) $\mathbf{r}(t) = \langle \cos at, \sin at, c \rangle$ *where a and c are positive constants.*

$\left.\begin{array}{l} x = \cos at \\ y = \sin at \end{array}\right\} \Rightarrow x^2 + y^2 = 1$

cylinder

$z = c$ plane

Check the direction of motion



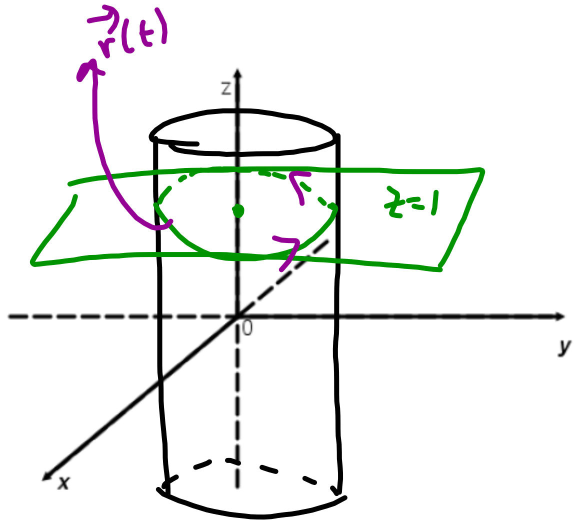

(c) $\mathbf{r}(t) = \langle \underbrace{2\cos t}_{x}, \underbrace{3\sin t}_{y}, \underbrace{1}_{z} \rangle$, $0 \le t \le 2\pi$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

elliptical cylinder

$z=1$ plane



(d) $\mathbf{r}(t)(\cancel{x}) = \langle \cos t, \sin t, t \rangle$

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \Rightarrow x^2 + y^2 = 1$$

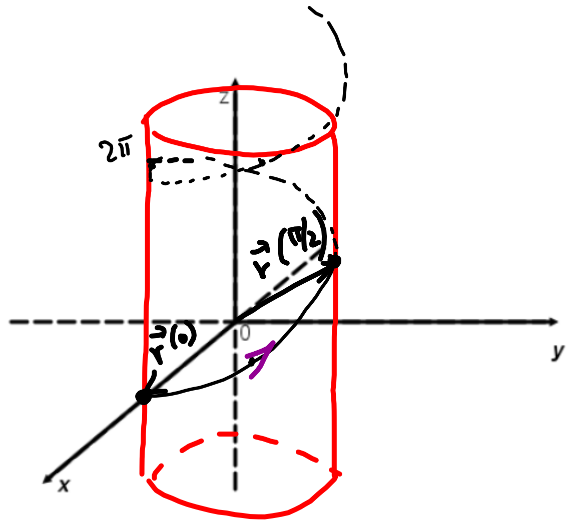
circular cylinder

$z = t$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, \frac{\pi}{2} \rangle$

$\vec{r}(2\pi) = \langle 1, 0, 2\pi \rangle$

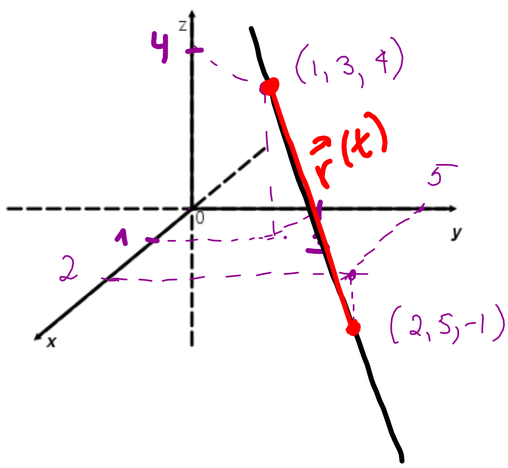



(e) $\mathbf{r}(t) = \langle 1 + t, 3 + 2t, 4 - 5t \rangle$, $0 \le t \le 1$.

Segment

$\vec{r}(0) = \langle 1, 3, 4 \rangle$

$\vec{r}(1) = \langle 2, 5, -1 \rangle$



EXAMPLE 3. *Show that the the curve given by*

$$\mathbf{r}(t) = \left\langle \sin t, 2\cos t, \sqrt{3}\sin t \right\rangle$$

*lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.*

$x = \sin t$ , $y = 2\cos t$ , $z = \sqrt{3}\sin t$

It follows that $\underbrace{z = \sqrt{3}\,x}_{\text{plane through origin}}$

$$x^2 + y^2 + z^2 = (\sin t)^2 + (2\cos t)^2 + (\sqrt{3}\sin t)^2$$
$$= \sin^2 t + 4\cos^2 t + 3\sin^2 t$$
$$= 4\sin^2 t + 4\cos^2 t = 4$$

$x^2 + y^2 + z^2 = 4$ sphere with $r = 2$ centered at origin.

The graph is the line of intersection between the sphere $x^2 + y^2 + z^2 = 4$ and plane $z = \sqrt{3}x$.

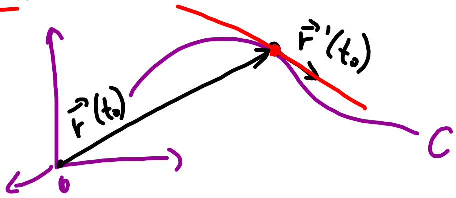
In other words, the graph is a circle.

The radius of this circle is 2 because the plane passes through its center.

**Derivatives:** *The derivative* $\mathbf{r}'$ *of a vector function* $\mathbf{r}$ *is defined just as for a real-valued function:*

$$\frac{\mathrm{d}\mathbf{r}(t_0)}{\mathrm{d}t} = \mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}$$

*if the limit exists. The derivative* $\mathbf{r}'(t_0)$ *is the* tangent vector *to the curve* $\mathbf{r}(t)$ *at the point* $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$.




THEOREM 4. *If the functions* $x(t), y(t), z(t)$ *are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

EXAMPLE 5. *Given* $\mathbf{r}(t) = \underbrace{(1+t)^2}_{=1}\mathbf{i} + \underbrace{e^t}_{=1}\mathbf{j} + \underbrace{\sin 3t}_{=0}\mathbf{k}$.

(a) *Find* $\mathbf{r}'(t)$

$$\vec{r}'(t) = \left\langle \frac{d}{dt}(1+t)^2, \frac{d}{dt}(e^t), \frac{d}{dt}(\sin 3t)\right\rangle = \langle 2(1+t), e^t, 3\cos 3t\rangle$$

(b) *Find a tangent vector to the curve at* $t = 0$.

$$\vec{r}'(0) = \langle 2, e^0, 3\cos 0\rangle = \langle 2, 1, 3\rangle$$

(c) *Find a tangent line to the curve at* $t = 0$.

$$\vec{r}(0) = \langle 1, e^0, \sin 0\rangle = \langle 1, 1, 0\rangle$$

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

$\vec{r}(0)$ $\quad$ $\vec{r}'(0)$

$$\boxed{\begin{aligned} x &= 1 + 2t \\ y &= 1 + t \\ z &= 0 + 3t \end{aligned}}$$

(c′) *Find a tangent line to the curve at the point* $\langle 1, 1, 0\rangle$.