

12.2: Limits and Continuity

Note: A more extensive study of these topics is usually given in advance calculus.

EXAMPLE 1. For the function $f(x, y) = \frac{xy}{x^2 + y^2}$ find the limits along **at** $(0, 0)$

(a) the x-axis $\langle t, 0 \rangle = \langle x(t), y(t) \rangle$

$$f(t, 0) = \frac{t \cdot 0}{t^2 + 0^2} = 0$$

(b) the y-axis

$$f(0, t) = 0$$

(c) the line $y = x$ $\langle t, t \rangle$

$$f(t, t) = \frac{t^2}{2t^2} = \frac{1}{2}$$

(d) the line $y = -x$

$$f(t, -t) = -\frac{t^2}{2t^2} = -\frac{1}{2}$$

(e) the parabola $y = x^2$

$$f(t, t^2) = \frac{t \cdot t^2}{t^2 + t^4} = \frac{\cancel{t} \cdot \cancel{t^2}}{\cancel{t}(1+t^2)} \xrightarrow[t \rightarrow 0]{} 0$$

Conclusion $\lim_{(x,y) \rightarrow 0} \frac{xy}{x^2+y^2}$ DNE

The statement

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

is intended to convey the idea that the value of the function $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0,y_0) .

We note without proof that the standard properties of limits hold for limits along curves and for general limits of functions of two variables, so that computations involving such limits can be performed in usual way.

EXAMPLE 2. *Find*

$$\lim_{(x,y) \rightarrow (1,3)} (4x^3y - 2014) = 4 \cdot 1^3 \cdot 3 - 2014 = -2002$$

THEOREM 3.

- If $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$ then $f(x,y) \rightarrow L$ as $(x,y) \rightarrow (x_0,y_0)$ along any smooth curve.
- If the limit of $f(x,y)$ fails to exist as $(x,y) \rightarrow (x_0,y_0)$ along some smooth curve, or if $f(x,y)$ has different limits as $(x,y) \rightarrow (x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ does not exist as $(x,y) \rightarrow (x_0,y_0)$.

EXAMPLE 4. For the function $f(x,y) = \frac{xy}{x^2+y^2}$ discuss its limit as $(x,y) \rightarrow (0,0)$.

along the x -axis $f(x,y) \rightarrow 0$
 $(x,y) \rightarrow (0,0)$

along $y = x$ $f(x,y) \rightarrow \frac{1}{2}$
 $(x,y) \rightarrow (0,0)$

Thus $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$ DNE

DEFINITION 5. A function $f(x, y)$ is continuous at the point (x_0, y_0) if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Roughly speaking, a function will be continuous at a point if the graph does not have any holes or breaks at that point.

All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, etc.

Recognizing Continuous functions:

- A polynomial or function of (x, y) is continuous.
- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

EXAMPLE 6. Confirm that the following functions are all continuous everywhere:

$$f(x, y) = ye^{x+y} + \sqrt[3]{x}, \quad g(x, y) = \frac{x^3y + 6}{1 + x^2 + y^2}, \quad h(x, y) = |1 - xy| + \sin(x^5y^4)$$

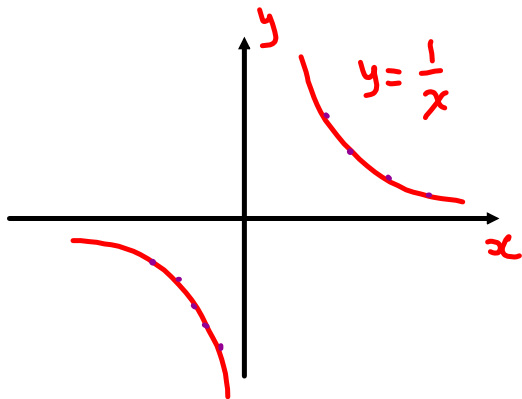
$$1 + x^2 + y^2 \neq 0$$

EXAMPLE 7. Evaluate $\lim_{(x,y) \rightarrow (-1,1)} \frac{xy}{x^2 + y^2} = \frac{-1 \cdot 1}{(-1)^2 + 1^2} = -\frac{1}{2}$

Continuous
at $(-1,1)$

EXAMPLE 8. Find all points where the function $f(x, y) = \frac{xy + 12y^4}{xy - 1}$ is continuous.

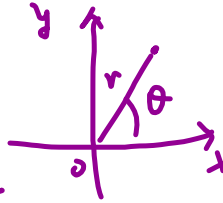
$$xy - 1 = 0 \Leftrightarrow xy = 1$$
$$y = \frac{1}{x}$$



$f(x, y)$ is continuous
everywhere except
of $(a, \frac{1}{a})$ $a \neq 0$

EXAMPLE 9. Evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$

Polar coordinates $x = r \cos \theta$, $y = r \sin \theta$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{(r \cos \theta)^2 r \sin \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$


$$= \lim_{r \rightarrow 0} \frac{r^3 \cos^2 \theta \sin \theta}{r^2} = \lim_{r \rightarrow 0} r \underbrace{\cos^2 \theta \sin \theta}_{\text{bounded}} = 0$$

