12.3: Partial Derivatives

DEFINITION 2. *If f is a function of two variables, its* **partial derivatives** *are the functions f_x and f_y defined by*

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Conclusion: $f_x(x,y)$ represents the *rate of change* of the function $f(x,y)$ as we change x and hold y fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change y and hold x fixed.

$\frac{\partial f(x,y)}{\partial x}$ $\partial \neq d$ $\frac{df(x)}{dx}$

partial

Notations for partial derivatives: If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$:

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 3. *If* $f(x, y) = x^3 + y^5 e^x$ *find* $f_x(0, 1)$ *and* $f_y(0, 1)$.

$$f_x = \frac{\partial}{\partial x}\left(x^3 + y^5 e^x\right) = 3x^2 + y^5 e^x$$

$$f_x(0,1) = 3 \cdot 0^2 + 1^5 e^0 = \boxed{1}$$

$$f_y = 0 + 5y^4 e^x$$

$$f_y(0,1) = 5 \cdot 1^4 \cdot e^0 = \boxed{5}$$

EXAMPLE 4. *Find all of the first order partial derivatives for the following functions:*

(a) $z(x,y) = x^3 \sin(xy)$

$$z_x \overset{PR}{=} 3x^2 \sin(xy) + x^3 \underbrace{\frac{\partial}{\partial x}(\sin xy)}_{CR}$$

$$= 3x^2 \sin(xy) + x^3 \cos(xy)\, y$$

$$z_y = x^3 \frac{\partial}{\partial y}(\sin xy) = x^3 \cdot \cos(xy) \cdot x$$

(b) $u(x,y,z) = ye^{xyz}$

$$u_x \overset{CR}{=} y e^{xyz} \frac{\partial}{\partial x}(xyz) = y e^{xyz} yz = y^2 z e^{xyz}$$

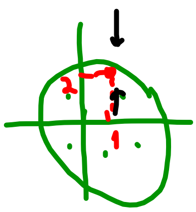
$$u_y \overset{PR+CR}{=} 1 \cdot e^{xyz} + y \cdot e^{xyz} \cdot xz = e^{xyz}(1 + xyz)$$

$$u_z \overset{\text{use symmetry}}{=} xy^2 e^{xyz}$$

EXAMPLE 5. *The temperature at a point* (x, y) *on a flat metal plate is given by*

$$T(x,y) = \frac{80}{1+x^2+y^2},$$

where T *is measured in* °C *and* x, y *in meters. Find the rate of change of temperature with respect to distance at the point* $(1,2)$ *in the* y*-direction.* $= \frac{\partial T}{\partial y}(1,2)$



$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(80\left(1+x^2+y^2\right)^{-1}\right)$$

$$= -\frac{80}{(1+x^2+y^2)^2}\,\frac{\partial}{\partial y}\left(1+x^2+y^2\right)$$

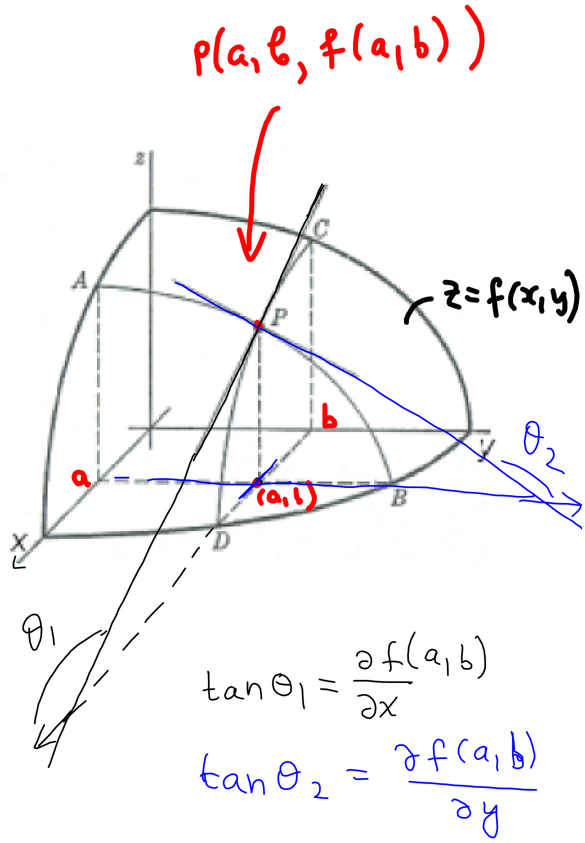
$$= -\frac{80}{(1+x^2+y^2)^2}\cdot 2y$$

$$\frac{\partial T(1,2)}{\partial y} = -\frac{80\cdot 2\cdot 2}{(1+1+4)^2} = -\frac{80\cdot \cancel{4}^{1}}{\cancel{36}_{9}} = -\frac{80}{9}\ {}^\circ\mathrm{C}/\mathrm{m}$$

Geometric interpretation of partial derivatives: Partial derivatives are the *slopes of traces*:

- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point (a,b).

- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at (a,b).



EXAMPLE 6. *If* $f(x,y)=\sqrt{4-x^2-4y^2}$, *find* $f_x(1,0)$ *and* $f_y(1,0)$ *and interpret these numbers as slopes. Illustrate with sketches.*

$$f(x,y)=\sqrt{4-x^2-4y^2}$$

$$f_x = \frac{1}{\cancel{2}\sqrt{4-x^2-4y^2}}\left(-\cancel{2}x\right) \Rightarrow f_x(1,0) = -\frac{1}{\sqrt{4-1}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} = \tan\theta_x$$

$$\theta_x = \frac{5\pi}{6}$$

$$f_y = \frac{1}{2\sqrt{4-x^2-4y^2}}\cdot(-8y) \Rightarrow f_y(1,0) = 0 = \tan\theta_y$$

$\theta_y = 0 \Rightarrow$ tangent line horizontal

Graph of $f(x,y)$:

$$z=\sqrt{4-x^2-4y^2}$$

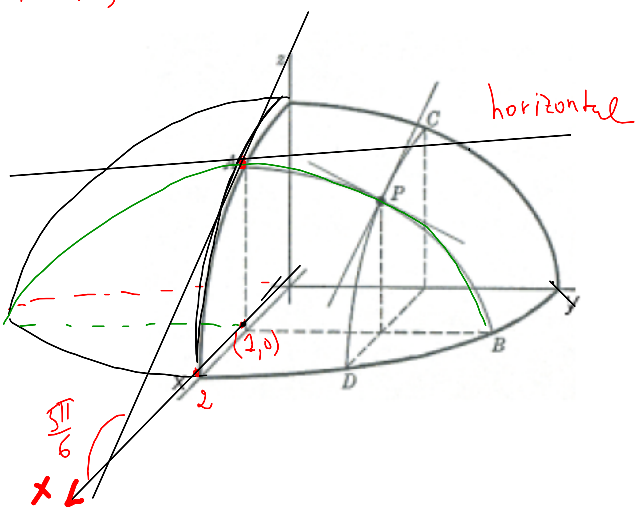
$$z^2 = 4-x^2-4y^2,\ z\geq 0$$

$$x^2+4y^2+z^2=4,\ z\geq 0$$

$$\frac{x^2}{4}+y^2+\frac{z^2}{4}=1,\ z\geq 0$$

upper half-ellipsoid

$A(1,0,\sqrt{3})$



$z = f(x, y)$

Higher derivatives: Since both of the first order partial derivatives for $f(x, y)$ are also functions of x and y, so we can in turn differentiate each with respect to x or y. We use the following notation:

$$
\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
\end{aligned}
$$

mixed second order partial derivatives

EXAMPLE 7. *Find the second partial derivatives of*

$$f(x,y) = y^3 + 5y^2e^{4x} - \cos(x^2).$$

$f_x = 20y^2e^{4x} + 2x\sin(x^2)$

$f_y = 3y^2 + 10ye^{4x}$

$f_{xx} = 80y^2e^{4x} + 2\sin(x^2) + 4x^2\cos(x^2)$

$f_{yy} = 6y + 10e^{4x}$

$f_{xy} = \frac{\partial}{\partial y}\left(20y^2e^{4x} + 2x\sin(x^2)\right) = 40ye^{4x}$

$\parallel$

$f_{yx} = \frac{\partial}{\partial x}\left(3y^2 + 10ye^{4x}\right) = 40ye^{4x}$

Clairaut's Theorem. *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial derivative of order three or higher can also be defined. For instance,

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions f_{yyx}, f_{xyy} and f_{yxy} are continuous then

EXAMPLE 8. *Find the indicated derivative for*

continuous and has

$$f(x, y, z) = \cos(xy + z).$$

cont. partial derivatives of all orders

(a) f_{xy}

$f_x = -y \sin(xy + z)$

$f_{xy} = \frac{\partial}{\partial y}(-y \sin(xy+z)) = -(\sin(xy+z) + xy\cos(xy+z))$

By the above theorem

(b) $f_{zxy} = f_{xyz} = \frac{\partial}{\partial z}(f_{xy})$

$= -(\cos(xy+z) - xy\sin(xy+z))$