12.7: Maximum and minimum values

Function $y = f(x)$ Calc 1

Function of two variables $z = f(x, y)$ Calc 3

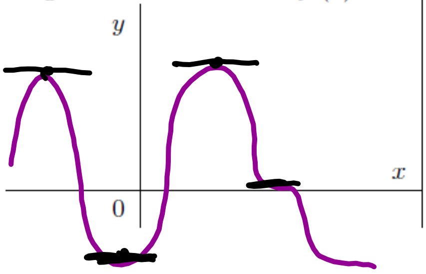
DEFINITION 1. *A function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ when x is near a (i.e. in a neighborhood of a). A function f has a local minimum at $x = a$ if $f(a) \leq f(x)$ when x is near a.*

DEFINITION 2. *A function $f(x, y)$ has a local maximum at $(x, y) = (a, b)$ if $f(a, b) \geq f(x, y)$ when (x, y) is near (a, b) (i.e. in a neighborhood of (a, b)). A function f has a local minimum at $(x, y) = (a, b)$ if $f(a, b) \leq f(x, y)$ when (x, y) is near (a, b).*

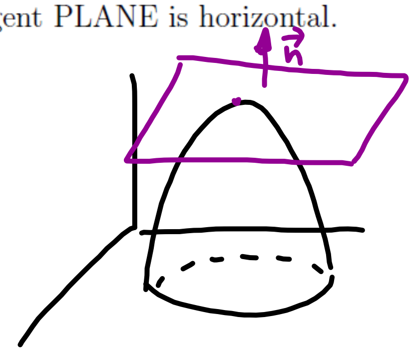
If the inequalities in this definition hold for ALL points x in the domain of f, then f has an **absolute max** (or **absolute min**) at a

If the inequalities in this definition hold for ALL points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

If the graph of f has a tangent line at a local extremum, then the tangent line is horizontal: $f'(a) = 0$.



If the graph of f has a tangent plane at a local extremum, then the tangent PLANE is horizontal.



THEOREM 3. *If f has a local extremum (that is, a local maximum or minimum) at (a, b) and first-order partial derivatives exist there, then*

$$f_x(a, b) = f_y(a, b) = 0 \quad (or,\ equivalently, \nabla f(a, b) = 0.)$$

Indeed, if tangent plane is horizontal then its normal vector must be $\parallel \hat{k} = \langle 0, 0, 1 \rangle$.

It means, $\langle f_x(a,b), f_y(a,b), -1 \rangle \parallel \langle 0, 0, 1 \rangle$

$\Updownarrow$

$f_x(a, b) = f_y(a, b) = 0$.

$\nabla f(a,b) = \bar{0}$

DEFINITION 4. *A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

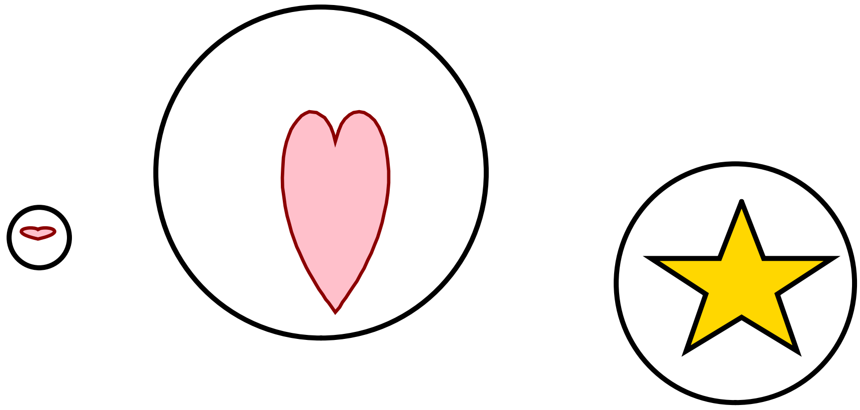
At a critical point, a function could have a local max or a local min, or neither.
We will be concerned with two important questions:

- Are there any local or absolute extrema?
- If so, where are they located?

SETS in $\mathbb{R}^2$

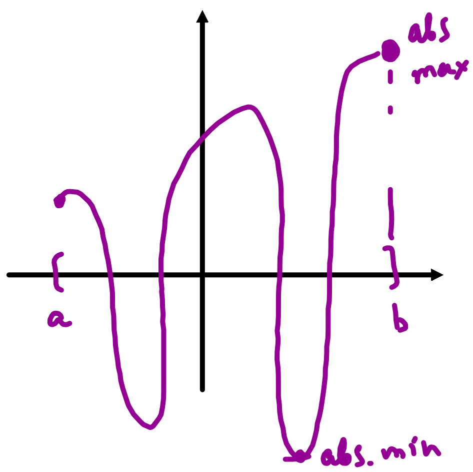
in $\mathbb{R}$	in $\mathbb{R}^2$
close interval $[a, b]$ $a \leq x \leq b$	close set $\{(x,y) \mid x^2+y^2 \leq 1\}$
open interval (a, b) $a < x < b$	open set $\{(x,y) \mid x^2+y^2 < 1\}$
end points of an interval $[a, b]$ $x=a, \; x=b$	boundary points $\{(x,y) \mid x^2+y^2=1\}$ boundary $D=\{(x,y) \mid x \geq 0\}$ $\partial D=\{(x,y) : x=0\}$

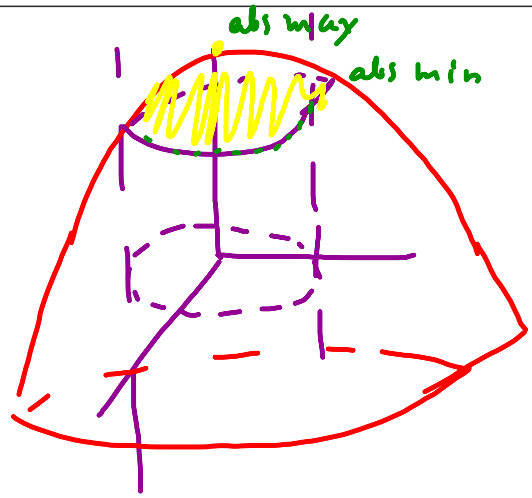
DEFINITION 5. *A* **bounded set** *in* $\mathbb{R}^2$ *is one that contained in some disk.*



THE EXTREME VALUE THEOREM:

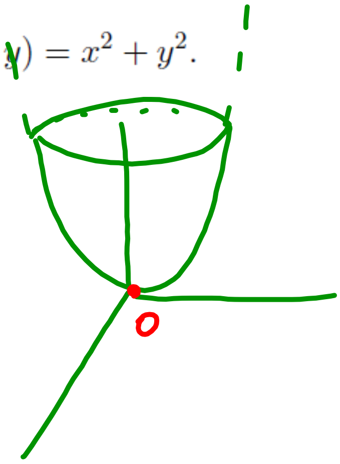
Function $y = f(x)$	Function of two variables $z = f(x, y)$
If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(x_1)$ and an absolute minimum value $f(x_2)$ at some points x_1 and x_2 in $[a, b]$.	If f is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.





EXAMPLE 6. *Find extreme values of* $f(x, y) = x^2 + y^2$.

	Local	Absolute
Maximum	NO	NO
Minimum	at (0, 0)	at (0, 0)

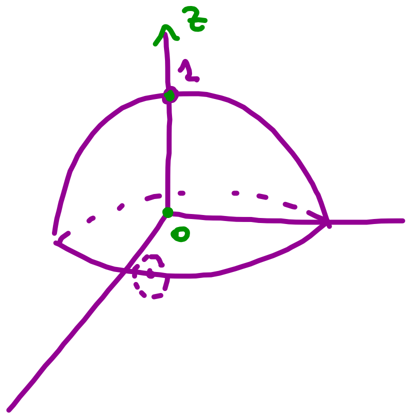


Domain: $\mathbb{R}^2$ unbdd
not close

EXAMPLE 7. *Find extreme values of* $f(x, y) = \sqrt{1 - x^2 - y^2}$.

	Local	Absolute
Maximum	at $(0,0)$	at $(0,0)$ $\max_D = f(0,0) = 1$
Minimum	NO	on ∂D $\min_D f(x,y) = 0$

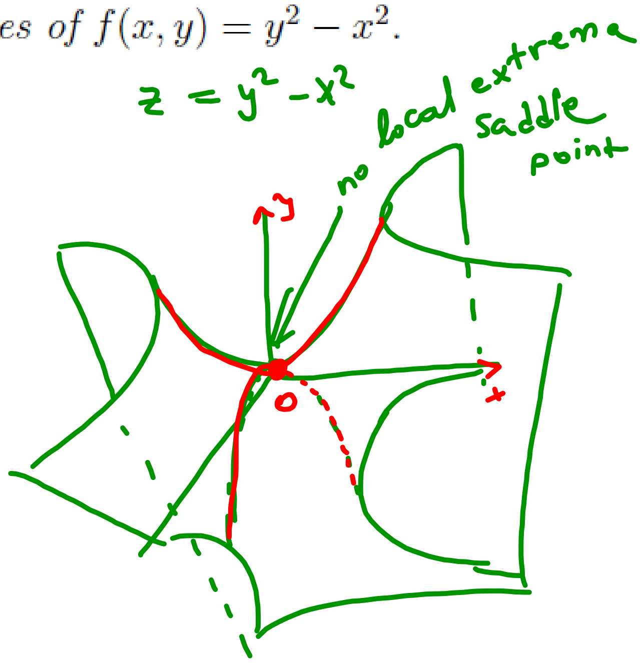
Domain: $\{(x,y) \mid x^2 + y^2 \le 1\} = D$



EXAMPLE 8. *Find extreme values of* $f(x,y) = y^2 - x^2$.

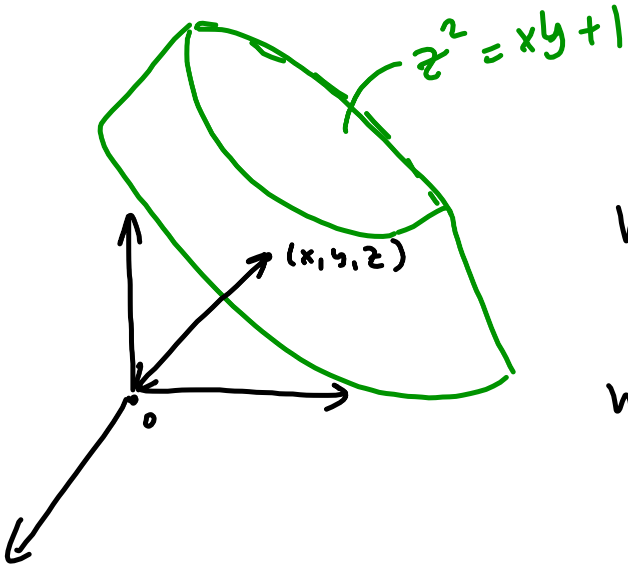
	Local	Absolute
Maximum	NO	NO
Minimum	NO	NO

Domain: $\mathbb{R}^2$



REMARK 9. Example 8 illustrates so called **saddle point** of f. Note that the graph of f crosses its tangent plane at (a,b).

EXAMPLE 10. *Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.*



Find abs. min
on unbdd open region
We cannot apply the extreme value theorem here.

Will be done
on recitation

ABSOLUTE MAXIMUM AND MINIMUM VALUES on a closed bounded set.

THE EXTREME VALUE THEOREM:

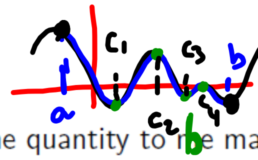
Calc 1 $y=f(x)$

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

1. Find the values of f at the critical points of f in (a,b).

2.Find the ~~extreme~~ values of f at the endpoints of the interval.

3.The largest of the values from steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.



$\boxed{\begin{matrix} f(c_1) & f(c_3) & f(a) \\ f(c_2) & f(c_4) & f(b) \end{matrix}}$

Calc 3

To find the absolute max and min values of a continuous function f on a closed bounded set D:

1. Find the values of f at the critical points of f in D.

2.Find the extreme values of f on the boundary of D.(This usually involves the Calculus I approach for this work.) +12.8

3. The largest of the values from steps 1&2 is the absolute maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.

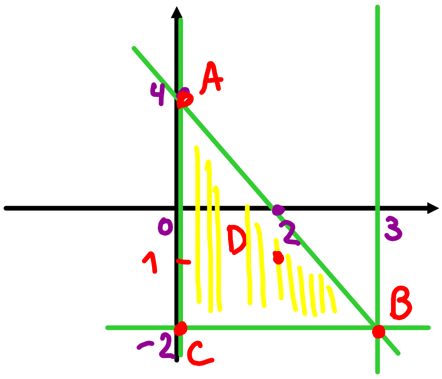
- The quantity to be maximized/minimized is expressed in terms of variables (as few as possible!)
- Any constraints that are presented in the problem are used to reduce the number of variables to the point they are independent,
- After computing partial derivatives and setting them equal to zero you get purely algebraic problem (but it may be hard.)
- Sort out extreme values to answer the original question.

EXAMPLE 11. *A lamina occupies the region* $D = \{(x, y) : \ 0 \le x \le 3, \ \ -2 \le y \le 4 - 2x\}$. *The temperature at each point of the lamina is given by*

$$T(x, y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

Find the hottest and coldest points of the lamina. (Find abs. max and abs. min of $T(x,y)$)

Label corners



① Find critical points in D

$\nabla T = \vec{0}$, OR

$$\begin{cases} T_x = 4(2x+y-3) = 0 \\ T_y = 4(\underbrace{x+4y+2) = 0}) \end{cases} \quad (\times -2)$$

$$\begin{array}{r} 2x + y = 3 \\ + \ -2x - 8y = 4 \\ \hline -7y = 7 \\ y = -1 \end{array}$$

$x = -4y - 2$

$x = -4\cdot(-1) - 2$

$x = 2$

Critical point $(2,-1)$

Note $(2,-1) \in D$

$T(2,-1) = 4(\cancel{4} - \cancel{2} + 2 - 6 - \cancel{2}) + 10$

$= -16 + 10 = \boxed{-6}$

$(2,-1)$	A	B	C	$(\frac{5}{2},-2)$	$(\frac{5}{2},-1)$
-6	170	2	26	1	35

$\max_D T = T(0,4) = 170$ degrees

$\min_D T = T(2,-1) = -6$

Boundary of D :

$\partial D = \overline{AC} \cup \overline{CB} \cup \overline{AB}$

Parameterize all parts of the Boundary separately and find critical points of T there + valued of T at end points.

Find values of T at corners:

$T(A) = f(0,4) = 4(32+8)+10 = \boxed{170}$

$T(B) = f(3,-2) = \boxed{2}$ mod Daniel

$T(C) = f(0,-2) = 4(8-4)+10 = \boxed{26}$

$\overline{AC}$: $x=0$, $-2 \le y \le 4$

$T\big|_{\overline{AC}} = T(0,y) = 4(2y^2+2y)+10 = h(y)$

Find critical values of $h(y)$ on $[-2,4]$

$h'(y) = 4(4y+2) = 0 \Rightarrow \underbrace{y=-2}_{\text{the same as the point } C}$

$\overline{CB}$: $y=-2$, $0 \le x \le 3$

$T\big|_{\overline{CB}} = T(x,-2) = 4(x^2-5x+4)+10$

$4(x^2-2x+8-3x-4)+10 = g(x)$

Find critical values of g on $(0,3)$

$g'(x) = 2x-5 = 0 \Rightarrow x = +\frac{5}{2}$ in D because $0 < \frac{5}{2} < 3$

$g\left(\frac{5}{2}\right) = 4\left(\frac{25}{4} - \frac{25}{2} + 4\right) + 10 = \boxed{1}$

$\overline{AB}$: $y = 4-2x$, $0 \le x \le 3$

$T\big|_{\overline{AB}} = T(x, 4-2x)$

$= 4(x^2 + x(4-2x) + 2\overbrace{(4-2x)^2}^{16-16x+4x^2} - 3x + 2(4-2x)) + 10$

$= 4(x^2 + 4\cancel{x} - \cancel{8}x^2 + 32 - 32x + 8x^2 - 3x + 8 - 4\cancel{x}) + 10$

$= 4(7x^2 - 35x + 40) + 10 = m(x)$

$m'(x) = 4(14x-35) = 0$

$x = \frac{35}{14} = \frac{5}{2}$ in D b/c $0 < \frac{5}{2} < 3$

$m\left(\frac{5}{2}\right) = 4\left(\frac{7 \cdot 25}{4} - \frac{35 \cdot 5}{2} + 40\right) + 10$

$= \boxed{35}$

$4 \cdot 5\left(\underbrace{\frac{35}{4} - \frac{35}{2}}_{-\frac{35}{4}} + 8\right) + 10$

$-\frac{35}{4} + 8$

Second derivatives test:

Suppose f'' is continuous near a and $f'(c) = 0$ (i.e. a is a critical point).

- If $f''(c) > 0$ then $f(c)$ is a local minimum.
- If $f''(c) < 0$ then $f(c)$ is a local maximum.

NOTE:

- If $f''(c) = 0$, then the test gives no information.

Suppose that the second partial derivatives of f are continuous near (a, b) and $\nabla f(a, b) = \mathbf{0}$ (i.e. (a, b) is a critical point). Let $\mathcal{D} = \mathcal{D}(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$

- If $\mathcal{D} > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
- If $\mathcal{D} > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
- If $\mathcal{D} < 0$ then $f(a, b)$ is not a local extremum (saddle point).

If $\mathcal{D} = 0$ or does not exist, then the test gives no information. fails.

To remember formula for $\mathcal{D}$:

$$\mathcal{D} = f_{xx} f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

EXAMPLE 12. *Use the Second Derivative Test to confirm that a local cold point of the lamina in the previous Example is* $(2, -1)$. critical

$$T(x, y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

$T_x = 4(2x+y-3)$:

$T_y = 4(\underbrace{x+4y+2})$

$T_{xx} = 4 \cdot 2 = 8$

$T_{xy} = 4$

$T_{yy} = 16$

$$D = \begin{vmatrix} T_{xx} & T_{xy} \\ T_{xy} & T_{yy} \end{vmatrix} = \begin{vmatrix} 8 & 4 \\ 4 & 16 \end{vmatrix}$$

$$= 8 \cdot 16 - 4 \cdot 4 > 0$$

$$T_{xx}(2,-1) = 8 > 0$$

$\Rightarrow$

$\Rightarrow (2,-1)$ is local min

$\Rightarrow (2,-1)$ is also a local cold point

EXAMPLE 13. *Find the local extrema of* $f(x, y) = x^3 + y^3 - 3xy$.

Solution: Find critical points:

$f_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y$

$f_y = 3y^2 - 3x = 0 \Rightarrow y^2 = x$

$(x^2)^2 = x \Rightarrow x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ OR $x^3 = 1$

$x = 1$

$y = x^2 \to y = 0$ $\qquad y = 1$

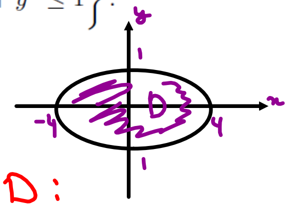
Calculate the second partial derivatives and D.

	$(0,0)$	$(1,1)$
$f_{xx} = 6x$	0	$6 > 0$
$f_{xy} = -3$	-3	-3
$f_{yy} = 6y$	0	6
D	$\begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$	$\begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 > 0$
	Saddle	local min

EXAMPLE 14. *The mountain is defined by* $z = xy$ *in the elliptical domain*

$$D = \left\{ (x, y) \mid \frac{x^2}{16} + y^2 \le 1 \right\}.$$

(a) *Find the top of the mountain.*



In other words, find absolute maximum of $z = xy$ on D.

First find critical points of z in D:

$z_x = y = 0$
$z_y = x = 0$ $\Rightarrow (0,0) \in D \Rightarrow z(0,0) = \boxed{0}$

Second, find critical values of z on the boundary of D!

$$\partial D = \left\{ (x,y) \mid \frac{x^2}{16} + y^2 = 1 \right\}$$

Method 1: parametrize ∂D

use Lagrange Multipliers (see the next section 12.8)

$x = x$, $y = \pm\sqrt{1 - \frac{x^2}{16}}$

another parametrization

$x = 4\cos\theta$
$y = \sin\theta$
$0 \le \theta \le 2\pi$

Note that $z(x,y) = z(-x,-y)$

Thus, one can consider $0 \le \theta \le \pi$

$$z\Big|_{\partial D} = xy\Big|_{\partial D} = 4\cos\theta\sin\theta = 2 \cdot \underbrace{2\cos\theta\sin\theta}_{\sin 2\theta} = 2\sin 2\theta = h(\theta)$$

Look for critical points of $h(\theta)$ on $(0, \pi)$

$h'(\theta) = 0 \Leftrightarrow 4\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$ or $2\theta = \frac{3\pi}{2}$

$\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$

$\boxed{2} = h\left(\frac{\pi}{4}\right) = z\left(4\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = z\left(2\sqrt{2}, \frac{\sqrt{2}}{2}\right) = z\left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$

$-2 = h\left(\frac{3\pi}{4}\right)$

End points $h(0) = h(2\pi) = 0$

Conclusion: the mountain has two tops at

$\left(2\sqrt{2}, \frac{\sqrt{2}}{2}, 2\right)$ and $\left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}, 2\right)$.

(b) *Is the critical point found in the previous item the highest or the lowest in its neighborhood?*

In other words, if $(0,0)$ is the local max or local min?

Use Second Derivative test:

$$\left.\begin{array}{l} z_{xx} = 0 \\ z_{xy} = 1 \\ z_{yy} = 0 \end{array}\right\} \Rightarrow D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} < 0$$

$(0,0)$ is saddle point

(it means that there is a pass at $(0,0,0)$).