13.2: Iterated integrals

Suppose that $f(x, y)$ is integrable over the rectangle $R = [a, b] \times [b, c]$.

Partial integration of f with respect to x: $\int_a^b f(x, y)\,\mathrm{d}x$

Partial integration of f with respect to y: $\int_c^d f(x, y)\,\mathrm{d}y$

EXAMPLE 1.

$$\int_0^4 (x + 3y^2)\,\mathrm{d}x = \left(\frac{x^2}{2} + 3y^2x\right)\Bigg|_{x=0}^{4} = \frac{16}{2} + 12y^2 - 0$$

$$= \underbrace{8 + 12y^2}$$

function of y.

$$\int_1^4 e^{xy}\,\mathrm{d}y = \frac{e^{xy}}{x}\Bigg|_{y=1}^{4} = \frac{1}{x}\left(e^{4x} - e^{x}\right)$$

function of x.

Iterated integrals:

$$\text{a number} = \int_a^b \Bigg[\overbrace{\int_c^d f(x,y)\,\mathrm{d}y}^{\text{a function of } x}\Bigg]\,\mathrm{d}x = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_c^d \Bigg[\int_a^b f(x,y)\,\mathrm{d}x\Bigg]\,\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 2. *Evaluate the integrals:*

$$I_1 = \int_0^1 \int_1^4 x\sqrt{y}\,\mathrm{d}y\mathrm{d}x, \quad I_2 = \int_1^4 \int_0^1 x\sqrt{y}\,\mathrm{d}x\mathrm{d}y$$

$$I_1 = \int_0^1 \left(\int_1^4 x\sqrt{y}\,dy\right)dx = \int_0^1 x\left(\int_1^4 \sqrt{y}\,dy\right)dx$$

$$= \left(\int_1^4 \sqrt{y}\,dy\right)\left(\int_0^1 x\,dx\right) = \frac{2}{3}y^{3/2}\Big|_1^4 \cdot \frac{x^2}{2}\Big|_0^1$$

$$= \frac{\cancel{2}}{3}\Big(\underbrace{(\sqrt{4})^3 - (\sqrt{1})^3}_{8-1}\Big)\cdot\frac{1}{\cancel{2}} = \boxed{\frac{7}{3}}$$

$$I_2 = \int_1^4 \left(\int_0^1 x\sqrt{y}\,dx\right)dy = \int_1^4 \sqrt{y}\left(\int_0^1 x\,dx\right)dy$$

$$= \left(\int_0^1 x\,dx\right)\left(\int_1^4 \sqrt{y}\,dy\right) = \frac{7}{3} \text{ (see above)}$$

FUBINI's THEOREM: *If f is continuous on on the rectangle*

$$R = [a,b] \times [c,d]$$

iterated integrals

then

$$\iint_R f(x,y)\,\mathrm{d}A = \underbrace{\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.}$$

EXAMPLE 3. *Evaluate*

$-\frac{\pi}{2} \le x \le \frac{\pi}{2} \quad 1 \le y \le 5$

$$\iint_R x\cos(xy)\,\mathrm{d}A$$

where $R = [-\pi/2, \pi/2] \times [1,5]$

By Fubini's Theorem

$$\iint_R x\cos(xy)\,dA = \int_1^5 \int_{-\pi/2}^{\pi/2} x\cos(xy)\,dx\,dy$$

requires integration by parts

OR

$$= \int_{-\pi/2}^{\pi/2}\left(\int_1^5 x\cos(xy)\,dy\right)dx =$$

$$\int_{-\pi/2}^{\pi/2} x\left(\int_1^5 \cos(xy)\,dy\right)dx = \int_{-\pi/2}^{\pi/2} \cancel{x}\,\frac{\sin(xy)}{\cancel{x}}\Bigg|_{y=1}^{5} dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(\sin(5x) - \sin x\right)dx = -\frac{\cos 5x}{5} + \cos x \Bigg|_{-\frac{\pi}{2}}^{\pi/2} = 0$$

EXAMPLE 4. ([redacted]) *Find the volume of the solid S lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.*

$\underbrace{z}_{\text{lid}}$ $\quad x^2+y^2 = f(x,y) \geq 0$

$$V = \iint_R f(x,y)\,dA = \iint_R (x^2+y^2)\,dA \overset{FT}{=}$$

$$\int_{-2}^{2}\left(\int_{-3}^{3}(x^2+y^2)\,dy\right)dx = \int_{-2}^{2}\left(x^2y + \frac{y^3}{3}\right)\Bigg|_{y=-3}^{3} dx$$

$$= \int_{-2}^{2}\left(3x^2+9-(-3x^2-9)\right)dx$$

$$= 2\int_{-2}^{2}(3x^2+9)\,dx = 2\left(x^3+9x\right)\Big|_{-2}^{2}$$

$$= 2\left(8+18-(-8-18)\right)$$

$$= 4\,(8+18) = 4\cdot 26 = \boxed{104}$$

FACT: *If g and h are continuous functions of one variable and $R = [a,b] \times [c,d]$ then*

$$\iint_R g(x)h(y)\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

EXAMPLE 5. *If $R = [0, \ln 2] \times [0, \ln 5]$ find* $\iint_R e^{2x-y}\,\mathrm{d}A$.

$$\boxed{e^{a+b} = e^a \cdot e^b}$$

$$\iint_R e^{2x-y}\,dA = \iint_R e^{2x} \cdot e^{-y}\,dA \overset{\text{by FACT}}{=} \left(\int_0^{\ln 2} e^{2x}\,dx\right)\left(\int_0^{\ln 5} e^{-y}\,dy\right)$$

$$= \frac{e^{2x}}{2}\Big|_0^{\ln 2} \left(-e^{-y}\right)\Big|_0^{\ln 5}$$

$$= -\frac{1}{2}(4-1)\left(\frac{1}{5}-1\right) = -\frac{3}{2}\cdot\left(-\frac{4}{5}\right) = \boxed{\frac{6}{5}}$$