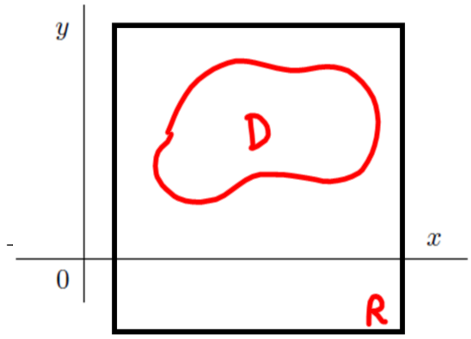
13.3: Double integrals over general regions

`All functions below are continuous on their domains.`

Let D be a bounded region enclosed in a rectangular region R. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$



If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of f over D** by

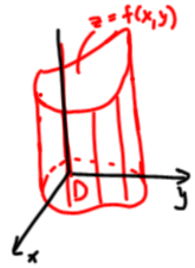
$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A$$

FACT: If $f(x,y) \geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$S = \{(x,y,z) \in \mathbb{R}^3 |\ 0 \leq z \leq f(x,y), (x,y) \in D\},$$

is

$$V = \iint_D f(x,y)\,\mathrm{d}A.$$



EXAMPLE 1. *Evaluate the integral*

$$\iint_D \sqrt{16 - x^2 - y^2}\,\mathrm{d}A$$

where $D = \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 16\}$ by identifying it as a volume of a solid.

$$z = f(x,y) = \sqrt{16 - x^2 - y^2}$$

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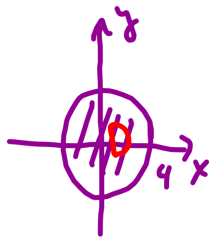
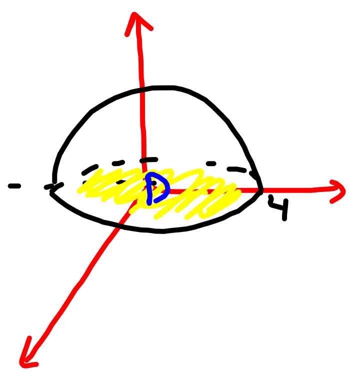
$$x^2 + y^2 + z^2 = 16, \quad z \geq 0$$





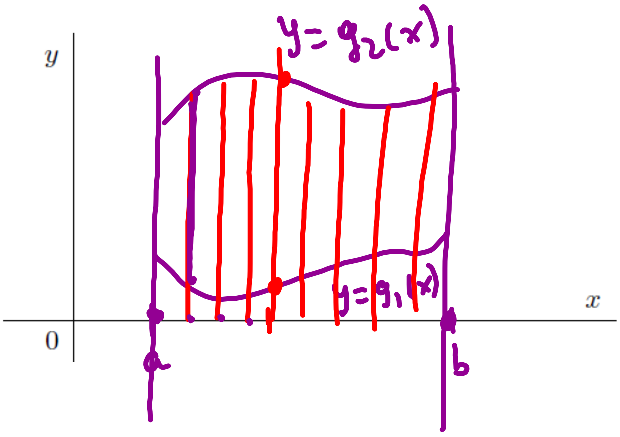
$$\iint_D \sqrt{16 - x^2 - y^2}\,dA = \frac{1}{2}\text{Volume of the sphere with } r = 4$$

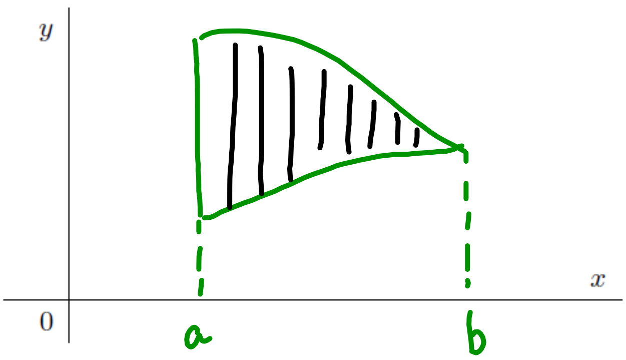
$$= \frac{1}{\cancel{2}} \cdot \frac{\overset{2}{\cancel{4}}}{3}\pi \cdot 4^3 = \boxed{\frac{128\pi}{3}}$$

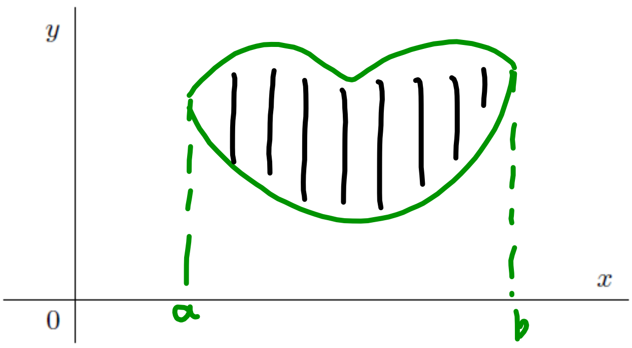
Computation of double integral:

A plain region of **TYPE I:** lower upper

$$D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}\,.$$

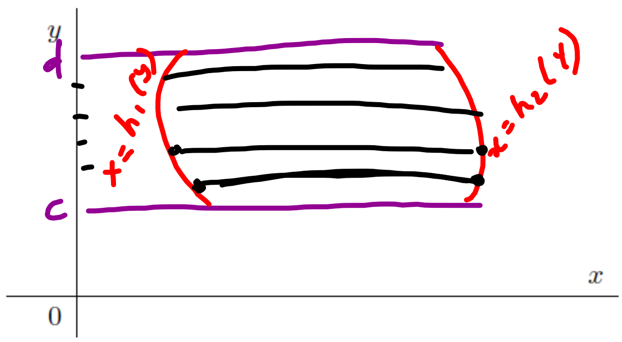


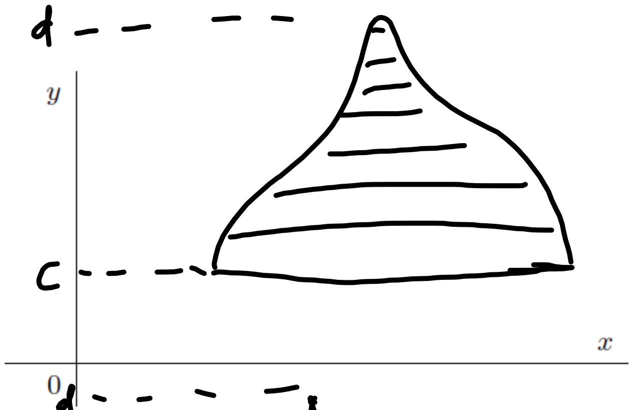


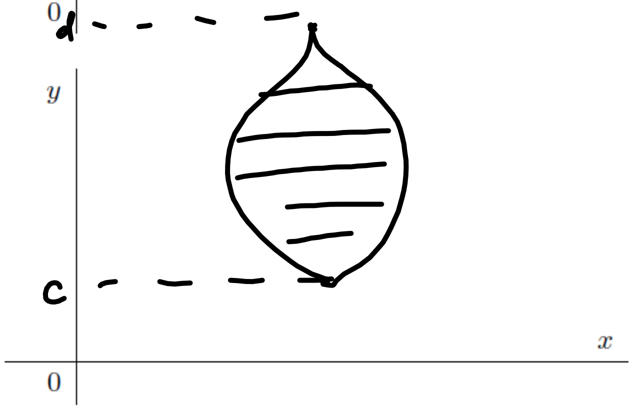


A plain region of **TYPE II:** left right

$$D = \{(x, y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}\,.$$







THEOREM 2. *If D is a region of type I such that $D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then*

upper curve

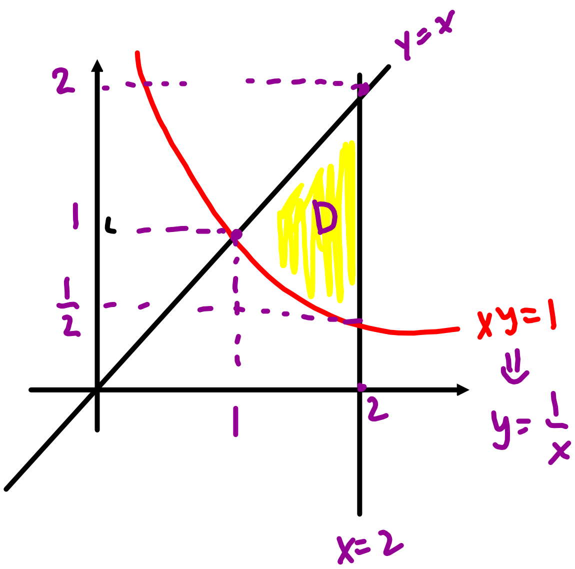
$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\mathrm{d}x.$$

lower curve

THEOREM 3. *If D is a region of type II s.t. $D = \{(x, y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then*

$$\iint_D f(x, y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 4. *Evaluate* $I = \iint_D (x+y)\,\mathrm{d}A$, *where* D *is the region bounded by the lines* $x = 2, y = x$ *and the hyperbola* $xy = 1$.



$$I = \int_1^2 \left(\int_{\frac{1}{x}}^{x} (x+y)\,dy \right) dx =$$

$$= \int_1^2 \left(xy + \frac{y^2}{2} \right) \Bigg|_{y=\frac{1}{x}}^{x} dx$$

$$= \int_1^2 \left(x^2 + \frac{x^2}{2} - \left(1 + \frac{1}{2x^2}\right)\right) dx$$

$$D = \left\{ (x,y) \,\middle|\, 1 \le x \le 2,\ \frac{1}{x} \le y \le x \right\}$$

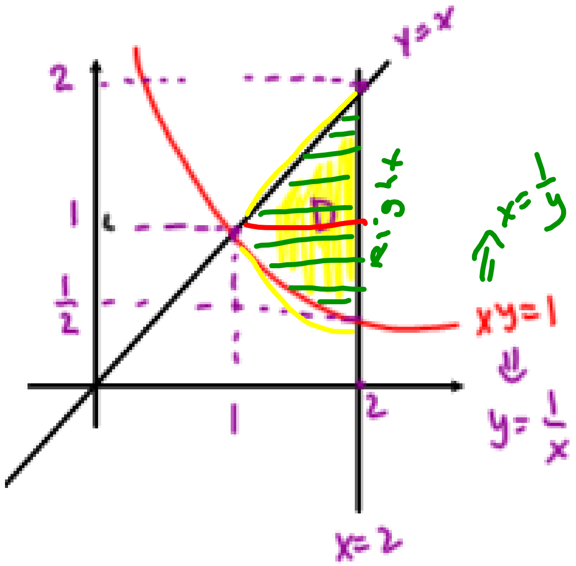
$$= \int_1^2 \left(\frac{3x^2}{2} - 1 - \frac{1}{2} x^{-2} \right) dx$$

$$= \left(\frac{x^3}{2} - x + \frac{1}{2x} \right) \Bigg|_1^2 = \frac{8}{2} - 2 + \frac{1}{4} - \underbrace{\left(\frac{1}{2} - 1 + \frac{1}{2} \right)}_{=0}$$

$$= 2 + \frac{1}{4} = \frac{9}{4}$$

Do Example 4 again reversing the order of integration,

$$I = \iint_D (x+y)\, dA = \int_{\frac{1}{2}}^{2} \int_{\text{left curve}}^{\text{right curve}} (x+y)\, dx\, dy$$



Right curve
$x = 2$

Left curve

$$x = \begin{cases} \frac{1}{y}, & \frac{1}{2} \le y \le 1 \\ y, & 1 \le y \le 2 \end{cases}$$

$$I = \int_{\frac{1}{2}}^{1} \int_{\frac{1}{y}}^{2} (x+y)\, dx\, dy + \int_{1}^{2} \int_{y}^{2} (x+y)\, dx\, dy$$

$$= \dots = \frac{9}{4}$$

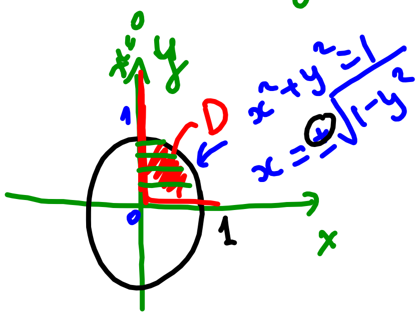
EXAMPLE 5. *Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $x = 0, y = z, z = 0$ in the first octant.*

$$V = \iint_{D} f(x,y)\, dA$$

the lid of the solid (part of the graph $z = f(x,y)$)

D: base, or projection of the solid onto the xy-plane

$z = y$ (the lid)



$$V = \iint_{D} y\, dA = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} y\, dx \right) dy$$

$$= \int_0^1 y \left(\int_0^{\sqrt{1-y^2}} dx \right) dy = \int_0^1 y\sqrt{1-y^2}\, dy$$

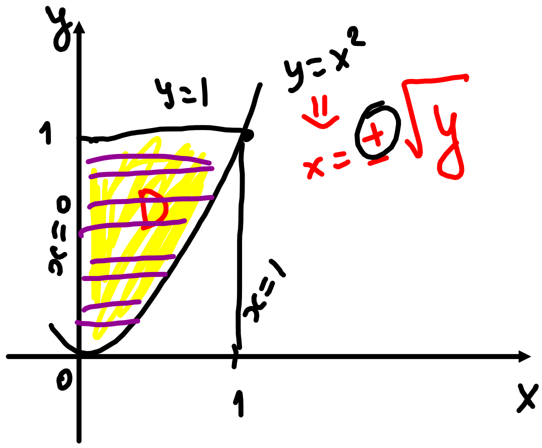
use u-sub.

$= \dots$

EXAMPLE 6. *Evaluate the integral by reversing the order of integration:*

$D = \{ 0 \le x \le 1,\ x^2 \le y \le 1 \}$

$$I = \underbrace{\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy dx.}_{\text{iterated integral}}$$



$$= \underbrace{\iint_D x^3 \sin y^3 \, dA}_{\text{double integral}}$$

Reverse the order of integration

$$I = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin y^3 \, dx\, dy$$

$$= \int_0^1 \sin y^3 \left(\underbrace{\int_0^{\sqrt{y}} x^3\, dx}\right) dy = \int_0^1 \sin y^3 \left. \frac{x^4}{4} \right|_{x=0}^{\sqrt{y}} dy$$

$$= \int_0^1 (\sin y^3)\, \frac{y^2}{4}\, dy \underset{\text{u-sub.}}{=\!=} \frac{1}{4} \int_0^1 \sin u \, \frac{du}{3}$$

$u = y^3$

$du = 3y^2 dy$

$0 \le u \le 1$

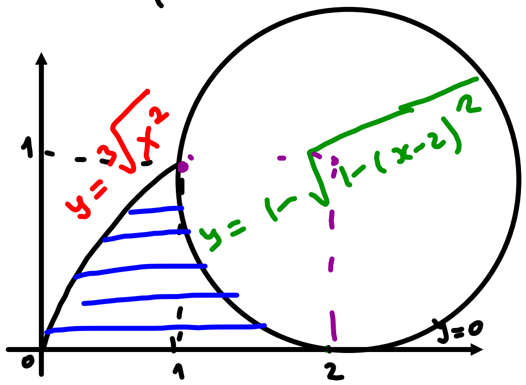
$$= \frac{1}{12} \left(-\cos u\right)\Big|_0^1 = -\frac{1}{12}\left(\cos 1 - \cos 0\right)$$

$$= \boxed{\frac{1}{12}\left(1 - \cos 1\right)}$$

EXAMPLE 7. *Sketch the region of integration and change the order of integration:*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_0^1 \int_0^{\sqrt[3]{x^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^{1-\sqrt{1-(x-2)^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x$$

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt[3]{x^2}\} \cup$
$\{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le 1 - \sqrt{1-(x-2)^2}\}$



$y = \sqrt[3]{x^2} \Rightarrow y^3 = x^2 \Rightarrow x = \pm\sqrt{y^3}$

$y = 1 - \sqrt{1-(x-2)^2}$
$(y-1)^2 = 1-(x-2)^2$
$(x-2)^2 + (y-1)^2 = 1$
$(x-2)^2 = 1 - (y-1)^2$
$x-2 = \pm\sqrt{1-(y-1)^2}$
$x = 2 \pm \sqrt{1-(y-1)^2}$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_{\text{left curve}}^{\text{Right curve}} f(x,y)\,dx\,dy$$

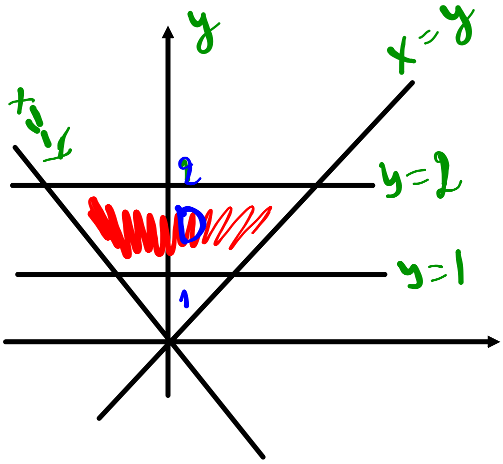
$$= \int_0^1 \int_{\sqrt{y^3}}^{2-\sqrt{1-(y-1)^2}} f(x,y)\,dx\,dy$$

EXAMPLE 8. *Evaluate the double integral*

$$I = \iint_D e^{\frac{x}{y}}\,\mathrm{d}A$$

where D is bounded by the lines

$$y = 1, y = 2, x = -y, x = y.$$



$$I = \int_1^2 \int_{-y}^{y} e^{\frac{x}{y}}\,dx\,dy$$

$$= \int_1^2 \frac{1}{\frac{1}{y}} e^{\frac{x}{y}} \Big|_{x=-y}^{y} dy$$

$$= \int_1^2 y\left(e^1 - e^{-1}\right) dy = \left(e - \frac{1}{e}\right) \int_1^2 y\,dy$$

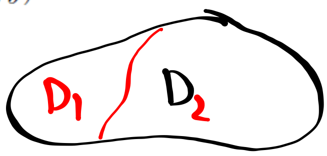
$$= \left(e - \frac{1}{e}\right) \frac{y^2}{2}\Big|_1^2 = \frac{1}{2}\left(e - \frac{1}{e}\right)(4-1)$$

$$= \boxed{\frac{3}{2}\left(e - \frac{1}{e}\right)}$$

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$



- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

- If we integrate the constant function $f(x,y) = 1$ over D, we get **area** of D:

$$\iint_D 1\,\mathrm{d}A = A(D).$$

EXAMPLE 9. *If* $D = \{(x, y) |\ x^2 + y^2 \leq 25\}$ *then*

$$\iint_D \mathrm{d}A = \text{Area}(D) = \pi 5^2 = 25\pi$$