12.8: Lagrange Multipliers

PROBLEM: Maximize/minimize a general function $u = f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

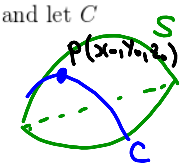
max / min $f(x, y, z)$
when $g(x, y, z) = k$

OR
max / min $f(x, y)$
when $g(x, y) = k$

SOLUTION:

Note that the equation $g(x, y, z) = k$ represents a surface in $\mathbb{R}^3$. Denote this surface by S.

Suppose that f has an extreme value at a point $P(x_0, y_0, z_0)$ on S and let C be a curve with vector equation:



$$C: \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

that lies on S and passes through P.

If t_0 is the parameter value corresponding to the point P then

$r(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle = \langle x_0, y_0, z_0 \rangle = \vec{OP}$

The values that f takes on the curve C:

$$f\big|_C = f(\vec{r}(t)) = f(x(t), y(t), z(t))$$

Since f has an extreme value at $P(x_0, y_0, z_0)$, it follows that

$$\frac{d f|_C}{dt}\Bigg|_{t=t_0} = 0 \iff f_x \cdot x' + f_y \cdot y' + f_z \cdot z' = 0$$

$$\langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle = 0 \text{ when } t = t_0$$

$$\nabla f \cdot \vec{r}\,'(t_0) = 0$$

$$\nabla f \perp \vec{r}\,'(t_0)$$

Since C is an arbitrary curve through P,

we conclude that $\nabla f(P) \perp S$ at C.
(in the sense that $\nabla f(P) \perp$ tangent plane at P)

On the other hand, S: $g(x, y, z) = K$

and we know that $\nabla g(P)$ is a normal to the tangent plane at P, i.e. $\nabla g(P) \perp S$.

We obtained:

$$\left.\begin{array}{l} \nabla f(P) \perp S \\ \nabla g(P) \perp S \end{array}\right\} \Rightarrow \nabla f(P) \parallel \nabla g(P)$$

$\Rightarrow$ there exists a real scalar λ such that

$$\nabla f(P) = \lambda \nabla g(P).$$

$$\nabla f(P) - \lambda \nabla g(P) = 0.$$

METHOD OF LAGRANGE MULTIPLIERS: *To Maximize/minimize a general function $u = f(x, y, z)$ subject to a constraint of the form $g(x, y, z) = k$ (assuming that these extreme values exist):*

1. *Find all values x, y, z and λ (a Lagrange multiplier) s.t.*

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. *Evaluate f at all points (x, y, z) that arise from the previous step. The largest of these values is the* max f*; the smallest is the* min f*.*

Rewrite the system

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$$

in component form:.

$$\begin{cases} f_x = \lambda \, g_x \\ f_y = \lambda \, g_y \\ f_z = \lambda \, g_z \\ g(x,y,z) = k \end{cases}$$

system with 4 equations and 4 unknowns: x, y, z, λ

Method of Lagrange Multipliers for function of two variables:

max / min $\quad z = f(x, y)$

under condition $\quad g(x, y) = k$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases} \xrightarrow[\text{form}]{\text{in component}} \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$$

EXAMPLE 2. *Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to $\underbrace{x^4 + y^4 = 1}_{g(x,y)=1}$.*

$$\boxed{\begin{array}{c}\nabla f = \lambda \nabla g \\ g(x,y) = k\end{array}}$$

$f_x = 2x \qquad g_x = 4x^3$

$f_y = 2y \qquad g_y = 4y^3$

$$\begin{cases} 2x = \lambda \cdot 4x^3 \\ 2y = \lambda \cdot 4y^3 \\ x^4 + y^4 = 1 \end{cases} \qquad \begin{cases} x - 2\lambda x^3 = 0 \\ y - 2\lambda y^3 = 0 \\ x^4 + y^4 = 1 \end{cases}$$

Factor

$$\begin{cases} x(1-2\lambda x^2) = 0 \\ y(1-2\lambda y^2) = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$$\begin{cases} x = 0 \\ y(1-2\lambda y^2) = 0 \\ x^4 + y^4 = 1 \end{cases} \quad \text{or} \quad \begin{cases} 1-2\lambda x^2 = 0 \\ y(1-2\lambda y^2) = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$0^4 + y^4 = 1 \Rightarrow y = \pm 1$

$\boxed{(0, \pm 1)}$

$f(0, \pm 1) = \boxed{1}$

$$\begin{cases} 1-2\lambda x^2 = 0 \\ y = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$x^4 = 1$

$x = \pm 1$

$\boxed{(\pm 1, 0)}$

$f(\pm 1, 0) = \boxed{1}$

$$\begin{cases} 1-2\lambda x^2 = 0 \\ 1-2\lambda y^2 = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$x^2 = y^2$

$x = \pm y$

$2x^4 = 1$

$x^4 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt[4]{2}}$

$\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right)$

$f\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right) = 2\left(\frac{1}{\sqrt[4]{2}}\right)^2 = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$

$\max\limits_{x^2+y^2=1} f = \sqrt{2}$

$\min\limits_{x^2+y^2=1} f = 1$

EXAMPLE 1. *Use Lagrange multipliers to solve Example 10 from Section 12.7:* Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

$$\min_{z^2=xy+1} d\big((x,y,z),(0,0,0)\big) = \sqrt{x^2+y^2+z^2}$$

Equivalently,

$$\min_{z^2=xy+1} x^2+y^2+z^2 = f(x,y,z)$$

$$g(x,y,z) = z^2 - xy = 1$$

$$\boxed{\begin{array}{c}\nabla f = \lambda \nabla g \\ g(x,y,z)=k\end{array}}$$

$f_x = 2x \qquad g_x = -y$

$f_y = 2y \qquad g_y = -x$

$f_z = 2z \qquad g_z = 2z$

$$\begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ 2z = 2\lambda z \iff z - \lambda z = 0 \iff z(1-\lambda) = 0 \\ z^2 = xy+1 \end{cases}$$

$z=0$ $\qquad$ $\lambda = 1$

$$\begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ z = 0 \\ xy+1 = 0 \end{cases} \qquad \begin{cases} 2x = -y \\ 2y = -x \\ \lambda = 1 \\ z^2 = xy+1 \end{cases}$$

$\Downarrow$ $\qquad$ $\Downarrow$

Note that $x \neq 0$, $y \neq 0$

$\Downarrow$

$$\begin{cases} -\lambda = \dfrac{2x}{y} \\ -\lambda = \dfrac{2y}{x} \\ z=0 \\ xy+1=0 \end{cases} \Rightarrow \frac{2x}{y} = \frac{2y}{x}$$

$$x^2 = y^2$$

$x = y$ OR $x = -y$

$x^2+1=0$

no solutions

$-x^2+1=0$

$x = \pm 1$

$y = \mp 1$

$(1,-1,0)$

$(-1,1,0)$

$f(-1,1,0) = f(1,-1,0) = \boxed{2}$

$\Downarrow \begin{cases} 2x = -y \\ 2y = -x \\ \lambda = 1 \\ z^2 = xy+1 \end{cases}$

$\Rightarrow y = 2x$

$\longrightarrow 2\cdot 2x = -x \Rightarrow x=0$

$\Downarrow$

$y=0$

$\Longrightarrow z^2 = 1 \Rightarrow z = \pm 1$

$$f(0,0,\pm 1) = \boxed{1}$$

$$\min_{z^2=xy+1} f = \boxed{1}$$

closest distance

$$d = \sqrt{f(0,0,\pm 1)} = \sqrt{1} = \boxed{1}$$