13.11: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_\alpha^\beta f(u)\,\mathrm{d}u.$$

- conversa~~tion~~ion to polar coordinates:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\mathrm{d}\theta.$$

- conversa~~tion~~ion to cylindrical coordinates:

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

$$= \frac{\partial(x,y,z)}{\partial(r,\theta,z)}$$

- conversa~~tion~~ion to spherical coordinates:

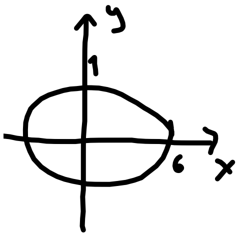
$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

$$= \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$$

We call the equations that define the change of variables a **transformation:**

$$x = x(u,v), \quad y = y(u,v).$$

EXAMPLE 1. *Determine the new region that we get by applying the transformation* $x = 3u, y = v/2$ *to the region* $D = \left\{(x, y) \mid \frac{x^2}{36} + y^2 \leq 1\right\}$.



$$\frac{x^2}{36} + y^2 \leq 1$$

$$\frac{(3u)^2}{36} + \left(\frac{v}{2}\right)^2 \leq 1$$

$$\frac{u^2}{4} + \frac{v^2}{4} \leq 1$$

$$D^* = \{(u, v) : u^2 + v^2 \leq 4\}$$

DEFINITION 2. *The* **Jacobian** *of the transformation* $x = x(u, v), y = y(u, v)$ *is*

$$J(u,v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 3. *Compute the Jacobian of the transformation* $x = r\cos\theta$, $y = r\sin\theta$.

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

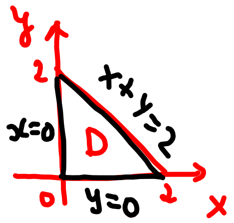
$$= r(\cos^2\theta + \sin^2\theta) = r$$

Change of variables for a double integral:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v.$$

EXAMPLE 4. *Evaluate*

$$\iint_D e^{\frac{y-x}{y+x}}\,\mathrm{d}A$$

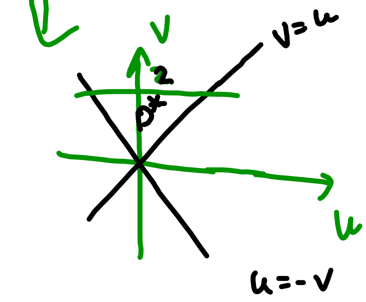


where D is triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.

$$u = y - x \quad (1)$$
$$\pm \quad v = y + x \quad (2)$$
$$(1)+(2) \quad u+v = 2y \implies y = \frac{u+v}{2}$$
$$(2)-(1) \quad v-u = 2x \implies x = \frac{v-u}{2}$$

$$x=0 \implies \frac{v-u}{2}=0 \implies v=u$$
$$y=0 \implies \frac{u+v}{2}=0 \implies u=-v$$
$$x+y=2 \implies v=2$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ +\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$$



$$\iint_D e^{\frac{y-x}{y+x}}\,dA = \iint_{D^*} e^{\frac{u}{v}}\,|J(u,v)|\,du\,dv$$
$$= \int_0^2\int_{-v}^{v} e^{\frac{u}{v}}\,\frac{1}{2}\,du\,dv$$
$$= \frac{1}{2}\int_0^2 \frac{1}{\frac{1}{v}}\,e^{\frac{u}{v}}\Big|_{u=-v}^{v} dv = \frac{1}{2}\int_0^2 v\left(e^{\frac{v}{v}}-e^{\frac{-v}{v}}\right)dv$$
$$= \frac{1}{2}\left(e-\frac{1}{e}\right)\int_0^2 v\,dv = \frac{1}{2}\left(e-\frac{1}{e}\right)\frac{v^2}{2}\Big|_0^2$$
$$= \boxed{e-\frac{1}{e}}$$

EXAMPLE 5. *Find mass of a lamina that occupies the region* 1st quadrant

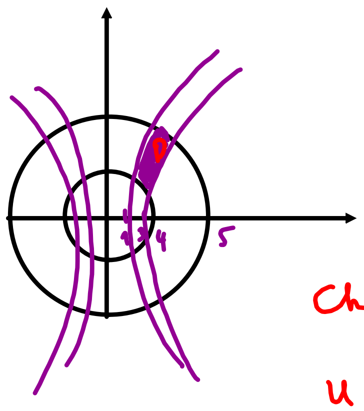
$$D=\{(x,y)|16\le \underbrace{x^2+y^2}_{u}\le 25,\quad 1\le \underbrace{x^2-y^2}_{v}\le 9,\quad x\ge 0, y\ge 0\}$$

with density $\rho(x,y)=8xy$.

$$m(D)=\iint_D \rho(x,y)\,dA$$

$$m(D)=\iint_D 8xy\,dA$$

$$m(D)=8\iint_D xy\,dA$$



Change of variables

$$u=x^2+y^2 \quad (1)$$

$$v=x^2-y^2 \quad (2)$$

$$(1)+(2):\quad u+v=2x^2 \Rightarrow x=\pm\sqrt{\frac{u+v}{2}}=\frac{\sqrt{u+v}}{\sqrt{2}}$$

$$(1)-(2):\quad u-v=2y^2 \Rightarrow y=\pm\sqrt{\frac{u-v}{2}}=\frac{\sqrt{u-v}}{\sqrt{2}}$$

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}=\begin{vmatrix} \frac{1}{\sqrt{2}}\frac{1}{2\sqrt{u+v}} & \frac{1}{\sqrt{2}}\frac{1}{2\sqrt{u+v}} \\ \frac{1}{\sqrt{2}}\frac{1}{2\sqrt{u-v}} & -\frac{1}{\sqrt{2}}\frac{1}{2\sqrt{u-v}} \end{vmatrix}$$

$$=-\frac{1}{8}\frac{1}{\sqrt{u^2-v^2}}-\frac{1}{8}\frac{1}{\sqrt{u^2-v^2}}=-\frac{1}{4}\cdot\frac{1}{\sqrt{u^2-v^2}}$$

$$m(D)=\cancel{8}\iint_{D^*}\frac{\cancel{\sqrt{u+v}}}{\cancel{\sqrt{2}}}\cdot\frac{\cancel{\sqrt{u-v}}}{\cancel{\sqrt{2}}}\cdot\left|-\frac{1}{\cancel{4}}\frac{1}{\cancel{\sqrt{u^2-v^2}}}\right|du\,dv$$

$$=\iint_{D^*}du\,dv=\text{Area}(D^*)=(25-16)(9-1)=\boxed{72}$$

where $D^*=\{(u,v)\,|\,16\le u\le 25,\ 1\le v\le 9\}$