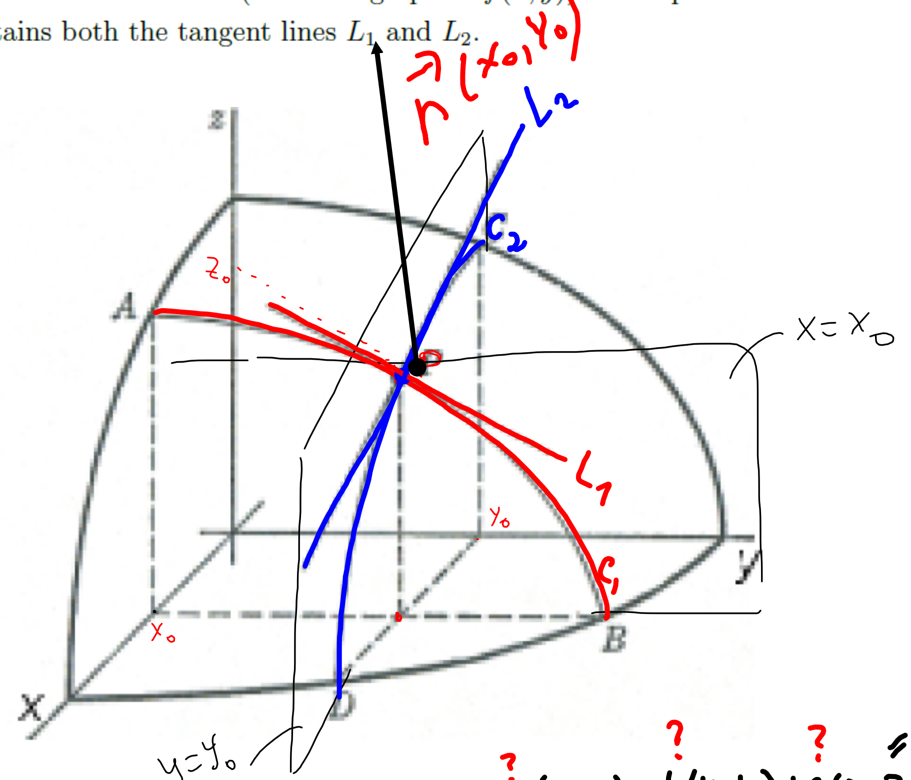
12.4: Tangent Planes to graph of $z = f(x,y)$ and Differentials

Suppose that $f(x,y)$ has continuous first partial derivatives and a surface S has equation $z = f(x,y)$. Let $P(x_0, y_0, z_0)$ be a point on S, i.e. $z_0 = f(x_0, y_0)$.

Denote by C_1 the trace to $f(x,y)$ for the plane $y = y_0$ and denote by C_2 the trace to $f(x,y)$ for the plane $x = x_0$. let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The **tangent plane** to the surface S (or to the graph of $f(x,y)$) at the point P is defined to be the plane that contains both the tangent lines L_1 and L_2.



$$\overset{?}{a}(x-x_0)+\overset{?}{b}(y-y_0)+\overset{?}{c}(z-\overset{f(x_0,y_0)}{\overset{\shortparallel}{z_0}})=0$$

$\vec{n}(a,b,c)$

Tangent plane

$$\vec{n}(x_0,y_0) = \langle \underbrace{f_x(x_0,y_0)}_{a}, \underbrace{f_y(x_0,y_0)}_{b}, \underbrace{-1}_{c} \rangle$$

$$\boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0}$$

$$\left.\begin{array}{l} z=f(x,y) \\ x=x_0 \end{array}\right\} \xRightarrow{\text{Curve}} \underline{z=f(x_0,y)}; C_1$$

$$\left.\begin{array}{l} z=f(x,y) \\ \cancel{y}=y_0 \end{array}\right\} \xRightarrow{\text{Curve}} z=f(x,y_0) : C_2$$

Parameterize C_1 and C_2

C_1: $\vec{r}_1(y)=\langle x_0, y, f(x_0,y)\rangle$

C_2: $\vec{r}_2(x)=\langle x, y_0, f(x,y_0)\rangle$

$\vec{n}(x_0,y_0) \perp \vec{r}_1{}'(y_0)$

$\vec{n}(x_0,y_0) \perp \vec{r}_2{}'(x_0)$

$$\vec{n}(x_0,y_0)=r_1'(y_0)\times\vec{r}_2{}'(x_0)$$

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & f_y(x_0,y_0) \\ 1 & 0 & f_x(x_0,y_0) \end{vmatrix}$$

$$=\underbrace{\langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle}$$

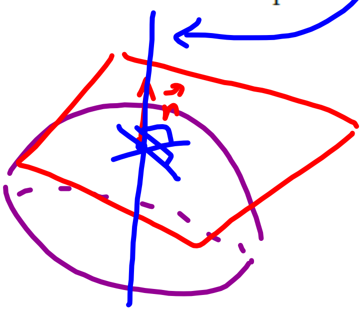
THEOREM 1. *An equation of the tangent plane to the graph of the function* $z = f(x, y)$ *at the point* $P(x_0, y_0, f(x_0, y_0))$ *is*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

*CONCLUSION:*A normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$$\mathbf{n} = \mathbf{n}(x_0, y_0) = \langle\ f_x(x_0,y_0),\ f_y(x_0,y_0),\ -1\ \rangle.$$

The line through the point $P(x_0, y_0, f(x_0, y_0))$ parallel to the vector $\mathbf{n}$ is perpendicular to the above tangent plane. This line is called **the normal line** to the surface $z = f(x, y)$ at P. It follows that this normal line can be expressed parametrically as



EXAMPLE 2. *Find an equation of the tangent plane to the graph of the function* $z = \underbrace{x^2+y^2+8}_{f(x,y)}$ *at the point* $(1,1) = (x_0, y_0)$

$$z_0 = f(x_0, y_0) = f(1,1) = 1^2 + 1^2 + 8 = 10$$

$$f_x = 2x \Rightarrow f_x(1,1) = 2$$

$$f_y = 2y \Rightarrow f_y(1,1) = 2$$

$$\Rightarrow \vec{n}(1,1) = \langle 2, 2, -1 \rangle$$

Tangent plane at $(1,1,10)$ with

$$2(x-1) + 2(y-1) - (z-10) = 0$$

$$2x - 2 + 2y - 2 - z + 10 = 0$$

$$\boxed{2x + 2y - z + 6 = 0}$$

Example 3 Find parametric equations for the normal line to the surface

$$z = e^{4y}\sin(4x)$$

at $P(\frac{\pi}{8}, 0, 1)$.

P $\quad \vec{n}(\frac{\pi}{8}, 0)$

$x = x_0 + at$
$y = y_0 + bt$
$z = z_0 + ct$

$z_x = 4e^{4y}\cos(4x)$ $\quad z_x(\frac{\pi}{8}, 0) = 0$

$z_y = 4e^{4y}\sin(4x)$ $\quad z_y(\frac{\pi}{8}, 0) = 4$

$\vec{n}(\frac{\pi}{8}, 0) = \langle 0, 4, -1 \rangle$

$$\boxed{\begin{aligned} x &= \frac{\pi}{8} \\ y &= 4t \\ z &= 1 - t \end{aligned}}$$

The **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables. The **differential** $\mathrm{d}z$ (or the **total differential**) is defined by

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y.$$

FACT: $\Delta z \approx \mathrm{d}z$.

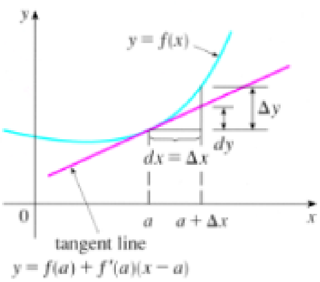
This implies:

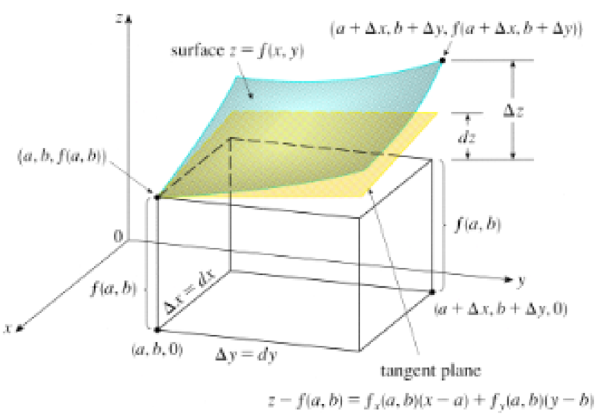
$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + \mathrm{d}z(a, b)$$

or

Differentials. Given $z = f(x, y)$. If Δx and Δy are given increments of $x = a$ and $y = b$ respectively, then the corresponding **increment** of z is

$$\Delta z(a, b) = f(a + \Delta x, b + \Delta y) - f(a, b). \tag{1}$$





[1]

[1]the pictures are from our textbook

The **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables. The **differential** $\mathrm{d}z$ (or the **tot**
differential) is defined by

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y. = z_x \, dx + z_y \, dy$$

FACT: $\Delta z \approx \mathrm{d}z$. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) \approx dz(a,b)$

This implies:

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + \mathrm{d}z(a,b)$$

$dx \approx \Delta x$

$dy \approx \Delta y$

or

$$\boxed{f(a+\Delta x, b+\Delta y) \approx f(a,b) + z_x \Delta x + z_y \Delta y}$$

EXAMPLE 4. *Use differentials to find an approximate value for* $\sqrt{1.03^2 + 1.98^3}$.

$$f(x,y) = \sqrt{x^2 + y^3} \qquad f(1,2) = \sqrt{1^2 + 2^3} = \sqrt{9} = 3$$

$$\sqrt{1.03^2 + 1.98^3} = f(1.03, 1.98) =$$

$$f(\underbrace{1}_{a} + \underbrace{0.03}_{\Delta x}, \underbrace{2}_{b} \underbrace{- 0.02}_{\Delta y}) \approx f(1,2) + z_x(1,2)\Delta x + z_y(1,2)\Delta y =$$

$$= 3 + \frac{1}{3} \cdot 0.03 - 2 \cdot 0.02$$

$$z_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^3}} \Rightarrow z_x(1,2) = \frac{1}{3}$$

$$= 3 + 0.01 - 0.04$$

$$= 3 - 0.03 = 2.97$$

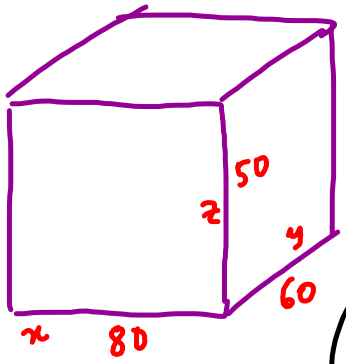
$$z_y = \frac{3y^2}{2\sqrt{x^2+y^3}} \Rightarrow z_y(1,2) = \frac{3 \cdot 2^2}{2 \cdot 3} = 2$$

By Calculator ≈ 2.970402

If $u = f(x, y, z)$ then the differential $\mathrm{d}u$ at the point $(x, y, z) = (a, b, c)$ is defined in terms of the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ of the independent variables:

$$\mathrm{d}u(a, b, c) = f_x(a, b, c)\mathrm{d}x + f_y(a, b, c)\mathrm{d}y + f_z(a, b, c)\mathrm{d}z.$$

EXAMPLE 5. *The dimensions of a closed rectangular box are measured as* 80 *cm,* 60 *cm and* 50 *cm, respectively, with a possible error of* 0.2 *cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.*



Surface area of the box with dimensions x, y, z:

$$S(x,y,z) = 2(xz + xy + yz)$$

$$\Delta x = \Delta y = \Delta z = 0.2$$

Find $\Delta S(80,60,50)$ – ?

$$\Delta S \simeq dS = S_x \Delta x + S_y \Delta y + S_z \Delta z = (S_x + S_y + S_z)\Delta x$$

$$S_x = 2(z+y) \Rightarrow S_x(80,60,50) = 2(60+50) = 220$$

$$S_y = 2(x+z) \Rightarrow S_y(80,60,50) = 2(80+50) = 260$$

$$S_z = 2(x+y) \Rightarrow S_z(80,60,50) = 2(80+60) = 280$$

$$\Delta S \simeq (220 + 260 + 280) \cdot 0.2$$

$$= \frac{76\cancel{0} \cdot 2}{\cancel{10}} = 140 + 12 = \boxed{152 \text{ cm}^2}$$

A function $f(x, y)$ is **differentiable** *at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).*

For example, all polynomial and rational functions are differentiable on their natural domains.

Let a surface S be a graph of a differentiable function f. As we zoom in toward a point on the surface S , the surface looks more and more like a plane (its tangent plane) and we can approximate the function f by a linear function of two variables.

$$z = f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) =: L(x, y).$$

If f_x and f_y are continuous at (a,b) $\Rightarrow$ $f(x,y)$ is differentiable at (a,b)

$\Downarrow$

there exist tengent plane at (a,b)