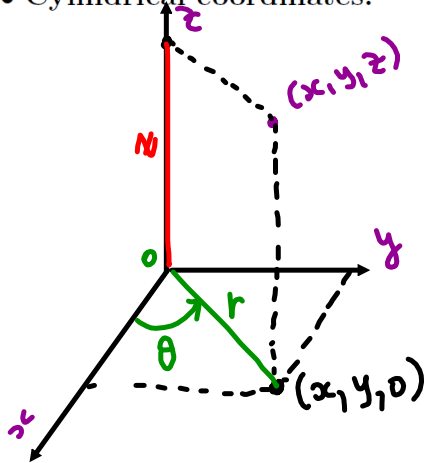


13.9-13.10: Part I

Triple integrals in cylindrical coordinates

- Cylindrical coordinates:



$$P(x, y, z) \in \mathbb{R}^3$$

In the cylindrical coordinates P is represented by the ordered triple (r, θ, z) , where r, θ are the polar coordinates of P_{xy} and z is the directed distance from the xy -plane to P :

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

where

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z.$$

• look for the correct octant

REMARK 1. The cylindrical coordinates

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\z &= z \\r &\geq 0, \quad 0 \leq \theta \leq 2\pi\end{aligned}$$

are useful in problems that involve symmetry about the z -axis.



EXAMPLE 2. Find an equation in cylindrical coordinates for the cone

$$z = \sqrt{x^2 + y^2}$$

$$z = \sqrt{r^2} \Rightarrow \boxed{z = r}$$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in cylindrical coordinates. Then*

$$\iiint_E f(x, y, z) \, dV = \iiint_{E^*} f(r \cos \theta, r \sin \theta, z) \, dV^*,$$

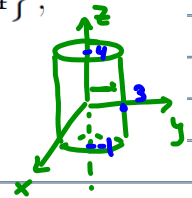
where

$$dV^* = \underbrace{r}_{\text{green}} \, dr \, dz \, d\theta.$$

EXAMPLE 4. The density at any point of the solid E ,

$$E = \{(x, y, z) : x^2 + y^2 \leq 9, -1 \leq z \leq 4\},$$

equals to its distance from the axis of E . Find the mass of E .



$$m(E) = \iiint_E \rho(x, y, z) dV, \text{ where } \rho(x, y, z) = \sqrt{x^2 + y^2}$$

$$m(E) = \iiint_E \sqrt{x^2 + y^2} dV = \iiint_{E^*} r dV^*,$$

$$\text{where } E^* = \{(r, \theta, z) \mid r^2 \leq 9, -1 \leq z \leq 4\}$$

or

$$E^* = \{(r, \theta, z) \mid 0 \leq r \leq 3, -1 \leq z \leq 4, 0 \leq \theta \leq 2\pi\}$$

$$m(E) = \int_{-1}^4 \int_0^{2\pi} \int_0^3 r r dr d\theta dz = 5 \cdot 2\pi \cdot \int_0^3 r^2 dr = 10\pi \left. \frac{r^3}{3} \right|_0^3 = \boxed{90\pi}$$

REMARK 5. If E is a solid region of type I, i.e.

$$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\},$$

where D is the projection of E onto the xy -plane then, as we know,

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{\phi_1(x, y)}^{\phi_2(x, y)} f(x, y, z) \, dz \right] dA.$$

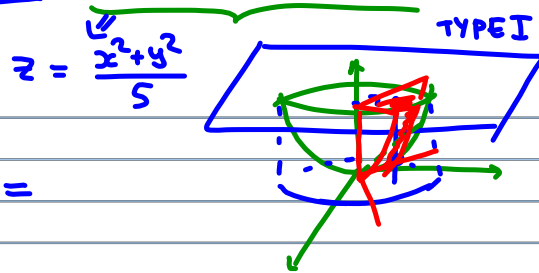
Passing to cylindrical coordinates here we actually have to replace D by its image D^* in polar coordinates and $dz \, dA$ by $r \, dz \, dr \, d\theta$.

EXAMPLE 6. Find the volume of the solid E bounded by the surfaces

$$y = x, \quad y = -x, \quad x^2 + y^2 = 5z, \quad z = 7$$

so that $y \geq 0$.

we use here Remark 5



$$V(E) = \iiint_E dV \stackrel{\text{we use here Remark 5}}{=} \iint_D \left[\int_{\frac{x^2+y^2}{5}}^7 dz \right] dA =$$

$$= \iint_D \left(7 - \frac{x^2+y^2}{5} \right) dA$$

where D is projection of E onto the xy -plane.

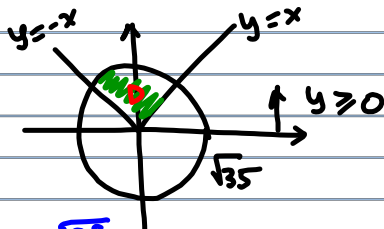
use polar coordinates

$$\left. \begin{array}{l} x^2 + y^2 = 5z \\ z = 7 \end{array} \right\} \Rightarrow x^2 + y^2 = 35$$

$$= \iint_{D^*} \left(7 - \frac{r^2}{5} \right) dA^*$$

where

$$D^* = \left\{ (r, \theta) \mid 0 \leq r \leq \sqrt{35}, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \right\}$$



$$V(E) = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{35}} \left(7 - \frac{r^2}{5} \right) r \, dr \, d\theta = \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \int_0^{\sqrt{35}} \left(7r - \frac{r^3}{5} \right) dr$$

$$= \frac{\pi}{2} \left(\frac{7r^2}{2} - \frac{r^4}{20} \right) \Big|_0^{\sqrt{35}} = \frac{\pi}{2} \left(\frac{7 \cdot 35}{2} - \frac{35^2}{20} \right)$$

$$= \frac{\pi}{2} \cdot 7 \cdot 35 \left(\frac{1}{2} - \frac{5}{20} \right) = \frac{245\pi}{8} \text{ unit}^3$$

