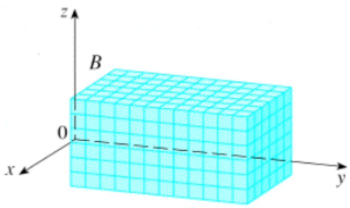
13.8: Triple Integrals

Mass problem: *Given a solid object, that occupies the region B in $\mathbb{R}^3$, with density $\rho(x, y, z)$. Find the mass of the object.*

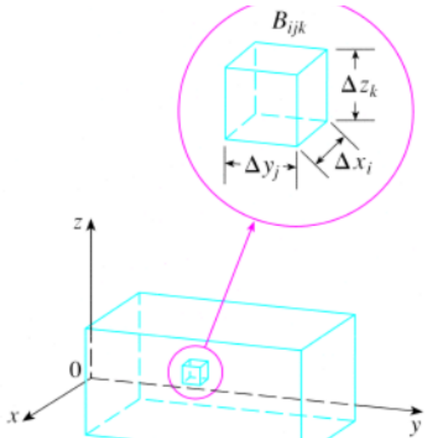
Solution: Let B be a rectangular box:



$$B = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

$= [a,b] \times [c,d] \times [r,s]$

Partition in sub-boxes:



$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\|\to 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z)\, \mathrm{d}V$$

FUBINI's THEOREM: *If f is continuous on the rectangular box $B=[a,b]\times[c,d]\times[r,s]$ then*

$$\iiint_B f(x,y,z)\,\mathrm{d}V=\int_r^s\int_c^d\int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}z$$

dy dx dz
dz dy dx ...

and there are 5 other possible orders in which we can integrate.

EXAMPLE 1. *Let $B=[0,1]\times[-1,3]\times[0,3]$. Evaluate*

$$I=\iiint_B xye^{yz}\,\mathrm{d}V$$

$$I=\int_{-1}^{3}\int_0^1\int_0^3 xye^{yz}\,dz\,dx\,dy=\int_{-1}^{3}\int_0^1 x\cancel{y}\frac{e^{yz}}{\cancel{y}}\Big|_{z=0}^{3}\,dx\,dy$$

$$=\int_{-1}^{3}\int_0^1\left(xe^{3y}-x\right)dx\,dy=\int_{-1}^{3}\int_0^1 x(e^{3y}-1)\,dx\,dy$$

$$=\int_{-1}^{3}(e^{3y}-1)\,dy\cdot\int_0^1 x\,dx=\left(\frac{e^{3y}}{3}-y\right)\Big|_{-1}^{3}\cdot\frac{1}{2}$$

$$=\frac{1}{2}\left[\frac{e^{9}}{3}-3-\left(\frac{e^{-3}}{3}+1\right)\right]$$

$$=\frac{1}{2}\left[\frac{e^{9}-e^{-3}}{3}-4\right]$$

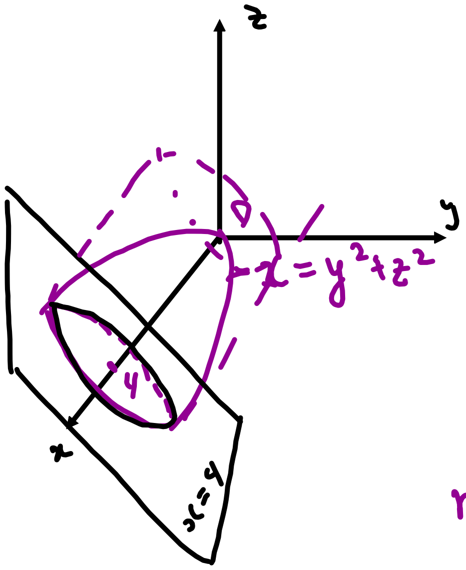
FACT: The volume of the solid E is given by the integral,

$$V = \iiint_E \mathrm{d}V.$$

FACT: The mass of the solid E with variable density $\rho(x, y, z)$ is given by the integral,

$$m = \iiint_E \rho(x, y, z)\, dV \,.$$

EXAMPLE 2. *Find the mass of the solid* E *bounded by* $x = y^2 + z^2$ *and the plane* $x = 4$ *if the density function is* $\rho(x, y, z) = \sqrt{y^2 + z^2}$. (TYPE I region)



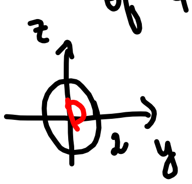
$$m = \iiint_E \rho(x, y, z)\, dV$$

$$m = \iiint_E \sqrt{y^2 + z^2}\, dV$$

$$m = \iint_D \left[\int_{y^2+z^2}^{4} \sqrt{y^2 + z^2}\, dx \right] dA$$

where D is the projection of the solid E onto the yz-plane.

∂D can be found as the line of intersection of two surfaces:



$$\left. \begin{array}{l} x = y^2 + z^2 \\ x = 4 \end{array} \right\} \Rightarrow \partial D = \{(y, z) \mid y^2 + z^2 = 4\}$$

$$m = \iint_D \sqrt{y^2+z^2}\left(\int_{y^2+z^2}^{4} dx\right) dA$$

$$= \iint_D \sqrt{y^2+z^2}\,\left(4-(y^2+z^2)\right) dA$$

$D = \{(y,z) : y^2+z^2 \le 4\}$

Use polar coordinates:

$y = r\cos\theta$, $z = r\sin\theta$

$D^* = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

Note $y^2+z^2 = r^2$

$$m = \iint_{D^*} r(4-r^2)\, dA^*$$

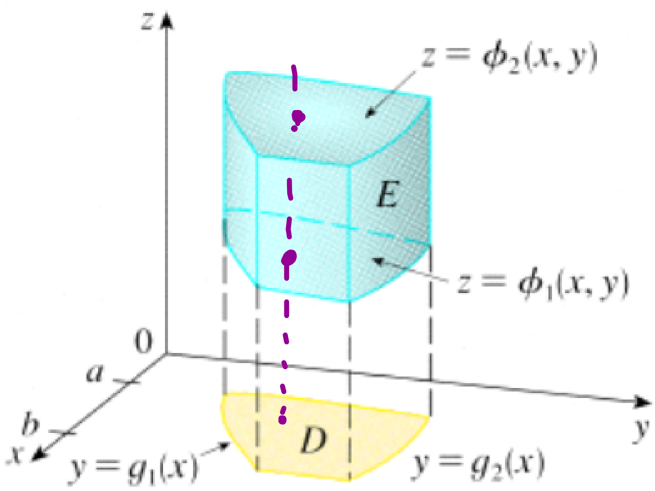
$$m = \int_0^{2\pi}\int_0^2 r(4-r^2)\, r\, dr\, d\theta = 2\pi \int_0^2 (4r^2 - r^4)\, dr$$

$$= 2\pi\left(\frac{4r^3}{3} - \frac{r^5}{5}\right)\Big|_0^2 = \ldots$$

A solid region of **TYPE I**:

$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \le z \le \phi_2(x, y)\}$ where D is the projection of E onto the xy-plane.

A type 1 solid region

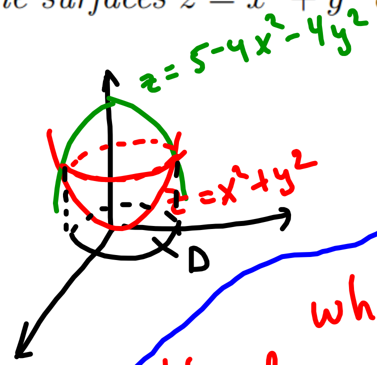


top surface

bottom surface

$$\iiint f(x, y, z)\, dV = \iint \left[\int^{\phi_2(x,y)} f(x, y, z)\, dz \right] dA$$

EXAMPLE 3. *Use a triple integral to find the volume of the solid* $\overset{E}{\smile}$ *bounded by the surfaces* $z = x^2 + y^2$ *and* $z = 5 - 4x^2 - 4y^2$.



$$V(E) = \iiint_E dV = \iint_D \left(\int_{x^2+y^2}^{5-4x^2-4y^2} dz \right) dA$$

where ∂D can be found as the line of intersection of the given surfaces:

$$\left.\begin{array}{l} z = x^2+y^2 \\ z = 5-4x^2-4y^2 \end{array}\right\} \Rightarrow \begin{array}{l} x^2+y^2 = 5-4x^2-4y^2 \\ 5x^2+5y^2 = 5 \\ \{x^2+y^2=1\} = \partial D \end{array}$$

Thus, $D = \{(x,y) \mid x^2+y^2 \le 1\}$ $\xrightarrow{\text{use polar coordinates}}$ $D^* = \{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

$$\Rightarrow V(E) = \iint_D z \Big|_{z=x^2+y^2}^{5-4(x^2+y^2)} dA$$

$$= \iint_D \left(5 - 4(x^2+y^2) - (x^2+y^2)\right) dA$$

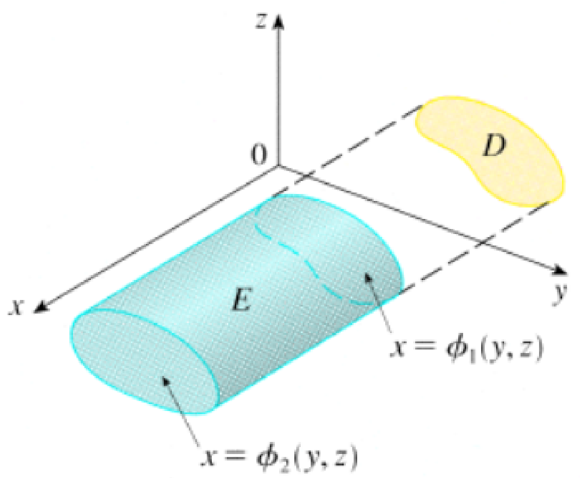
$$= 5 \iint_D \left(1 - (x^2+y^2)\right) dA = 5 \iint_{D^*} (1-r^2)\, dA^*$$

$$= 5 \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta = 5 \cdot 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1$$

$$= 10\pi \cdot \frac{1}{4} = \boxed{\frac{5\pi}{2}} \text{ unit}^3$$

A solid region of **TYPE II**:

$E = \{(x, y, z) | (y, z) \in D, \phi_1(y, z) \leq x \leq \phi_2(y, z)\}$
where D is the projection of E onto the yz-plane.

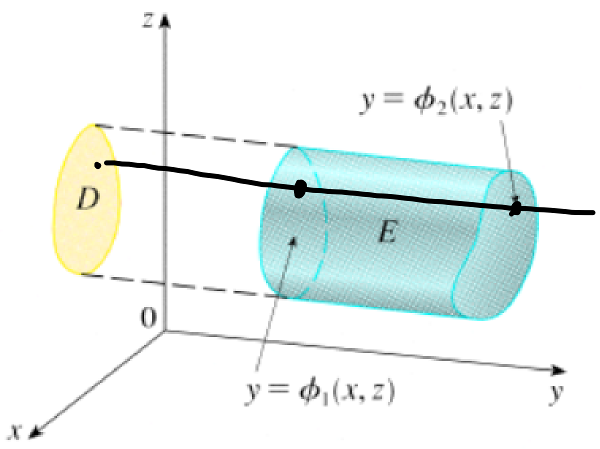


A type 2 region

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \iint_D \left[\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x, y, z)\,\mathrm{d}x\right]\mathrm{d}A$$

A solid region of **TYPE III**:

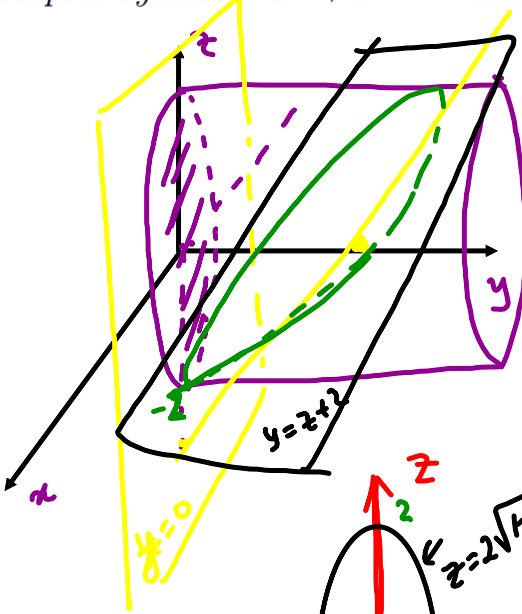
$E = \{(x, y, z) | (x, z) \in D, \phi_1(x, z) \leq y \leq \phi_2(x, z)\}$
where D is the projection of E onto the xz-plane.



A type 3 region

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_D \left(\int_{\varphi_1(x,z)}^{\varphi_2(x,z)} f(x, y, z)\, dy \right) dA$$

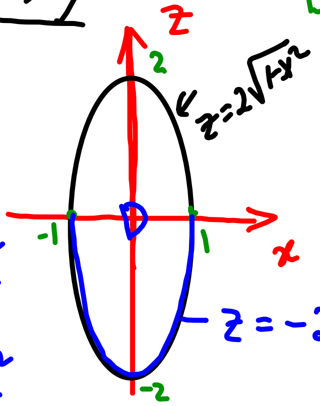
EXAMPLE 4. *Use a triple integral to find the volume of the solid bounded by the elliptic cylinder $4x^2+z^2=4$ and the planes $y=0$ and $y=z+2$.* TYPE III



$$V=\iiint_E dV = \iint_D \left(\int_0^{z+2} dy\right) dA$$

where D is projection of E onto the xz-plane.

$$D=\{(x,z) \mid 4x^2+z^2\le 4\}$$



$$V=\iint_D (z+2)\,dA$$

$$V=\int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}(z+2)\,dz\,dx$$

$$=\int_{-1}^{1}\left(\frac{z^2}{2}+2z\right)\Bigg|_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx$$

2D: $4x^2+z^2=4$

$\Downarrow$

$z=\pm\sqrt{4-4x^2}$

or $z=\pm 2\sqrt{1-x^2}$

$$V=\int_{-1}^{1}\left(0+2\cdot\left(2\sqrt{1-x^2}-\left(-2\sqrt{1-x^2}\right)\right)\right)dx$$

$$V=8\int_{-1}^{1}\sqrt{1-x^2}\,dx = 8\cdot\left(\text{Area of half unit disk}\right)$$

$$=8\,\frac{\pi}{2}=\boxed{4\pi}\ \text{unit}^3$$