12.6: Directional Derivatives and the Gradient Vector

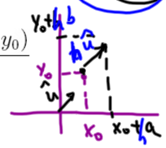
Recall that the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y)$ represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. In other words, $f_x(x,y)$ and $f_y(x,y)$ represent the rate of change of f in the directions of the unit vectors **i** and **j** respectively. Let's consider how to find the rate of change of f if we allow both x and y to change simultaneously, or how to find the rate of change of f in the direction of an arbitrary vector **u**.

DEFINITION 1. *The rate of change of $f(x,y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is called the* **directional derivative** *and it is denoted by $D_{\mathbf{u}}f(x,y)$.*

$(x_0, y_0) \longrightarrow \langle x_0, y_0\rangle + \hat{u}$

The **directional derivative** *of f at (x_0,y_0) in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is*

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$



if this limit exists.

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}, \quad f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$$

$\hat{u} = \langle 0, 1\rangle = \hat{j}$

we see that $\hat{u} = \langle 1, 0\rangle = \hat{i}$

$f_x(x_0,y_0) = D_{\hat{i}}\, f(x_0, y_0)$ and $f_y(x_0,y_0) = D_{\hat{j}}\, f(x_0, y_0)$

For computational purposes use the following theorem.

means that there is tangent plane at (x,y)

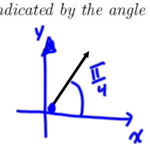
THEOREM 3. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ and*

$a^2 + b^2 = 1$

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

EXAMPLE 4. *Find the rate of change $f(x,y) = x^3 + \sin(xy)$ at the point $(1, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*

"directional derivative



$\hat{u} = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}\rangle = \langle \underbrace{\tfrac{\sqrt{2}}{2}}_{a}, \underbrace{\tfrac{\sqrt{2}}{2}}_{b}\rangle$

$f_x = 3x^2 + y\cos(xy)$

$f_y = x\cos(xy)$

$f_x(1, \frac{\pi}{2}) = 3 + \frac{\pi}{2}\cos\frac{\pi}{2} = 3$

$f_y(1, \frac{\pi}{2}) = 0$

$D_{\vec{u}}f(1, \frac{\pi}{2}) = f_x(1, \frac{\pi}{2})a + f_y(1, \frac{\pi}{2})b$

$= 3 \cdot \frac{\sqrt{2}}{2} + 0 \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{3\sqrt{2}}{2}}$

The Directional Derivative As The Dot Product Of Two Vectors. Gradient.

DEFINITION 5. *The* gradient *of $f(x,y)$ is the vector function ∇f defined by*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notations for gradient: grad$f$ or ∇f which is read "del f". nabla

EXAMPLE 6. *Find the gradient of $f = \cos(xy) + e^x$ at (0,3).*

$$f_x = -y\sin(xy) + e^x \Big|_{(0,3)} = 0 + e^0 = 1$$

$$f_y = -x\sin(xy) \Big|_{(0,3)} = 0$$

$$\nabla f(0,3) = \langle 1, 0\rangle = \hat{i}$$

By Theorem 3 we have:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x, f_y\rangle \cdot \langle a, b\rangle$$

Formula for the directional derivative using the gradient vector:

$$\boxed{D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \hat{\mathbf{u}}.}$$

unit

EXAMPLE 7. *Find the directional derivative for f from Example 6 at $(0,3)$ in the directio*
$\langle 3,4\rangle = \vec{u}$

By Ex.6 $\nabla f(0,3) = \langle 1,0\rangle$

$$D_{\vec{u}} f(0,3) = \nabla f(0,3)\cdot \hat{u} = \langle 1,0\rangle \cdot \frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}$$

$$= \frac{1\cdot 3 + 0\cdot 4}{\sqrt{3^2+4^2}} = \boxed{\frac{3}{5}}$$

Title : Sep 25-4:29 PM (Page 2 of 7)

The directional derivative of function of *three* variables

THEOREM 8. *If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector* $\hat{\mathbf{u}} = \langle a, b, c\rangle$ *and*

$$D_{\mathbf{u}} f(x,y) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \underbrace{\nabla f \cdot \hat{\mathbf{u}}},$$

where

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is the gradient vector of $f(x,y,z)$.

EXAMPLE 9. *Find the directional derivative of* $f(x,y,z) = z^3 - x^2 y$ *at the point* $(1,6,2)$ *in the direction* $\mathbf{u} = \langle 1, -2, 3\rangle$.

$$\left.\begin{aligned} f_x &= -2xy \Big|_{(1,6,2)} = -12 \\ f_y &= -x^2 \Big|_{(1,6,2)} = -1 \\ f_z &= 3z^2 \Big|_{(1,6,2)} = 12 \end{aligned}\right\} \Rightarrow \nabla f(1,6,2) = \langle -12, -1, 12\rangle$$

$$\left.\begin{aligned} &\text{Normalize } \hat{u} = \frac{\vec{u}}{|\vec{u}|} \\ &|\vec{u}| = \sqrt{1^2 + 4 + 9} = \sqrt{14} \end{aligned}\right\} \Rightarrow \hat{u} = \frac{\langle 1, -2, 3\rangle}{\sqrt{14}}$$

$$D_u f(1,6,2) = \nabla f(1,6,2) \cdot \hat{u} = \frac{\langle -12, -1, 12\rangle \cdot \langle 1, -2, 3\rangle}{\sqrt{14}}$$

$$= \frac{-12 + 2 + 36}{\sqrt{14}} = \boxed{\frac{26}{\sqrt{14}}}$$

QUESTION: In which of all possible directions does f change fastest and what is the maximum rate of change.

ANSWER is provided by the following theorem:

THEOREM 10. *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector ∇f.*

Proof. see Ex. 11 (Sec. 11.1-11.3)

$$\max_{\vec{u}} D_{\vec{u}} f = |\nabla f| = D_{\nabla f} f$$

EXAMPLE 11. *Suppose that the temperature at a point (x, y, z) in the space is given by*

$$T(x,y,z) = \frac{100}{1+x^2+y^2+z^2},$$

where T is measured in $°C$ and x, y, z in meters.

(a) *In which direction does the temperature increase fastest at the point $(1, 1, -1)$?*

in the direction of the gradient at $(1,1,-1)$

$$\nabla T = \langle T_x, T_y, T_z \rangle = \left\langle -\frac{200x}{(1+x^2+y^2+z^2)^2}, -\frac{200y}{(1+x^2+y^2+z^2)^2}, -\frac{200z}{(1+x^2+y^2+z^2)^2} \right\rangle$$

$$\left(\frac{1}{u}\right)' = -\frac{1}{u^2}$$

$$\nabla T(1,1,-1) = \left\langle \frac{-200}{16}, -\frac{200}{16}, \frac{200}{16} \right\rangle$$

$$= \left\langle -\frac{25}{2}, -\frac{25}{2}, \frac{25}{2} \right\rangle = -\frac{25}{2}\langle 1,1,-1 \rangle$$

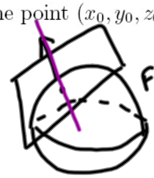
(b) *What is the maximum rate of increase?*

$$|\nabla T(1,1,-1)| = \left|-\frac{25}{2}\langle 1,1,-1\rangle\right| = \frac{25}{2}|\langle 1,1,-1\rangle|$$

$$= \frac{25}{2}\sqrt{3} \ °C/m$$

Tangent planes to level surfaces:

FACT: The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $F(x,y,z) = k$ at the point (x_0, y_0, z_0).



$F(x,y,z)=k$

$\vec{n} = \nabla F$ at point P

$\parallel$

$\langle F_x, F_y, F_z \rangle$

So, the *tangent plane* to the surface $f(x,y,z) = k$ at the point (x_0, y_0, z_0) has the equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to the surface at the point (x_0, y_0, z_0) is the line passing through (x_0, y_0, z_0) and perpendicular to the tangent plane. Therefore its direction is given by the $\vec{v} = \nabla F(x_0, y_0, z_0)$ vector

EXAMPLE 13. *Find the equation of the tangent plane and normal line at the point* $\underbrace{(1,0,5)}_{(x_0,y_0,z_0)}$ *to the surface* $xe^{yz} = 1$.

$F(x,y,z) = xe^{yz}$

$\vec{n}(x,y,z) = \nabla F = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$ $\parallel \langle 1, xz, xy \rangle$ BTW one can use it as a normal

$\vec{n}(1,0,5) = \langle 1, 5, 0 \rangle = \vec{v}$ a direction vector for normal line

$1 = a,\ 5 = b,\ 0 = c$

Tangent plane

$1\cdot(x-1) + 5(y-0) + 0(z-5) = 0$

$$\boxed{x + 5y = 1}$$

Normal line $\boxed{\frac{x-1}{1} = \frac{y-0}{5},\ z = 5}$

$\nabla f(x_0, y_0, z_0) \perp$ to the level surface $f(x, y, z) = k$.

Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x, y) = k$ at the point (x_0, y_0).

$\nabla f(x_0, y_0) \perp$ to the level curve $f(x, y) = k$

Consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y). Draw a curve of steepest ascent.