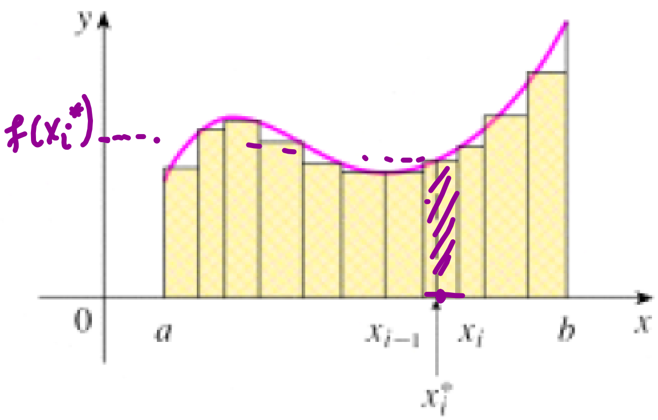
# 13.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:




$f \geq 0$

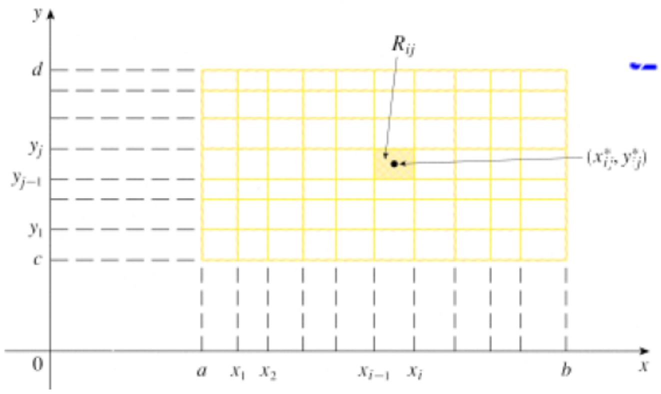
The exact area is also the definition of the definite integral:

$$\text{Area} = \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

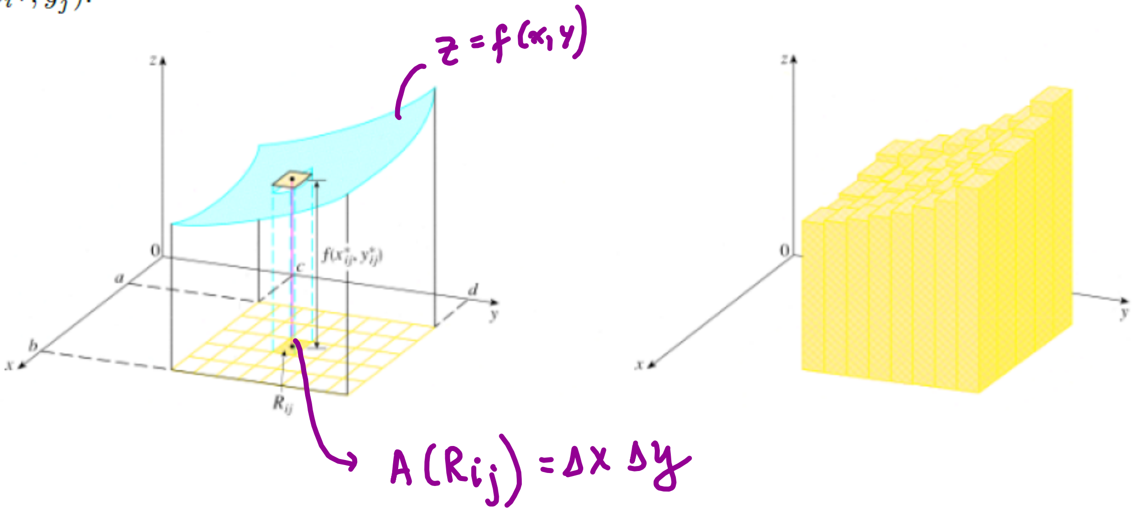
**Problem:** Assume that $f(x, y)$ is defined on a closed rectangle

$R = [a, b] \times [b, c] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$ and $f(x, y) \geq 0$ over $R$. Denote b
$S$ the part of the surface $z = f(x, y)$ over the rectangle $R$. What the volume of the regio
under $S$ and above the $xy$-plane is?

**Solution:** Approximate the volume. Divide up $a \leq x \leq b$ into $n$ subintervals and divide u
$c \leq y \leq d$ into $m$ subintervals. From each of these smaller rectangles choose a point $(x_i^*, y_j^*$



Over each of these smaller rectangles we will construct a box whose height is given b
$f(x_i*, y_j^*)$.




The volume is given by

$$\lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\overbrace{f(x_i^*, y_j^*)}^{\text{height}}\overbrace{\Delta x\Delta y}^{\text{base area}}$$

which is also the definition of a double integral

$$\parallel$$

$$\iint_R f(x,y)\mathrm{d}A.$$

Another notation: $\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$ (dA = dx dy only for rectangle R).

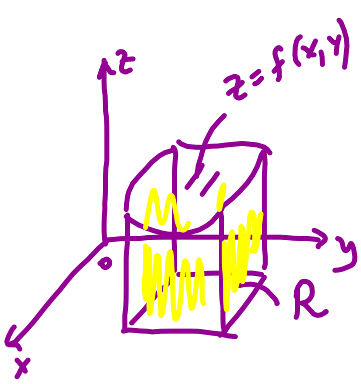
THEOREM 1. *If $f$ is continuous on $R$ then $f$ is integrable over $R$.*

THEOREM 2. *If $f(x,y)\geq 0$ and $f$ is continuous on the rectangle $R=[a,b]\times[c,d]$, then the volume $V$ of the solid $S$ that lies above $R$ and inder the graph of $f$, i.e.*

$$S=\left\{(x,y,z)\in\mathbb{R}^3|(x,y)\in R, 0\leq z\leq f(x,y), (x,y)\in R\right\},$$

*is*

$$V=\iint_R f(x,y)\,\mathrm{d}A.$$



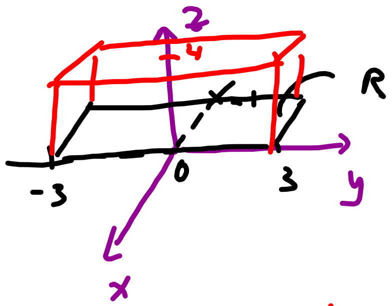
EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4\,\mathrm{d}A$$

*where* $\underbrace{R = [-1, 0] \times [-3, 3]}$ *by identifying it as a volume of a solid.*

$f(x, y) = 4$

$z = 4$ (lid)




$\iint_R 4\, dR =$ Volume of the parallelepiped with base R and height 4

$= A(R) \cdot 4 = 6 \cdot 1 \cdot 4 = 24$ units$^3$