12.1: Functions of Several Variables

Consider the following formulas:

two variables x, y

$$z = \boxed{2 - x - 4y = f(x,y)} \qquad (1)$$

$$z = x^2 + y^2 \qquad (2)$$

$z = f(1,2) = 2-1-4\cdot 2 = -7$

$$z = \sqrt{x^2 + y^2} \qquad (3)$$

$$z = \sqrt{1 - x^2 - y^2} \qquad (4)$$

ordered pair of independent variables (x, y) (input) $\longrightarrow$ $f(x,y)$ $\longrightarrow$ dependent variable z (output) $z = f(x,y)$

DEFINITION 1. *Let $D \subset \mathbb{R}^2$. A* **function** f **of two variables** *is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$.*

The set D is the **domain** *of f and the* **range** *of f is the set of values that f takes on, that is $\{f(x, y)|(x, y) \in D\}$.*

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

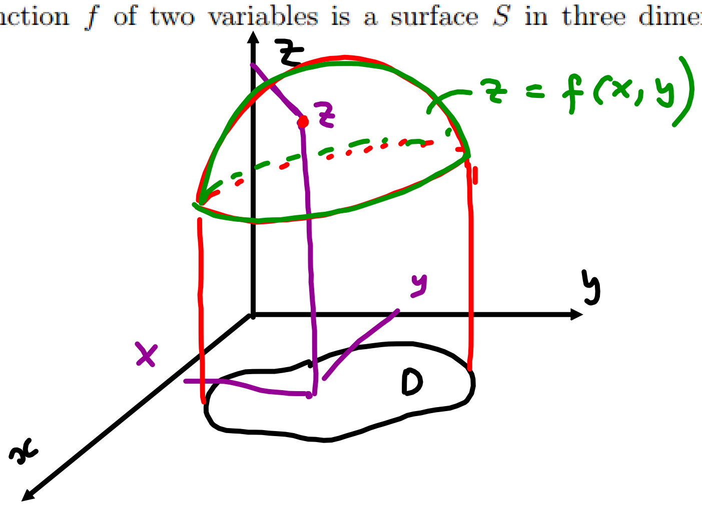
- **GRAPH of** $f(x, y)$.

Recall that a graph of a function f of one variable is a curve C with equation $y = f(x)$.

DEFINITION 3. *The* **graph** *of f with domain D is the set:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y). \;\; (x, y) \in D\right\}.$$

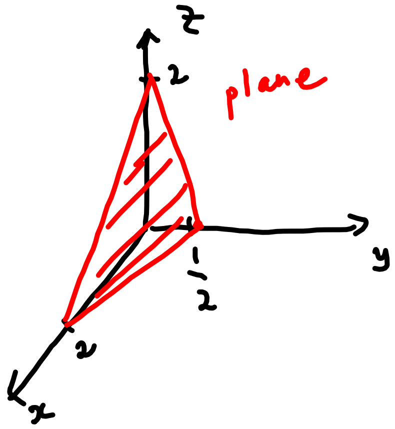
The graph of a function f of two variables is a surface S in three dimensional space with equation $z = f(x, y)$.



EXAMPLE 4. *Find the domain and sketch the graph of the functions (1)-(4). What is the range?*

(1) $z = 2 - x - 4y$

$D = \mathbb{R}^2$

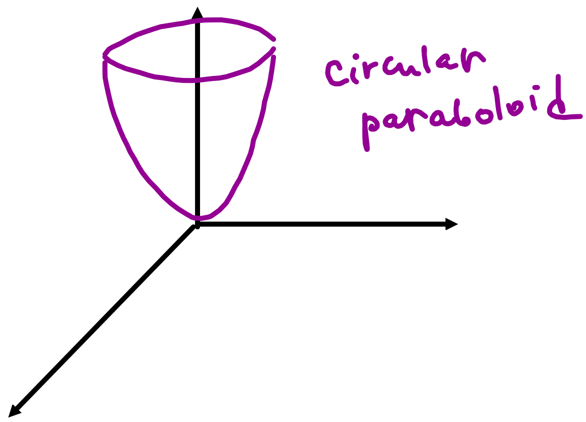


Range $\mathbb{R}$

(2) $z = x^2 + y^2$

$D = \mathbb{R}^2$

Range $= \{z \mid z \geq 0\} = [0, +\infty)$

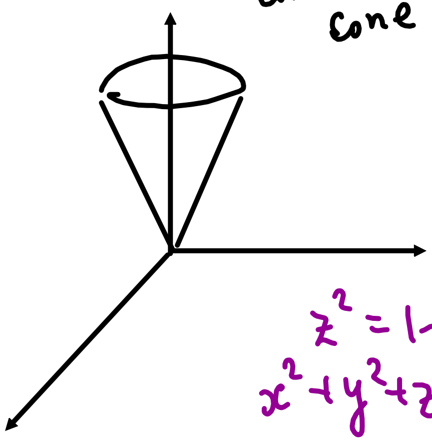


(3) $z = \sqrt{x^2+y^2}$

$D = \mathbb{R}^2$

Range $= [0, +\infty)$

circular cone



$z^2 = 1 - x^2 - y^2, \ z \geq 0$

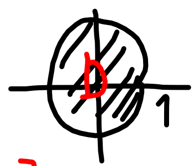
$x^2 + y^2 + z^2 = 1, \ z \geq 0$

(4) $z = \sqrt{1-x^2-y^2}$

$D = \{(x,y) \mid x^2+y^2 \leq 1\}$

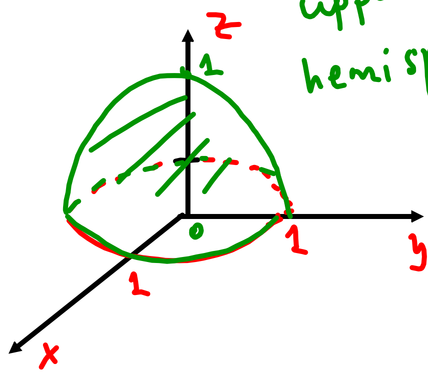
$1 - x^2 - y^2 \geq 0$

$1 \geq x^2 + y^2$



Range $= [0, 1]$

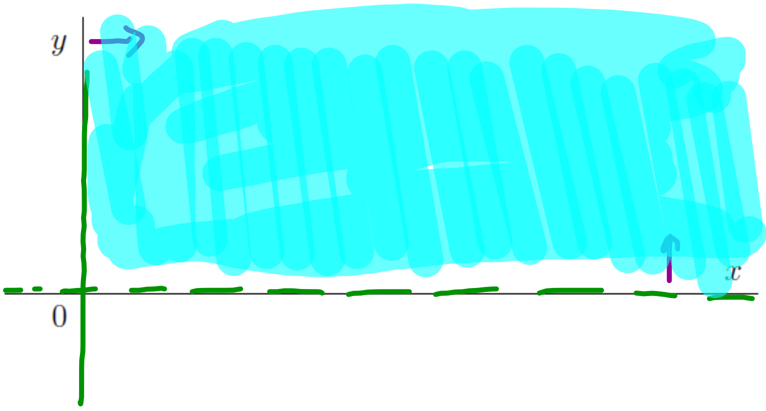
upper hemisphere



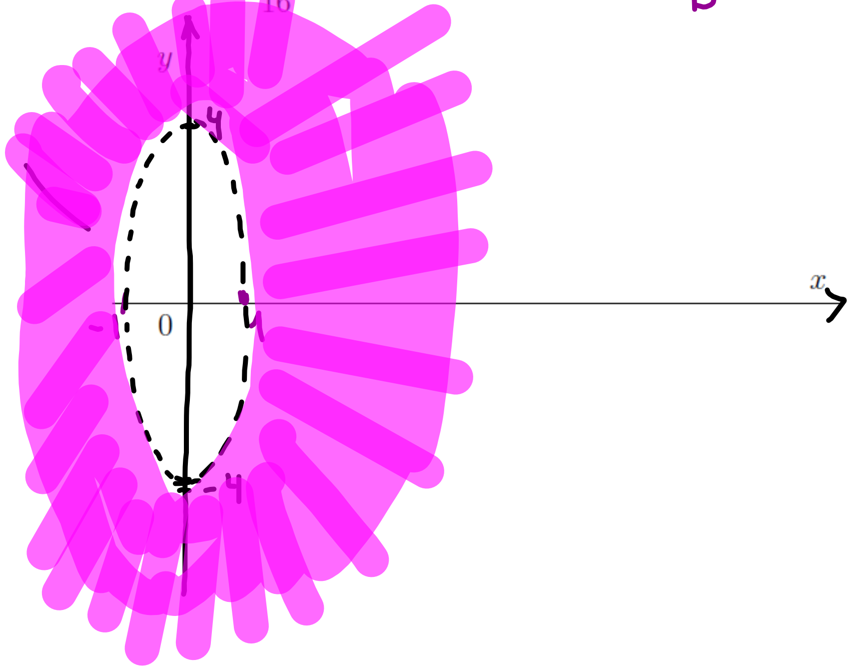
EXAMPLE 5. *Sketch the domain of each of the following:*

(a) $z = \sqrt{x} - \dfrac{5}{\sqrt{y}}$

$D = \{(x,y) \mid x \geq 0 \ , \ y > 0\}$



(b) $z = \ln(\underbrace{x^2 + \frac{y^2}{16} - 1}^{u})$



$D(\ln u) = \{ u > 0 \}$

$D(z) = \{ x^2 + \frac{y^2}{16} - 1 > 0 \}$

$x^2 + \frac{y^2}{16} = 1$

• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

where k is a constant in the range of f.

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

Sketch the contour map for....

EXAMPLE 7. *Sketch the level curves of the functions (2) and (3) for the values* $k = 0, 1, 2, 3, 4$*:*

(2) $z = x^2 + y^2$ (3) $z = \sqrt{x^2 + y^2}$

Find the general shape of level curves

$k = x^2 + y^2$

$K = 0$ $(0,0)$

$K > 0$ circles with radius $\sqrt{K}$

$k = \sqrt{x^2 + y^2}$

$x^2 + y^2 = K^2$, $K \geq 0$

$K = 0$ $(0,0)$

$K > 0$ circles with $r = K$

$K = 1$ $x^2 + y^2 = 1$

2

3

4

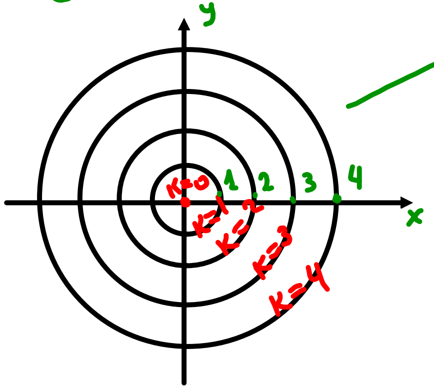
$K = 1$ $x^2 + y^2 = 1$

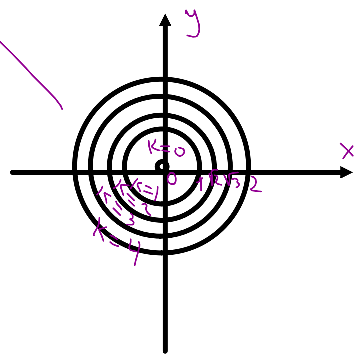
$K = 2$ $x^2 + y^2 = 4$

$K = 3$ $x^2 + y^2 = 9$

$K = 4$ $x^2 + y^2 = 16$

contour map





- **Functions of three variables.**

DEFINITION 8. *Let* $D \subset \mathbb{R}^3$. *A* **function** f **of three variables** *is a rule that assigns to each ordered pair* (x, y, z) *in* D *a unique real number denoted by* $f(x, y, z)$.

$(x, y, z) \longrightarrow f(x, y, z) = u$

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12s_2 - s_3^{-5}.$$

Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. *Find the domain of the following function:*

$$f(x,y,z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

$D(\ln u) = \{u > 0\} = \{(x,y,z) \mid 36 - x^2 - y^2 - z^2 > 0\}$

$D\left(\frac{1}{\sqrt{u}}\right) = \{u > 0\} = \{(x,y,z) \mid x^2 + y^2 + z^2 - 25 > 0\}$

$x^2 + y^2 + z^2 > 25$

$36 > x^2 + y^2 + z^2$

$D(f) = \{(x,y,z) \mid 25 < x^2 + y^2 + z^2 < 36\}$

Region between (but not on) two concentric spheres centered at origin with radii 5 and 6.

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

$$f(x, y, z) = k$$

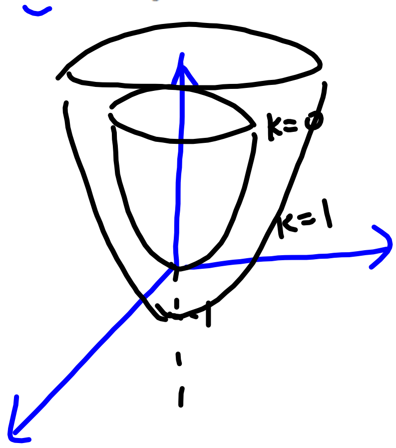
where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. *Find the level surfaces of the function $u = x^2 + y^2 - z$.*

$x^2 + y^2 - z = k$

$z = x^2 + y^2 - k$

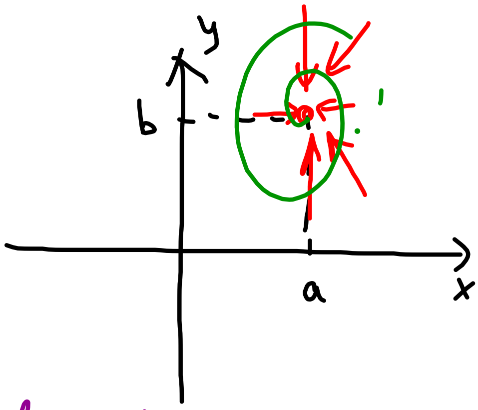
circular paraboloid



REMARK 11. For any function there exist a unique level curve (surface) through given point!!!

12.2 Limits and continuity

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



Composition of polynomials with common functions we usually can use the direct substitution Rule

$$\lim_{(x,y)\to(1,3)} (x^2+y^2) = 1^2+3^2=10$$

$$\lim_{(x,y)\to(0,-1)} \ln|x+y^3| = \ln|0+(-1)^3| = \ln 1 = 0$$

<u>Def</u> $f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Thus, $f(x,y)=x^2+y^2$ and all other polynomial in two variables are continuous everythere.

Similarly, their composition with common functions gives continuous functions.