12.5: The Chain Rule

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where f and g ar
differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dg}{dt}.$$

EXAMPLE 1. *Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.*

$$z = x^y = (t^2)^{\sin t} = t^{2\sin t}$$

$$z(t) = t^{2\sin t}$$

$$\ln z(t) = \ln t^{2\sin t}$$

$$(\ln z(t))' = (2\sin t \ln t)'$$

$$\frac{z'}{z} = \boxed{}$$

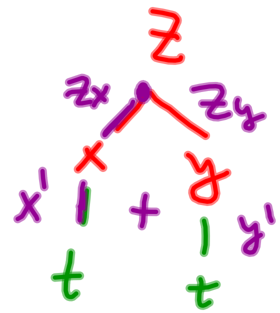
Assume that all functions below have continuous derivatives (ordinary or partial).

TREE DIAGRAM

- CASE 1: $z = f(x, y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

  Chain Rule:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$



SOLUTION OF EXAMPLE 1:

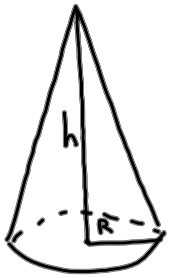
Let $z = x^y$, *where* $x = t^2$, $y = \sin t$. *Compute* $z'(t)$.

$$\frac{dz}{dt} = z_x\, x' + z_y \cdot y'$$

$$= y\, x^{y-1}\, 2t + x^y \ln x \cos t$$

$$= (\sin t)\left(t^2\right)^{\sin t - 1} \cdot 2t + t^{2\sin t} \ln t^2 \cos t$$

EXAMPLE 2. *The radius of a right circular cone is increasing at a rate of* 1.8 *cm/s while its height is decreasing at a rate* 2.5 *cm/s. At what rate is the volume of the cone changing when the radius is* 120 *cm and the height is* 140 *cm.*



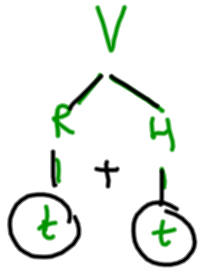
$$\frac{dR}{dt} = 1.8 \ \text{cm/s}$$

$$\frac{dh}{dt} = -2.5 \ \text{cm/s}$$

$$\left.\frac{dV}{dt}\right|_{\substack{R=120\text{ cm}\\ H=140\text{ cm}}} - ?$$

$R = R(t)$

$H = H(t)$



$$V(R,H) = \frac{1}{3}\pi R^2 H$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial R}\frac{dR}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt}$$

$$= \frac{2}{3}\pi R H \cdot 1.8 + \frac{1}{3}\pi R^2 (-2.5)$$

$$\left.\frac{dV}{dt}\right|_{\substack{R=120\\ H=140}} = \frac{2}{\cancel{3}}\pi \cdot 120 \cdot 140 \ \frac{\cancel{9}\,3}{5} - \frac{1}{3}\pi \ 120^2 \ \frac{5}{2}$$

$$120\pi\left(\frac{\overset{56}{\cancel{280}} \cdot 3}{\underset{1}{\cancel{5}}} - \frac{\overset{20}{\cancel{120}} \cdot 5}{\underset{1}{\cancel{6}}}\right)$$

$$120\pi\,(150 + 18 - 100) = 120\pi \cdot 68$$

$$= 8160\pi \ \frac{\text{cm}^3}{\text{s}}$$ (according to Daniel).

$z = f(x(s,t), y(s,t))$

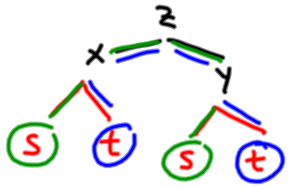
- CASE 2: $z = f(x, y)$, where $x = x(s,t)$, $y = y(s,t)$ and compute z_s and z_t.

Chain Rule:

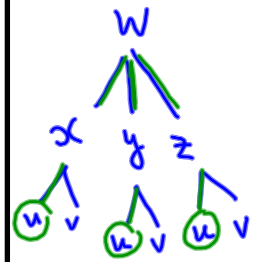
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tree diagram:



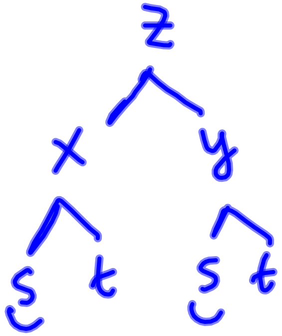
EXAMPLE 3. *Write out the Chain Rule for the case where* $w = f(x, y, z)$ *and* $x = x(u, v)$, $y = y(u, v)$ *and* $z = z(u, v)$.



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

EXAMPLE 4. *If* $z = \sin x \cos y$, *where* $x = (s-t)^2$, $y = s^2 - t^2$ *find* $z_s + z_t$.



$$
\begin{aligned}
&z_s = z_x x_s + z_y y_s \\
+\ &z_t = z_x x_t + z_y y_t
\end{aligned}
$$

$$z_s + z_t = z_x (x_s + x_t) + z_y (y_s + y_t)$$

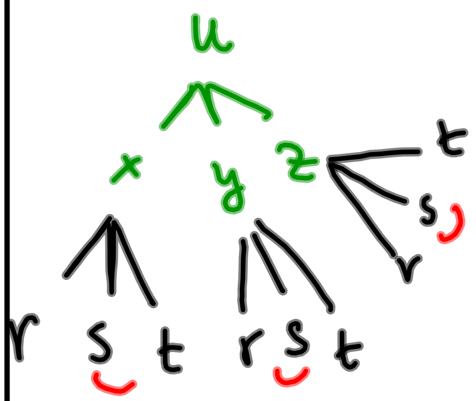
$$= \cos x \cos y \left(\underbrace{2(s-t) - 2(s-t)}_{=0} \right)$$

$$+ \sin x(-\sin y)(2s - 2t)$$

$$= 2 \sin x \sin y \,(t - s)$$

$$= 2 \sin (s-t)^2 \sin(s^2 - t^2)\,(t - s)$$

EXAMPLE 5. *If* $u = x^2y + y^3z^2$ *where* $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs\sin t$, *find* u_s *when* $(r, s, t) = (1, 2, 0)$



$$u_s = u_x x_s + u_y y_s + u_z z_s$$

$$= 2xy\, re^t + (x^2 + 3y^2z^2)2se^{-t} + 2y^3z\, r\sin t$$

$x(1,2,0) = 1 \cdot 2e^0 = 2$

$y(1,2,0) = 1 + 2^2e^{-0} = 5$

$z(1,2,0) = 1 \cdot 2\sin 0 = 0$

$$u_s\Big|_{(r,s,t)=(1,2,0)} = 2 \cdot 2 \cdot 5 \cdot 1 \cdot e^0 + (2^2 + 0) \cdot 2 \cdot 2e^{-0}$$

$$= 20 + 4 \cdot 4 = \boxed{36}$$

Implicit differentiation: Suppose that an equation

$$F(x,y)=0$$

defines y implicitly as a differentiable function of x, i.e. $y=y(x)$, where $F(x,y(x))=0$ for all x in the domain of $y(x)$. Find y':

$$\begin{tikzcd} & F \arrow[dl, dash] \arrow[dr, dash] & \\ x & & y \arrow[d, dash] \\ & & x \end{tikzcd}$$

$$F(x, y(x)) = 0$$

$$F_x + F_y\, y' = 0$$

$$\boxed{y' = -\frac{F_x}{F_y}}$$

EXAMPLE 6. *Find y' if $x^4+y^3=6e^{xy}$.*

$$F(x,y) = x^4 + y^3 - 6e^{xy} = 0$$

$$y' = -\frac{F_x}{F_y} = -\frac{4x^3 - 6ye^{xy}}{3y^2 - 6xe^{xy}}$$

Suppose that an equation

$$F(x, y, z) = 0 \qquad \left(\text{or } F(x,y,z) = k \right)$$

defines z implicitly as a differentiable function of x and y, i.e. $z = z(x, y)$, where

$$F(x, y, z(x, y)) = 0$$

for all (x, y) in the domain of z. Find the partial derivatives z_x and z_y:

$$\frac{\partial}{\partial x}\left[F(x, y, z(x, y))\right] = \frac{\partial}{\partial x}(0)$$

$$F_x + F_z \cdot z_x = 0$$

$$z_x = -\frac{F_x}{F_z} = \frac{\partial z}{\partial x}$$

Similarly, $z_y = -\dfrac{F_y}{F_z} = \dfrac{\partial z}{\partial y}$

EXAMPLE 7. *If* $\underbrace{x^4 + y^3 + z^2 + xye^z}_{F(x,y,z)} = 10$ *find*

(a) z_x *and* z_y

$$F_x = 4x^3 + ye^z$$

$$F_y = 3y^2 + xe^z$$

$$F_z = 2z + xye^z$$

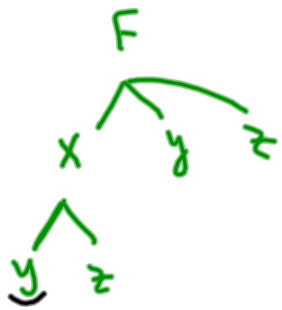
$$z_x = -\frac{F_x}{F_z} = -\frac{4x^3 + ye^z}{2z + xye^z}$$

$$z_y = -\frac{F_y}{F_z} = -\frac{3y^2 + xe^z}{2z + xye^z}$$

plug in

(b) x_y *and* x_z $\quad x = x(y,z)$

$$F(x(y,z), y, z) = 10$$



$$F_x x_y + F_y = 0 \Rightarrow x_y = \frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$$

$$F_x x_z + F_z = 0 \Rightarrow x_z = \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}$$