

### 13.11: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule

$$\int_a^b f(g(x))g'(x) dx = \int_\alpha^\beta f(u) du.$$

- conversion to polar coordinates:

$$\iint_D f(x, y) dA = \iint_{D^*} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

- conversion to cylindrical coordinates:

$$\iiint_E f(x, y, z) dV = \iiint_{E^*} f(r \cos \theta, r \sin \theta, z) r dr dz d\theta$$

- conversion to spherical coordinates:

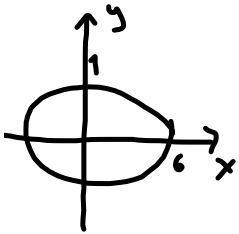
$$\iiint_E f(x, y, z) dV = \iiint_{E^*} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi.$$

We call the equations that define the change of variables a **transformation**:

$$x = x(u, v), \quad y = y(u, v).$$

*Handwritten notes in red ink:*  
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 - The word "ion" is written above the second dot in the list.  
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 - The word "ion" is written above the eighty-ninth dot in the list.  
 - The word "ion" is written above the ninetieth dot in the list.  
 - The word "ion" is written above the hundredth dot in the list.

EXAMPLE 1. Determine the new region that we get by applying the transformation  $x = 3u, y = v/2$  to the region  $D = \left\{ (x, y) \mid \frac{x^2}{36} + y^2 \leq 1 \right\}$ .



$$\frac{x^2}{36} + y^2 \leq 1$$

$$\frac{(3u)^2}{36} + \left(\frac{v}{2}\right)^2 \leq 1$$

$$\frac{u^2}{4} + \frac{v^2}{4} \leq 1$$

$$D^* = \left\{ (u, v) : u^2 + v^2 \leq 4 \right\}$$

DEFINITION 2. The **Jacobian** of the transformation  $x = x(u, v), y = y(u, v)$  is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 3. Compute the Jacobian of the transformation  $x = r \cos \theta, y = r \sin \theta$ .

$$\begin{aligned} J(r, \theta) &= \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta \\ &= r (\cos^2 \theta + \sin^2 \theta) = r \end{aligned}$$

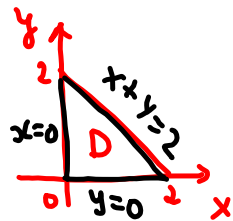
Change of variables for a double integral:

$$\iint_D f(x, y) dA = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

EXAMPLE 4. Evaluate

$$\iint_D e^{\frac{y-x}{y+x}} dA$$

where  $D$  is triangle with vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(0, 2)$ .



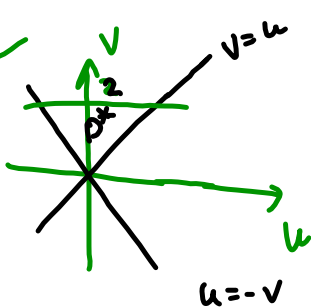
$$\begin{aligned} + & u = y - x & (1) \\ - & v = y + x & (2) \end{aligned}$$

$$(1)+(2) \quad u+v = 2y \Rightarrow y = \frac{u+v}{2}$$

$$(2)-(1) \quad v-u = 2x \Rightarrow x = \frac{v-u}{2}$$

$$\begin{cases} x=0 \Rightarrow \frac{v-u}{2} = 0 \Rightarrow v=u \\ y=0 \Rightarrow \frac{u+v}{2} = 0 \Rightarrow u=-v \\ x+y=2 \Rightarrow v=2 \end{cases}$$

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ +\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$



$$\iint_D e^{\frac{y-x}{y+x}} dA = \iint_{D^*} e^{\frac{u}{v}} |J(u, v)| du dv$$

$$= \int_0^2 \int_{-v}^v e^{\frac{u}{v}} \frac{1}{2} du dv$$

$$= \frac{1}{2} \int_0^2 \frac{1}{v} e^{\frac{u}{v}} \Big|_{u=-v}^v dv = \frac{1}{2} \int_0^2 v (e^{\frac{v}{v}} - e^{-\frac{v}{v}}) dv$$

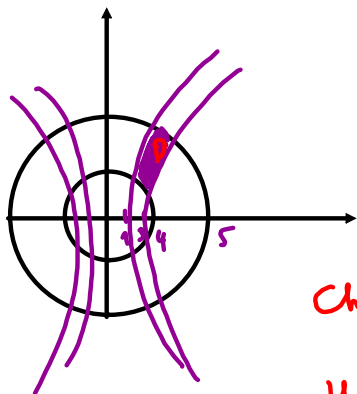
$$= \frac{1}{2} (e - \frac{1}{e}) \int_0^2 v dv = \frac{1}{2} (e - \frac{1}{e}) \frac{v^2}{2} \Big|_0^2$$

$$= \boxed{e - \frac{1}{e}}$$

EXAMPLE 5. Find mass of a lamina that occupies the region 1st quadrant

$$D = \{(x, y) \mid 16 \leq \underbrace{x^2 + y^2}_u \leq 25, \quad 1 \leq \underbrace{x^2 - y^2}_v \leq 9, \quad x \geq 0, y \geq 0\}$$

with density  $\rho(x, y) = 8xy$ .



$$m(D) = \iint_D \rho(x, y) \, dA$$

$$m(D) = \iint_D 8xy \, dA$$

$$m(D) = 8 \iint_D xy \, dA$$

Change of variables

$$u = x^2 + y^2 \quad (1)$$

$$v = x^2 - y^2 \quad (2)$$

$$(1) + (2): \quad u + v = 2x^2 \Rightarrow x = \frac{+}{-} \sqrt{\frac{u+v}{2}} = \frac{\sqrt{u+v}}{\sqrt{2}}$$

$$(1) - (2): \quad u - v = 2y^2 \Rightarrow y = \frac{+}{-} \sqrt{\frac{u-v}{2}} = \frac{\sqrt{u-v}}{\sqrt{2}}$$

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} \frac{1}{2\sqrt{u+v}} & \frac{1}{\sqrt{2}} \frac{1}{2\sqrt{u+v}} \\ \frac{1}{\sqrt{2}} \frac{1}{2\sqrt{u-v}} & -\frac{1}{\sqrt{2}} \frac{1}{2\sqrt{u-v}} \end{vmatrix}$$

$$= -\frac{1}{8} \frac{1}{\sqrt{u^2 - v^2}} - \frac{1}{8} \frac{1}{\sqrt{u^2 - v^2}} = -\frac{1}{4} \frac{1}{\sqrt{u^2 - v^2}}$$

$$m(D) = 8 \iint_{D^*} \frac{\sqrt{u+v}}{\sqrt{2}} \cdot \frac{\sqrt{u-v}}{\sqrt{2}} \cdot \left| -\frac{1}{4} \frac{1}{\sqrt{u^2 - v^2}} \right| \, du \, dv$$

$$= \iint_{D^*} du \, dv = \text{Area}(D^*) = (25 - 16)(9 - 1) = \boxed{72}$$

where  $D^* = \{(u, v) \mid 16 \leq u \leq 25, \quad 1 \leq v \leq 9\}$