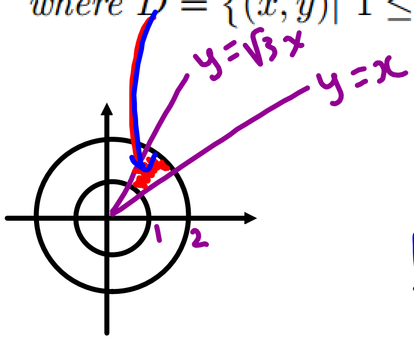
13.5: Double integrals in polar coordinates

EXAMPLE 1. *Evaluate*

$$I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$$

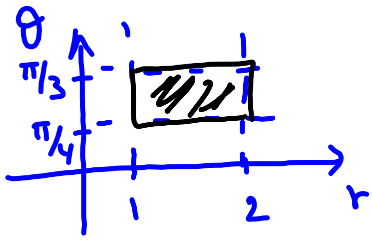
where $D = \left\{ (x, y) \mid \overbrace{1 \le x^2 + y^2}^{r^2} \le 4, x \le y \le \underbrace{\sqrt{3}x}, x \ge 0 \right\}.$

$\frac{y}{x} = \sqrt{3} = \tan\theta$



$$\Big\downarrow$$

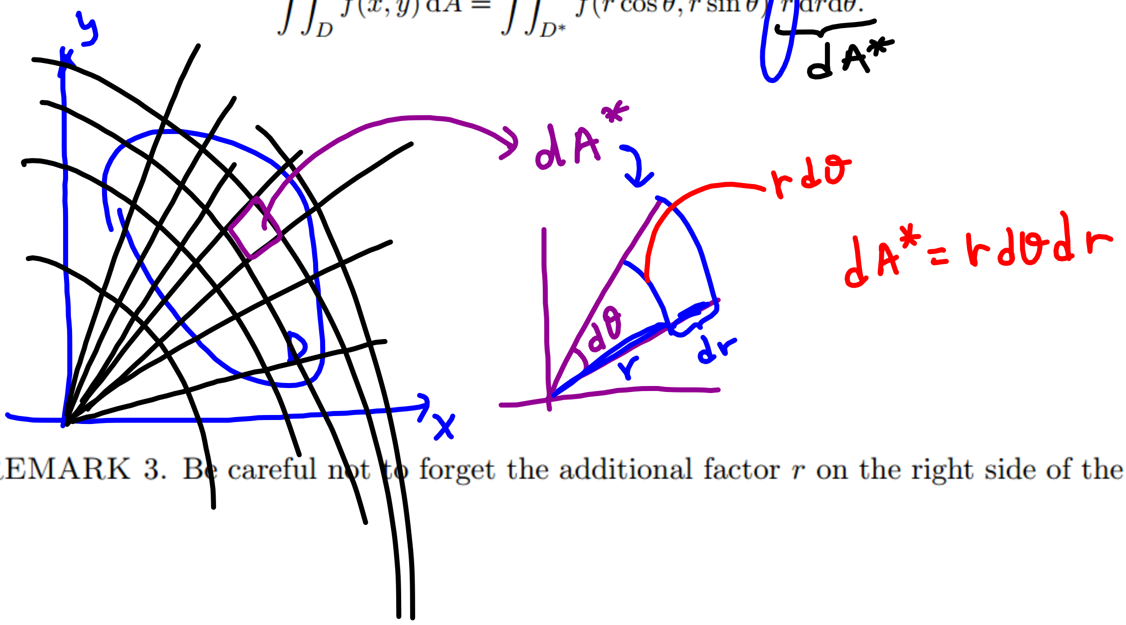
$$D^* = \left\{ (r, \theta) : 1 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{3} \right\}$$



$$\iint_{D^*} f(r, \theta) \, dA^*$$

THEOREM 2. *Change to polar coordinates in a double integral: Let f be a continuous on the region D. Denote by D^* the region representing D in the polar coordinates (r, θ). Then*

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,\underbrace{r\,\mathrm{d}r\mathrm{d}\theta}_{dA^*}.$$



REMARK 3. Be careful not to forget the additional factor r on the right side of the formula.

Solution of Example 1: Evaluate $I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$ where $D = \left\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, x \leq y \leq \sqrt{3}x, x \geq 0\right\}$.

$$D^* = \left\{(r, \theta) \mid 1 \leq r \leq 2, \ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}\right\}$$

$$I = \iint_{D^*} \arctan(\tan\theta) \, dA^* = \iint_{D^*} \arctan(\tan\theta) \, \underline{r\,dr\,d\theta}$$

$$= \int_{\pi/4}^{\pi/3} \int_1^2 \theta \cdot r \, dr \, d\theta = \left(\int_{\pi/4}^{\pi/3} \theta \, d\theta\right)\left(\int_1^2 r \, dr\right)$$

$$= \frac{\theta^2}{2}\Bigg|_{\theta=\pi/4}^{\pi/3} \frac{r^2}{2}\Bigg|_1^2 = \frac{\pi^2}{4}\left(\frac{1}{9} - \frac{1}{16}\right) \cdot (4-1)$$

$$= \ldots$$

EXAMPLE 4. *Find the volume of the solid that lies under the paraboloid* $z = x^2 + y^2$, *above the xy-plane and inside the cylinder* $\underbrace{x^2 + y^2 = 2x}$.

$z = f(x,y)$

$D = \{x^2+y^2 \le 2x\}$

$$V = \iint_D f(x,y)\,dA = \iint_D (x^2+y^2)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$

$$D^* = \{(r,\theta) : r^2 \le 2r\cos\theta\}$$

$$D^* = \{(r,\theta) : 0 \le r \le 2\cos\theta,\ \underbrace{-\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}}_{\cos\theta \ge 0}\}$$

$$V = \iint_{D^*} r^2\,dA^* = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} r^2\,r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2}\left(\int_0^{2\cos\theta} r^3\,dr\right)d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Bigg|_0^{2\cos\theta} d\theta$$

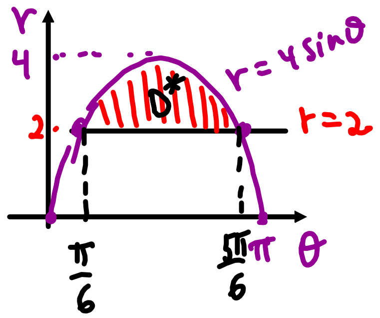
$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\pi/2} 2^4 \cos^4\theta\,d\theta = 4\int_{-\pi/2}^{\pi/2} (\cos^2\theta)^2\,d\theta$$

$$(\cos^2\theta)^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$$

$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$

$$V = \int_{-\pi/2}^{\pi/2}\left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)d\theta = \ldots$$

EXAMPLE 5. *Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.*



Find intersection points

$$\left.\begin{array}{l} r = 4\sin\theta \\ r = 2 \end{array}\right\} \Rightarrow 4\sin\theta = 2$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \iint_D dA = \iint_{D^*} dA^* = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_2^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left.\frac{r^2}{2}\right|_{r=2}^{4\sin\theta} d\theta = \int_{\frac{\pi}{6}}^{5\pi/6} \left(8\sin^2\theta - 2\right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 4(1-\cos 2\theta) - 2 \;\; d\theta = \ldots$$

Remark See WIR #7 for another solution of this problem.