13.9-13.10: Part II

Triple integrals in spherical coordinates

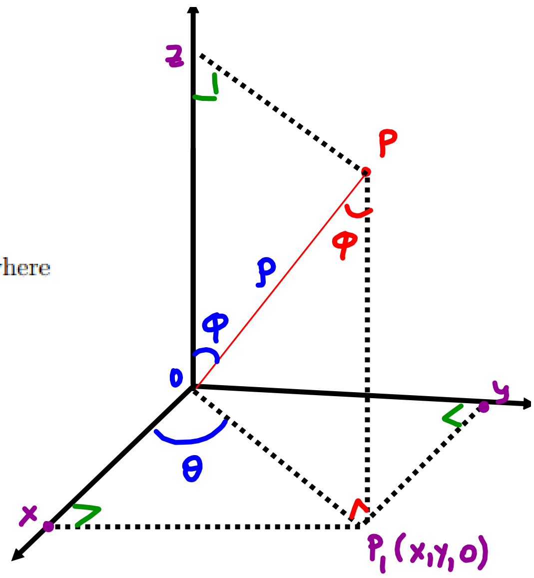
- **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

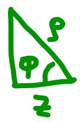
$$x = |OP_1| \cdot \cos\theta = \rho \sin\varphi \cos\theta$$

$$y = |OP_1| \sin\theta = \rho \sin\varphi \sin\theta$$

$$|OP_1| = \rho \sin\varphi$$

$$z = \rho\cos\varphi$$





We have

$$\boxed{\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho\cos\varphi \end{aligned}}$$

REMARK 1. The spherical coordinates

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\phi \\
\rho &\geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi
\end{aligned}$$

are especially useful in problems where there is *symmetry about the origin.*

Note that

$$\begin{aligned}
x^2 + y^2 + z^2 &= \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta + \rho^2 \cos^2\varphi \\
&= \rho^2 \left(\sin^2\varphi \cos^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\varphi \right) \\
&= \rho^2 \Big(\underbrace{\sin^2\varphi \big(\underbrace{\cos^2\theta + \sin^2\theta}_{1} \big) + \cos^2\varphi}_{1} \Big) = \rho^2
\end{aligned}$$

Finally, $\boxed{x^2 + y^2 + z^2 = \rho^2}$

EXAMPLE 2. *Find equation in spherical coordinates for the following surfaces.*

(a) $x^2 + y^2 + z^2 = 16$ $\Rightarrow$ $\rho^2 = 16$ $\Rightarrow$ $\boxed{\rho = 4}$

$\rho \geq 0$

(b) $z = \sqrt{x^2 + y^2}$

$\sqrt{\sin^2\varphi} = \sin\varphi$

$z = \sqrt{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}$

$= \rho|\sin\varphi| \sqrt{\cos^2\theta + \sin^2\theta} = \rho \sin\varphi$

$0 \leq \varphi \leq \pi$

$z = \rho \sin\varphi$

$\rho \cos\varphi = \rho \sin\varphi$

$\tan\varphi = 1$

$\underbrace{\varphi = \frac{\pi}{4}}_{z \geq 0}$ or $\underbrace{\frac{5\pi}{4}}_{z \leq 0}$

$\boxed{\varphi = \frac{\pi}{4}}$

(c) $z = \sqrt{3x^2 + 3y^2} = \sqrt{3}\,\sqrt{x^2+y^2}$ By (b)

$\rho\cos\varphi = \sqrt{3}\,\underbrace{\rho\sin\varphi}$

$\tan\varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \boxed{\varphi = \frac{\pi}{6}}$

(d) $x = y$

$\cancel{\rho}\,\cancel{\sin\varphi}\cos\theta = \cancel{\rho}\,\cancel{\sin\varphi}\sin\theta$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

•**Triple integrals in spherical coordinates**

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

$$(x, y, z) \quad \rightarrow \quad (r, \theta, z) \quad \rightarrow \quad (\rho, \theta, \phi)$$

$$x = r\cos\theta \qquad z = \rho\cos\phi$$
$$y = r\sin\theta \qquad r = \rho\sin\phi$$
$$z = z \qquad \theta = \theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,dr\,d\theta\,dz = r\rho\,d\rho\,d\theta\,d\varphi = \rho\sin\varphi\,\rho\,d\rho\,d\theta\,d\varphi$$

$$dV = \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi$$

THEOREM 3. *Let $f(x,y,z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then*

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\underbrace{\rho^2\sin\phi}\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

EXAMPLE 4. *Evaluate* $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\,\mathrm{d}V$ *where* $E = \{(x,y,z) : 9 \le \overbrace{x^2+y^2+z^2}^{\rho^2} \le 16\}$

$E^* = \{(\rho,\theta,\varphi) \mid 3 \le \rho \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

$$I = \iiint_{E^*} e^{\sqrt{(\rho^2)^3}} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_3^4 \int_0^{2\pi} \int_0^{\pi} \rho^2 e^{\rho^3} \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= 2\pi \Big(\underbrace{\int_3^4 \rho^2 e^{\rho^3} d\rho}_{\text{u-sub.}} \Big) \int_0^{\pi} \sin\varphi \, d\varphi = \ldots$$

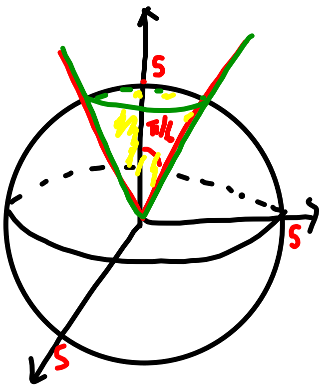
EXAMPLE 5. *Write the integral* $\iiint_E f(x,y,z)\,\mathrm{d}V$ *in spherical coordinates where*

(a) $E = \{(x,y,z) : x^2+y^2+z^2 \le 1, y \ge 0, z \ge 0\}$.

$E^* = \{ (\rho, \theta, \varphi): 0 \le \rho \le 1, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le \frac{\pi}{2} \}$

$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{\pi} \int_0^{\pi/2} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\, d\varphi\, d\theta\, d\rho$

(b) *E is the icecream cone-shaped solid, which is cut from the sphere of radius* 5 *by the cone* $\phi = \pi/6$.



$0 \le \varphi \le \frac{\pi}{6}$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 5$

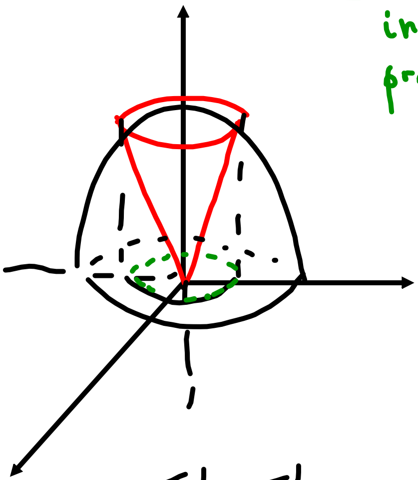
$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/6} \int_0^{2\pi} \int_0^{5} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$

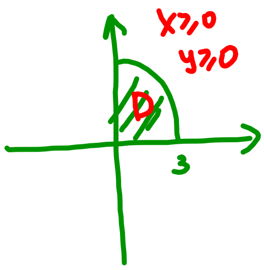
EXAMPLE 6. *Evaluate the integral by changing to spherical coordinates:*

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \iiint_E (x^2+y^2+z^2)\,dV$$

where

$$E = \Big\{(x,y,z) \,\Big|\, \underbrace{0 \le y \le 3,\ 0 \le x \le \sqrt{9-y^2}}_{\text{information about projection } D \text{ of } E \text{ onto the xy-plane.}},\ \underset{\substack{\downarrow \\ \varphi=\pi/4}}{\sqrt{x^2+y^2}} \le z \le \underbrace{\sqrt{18-x^2-y^2}}_{\substack{x^2+y^2+z^2=18 \\ z\ge 0}}\Big\}$$





$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le \varphi \le \frac{\pi}{4}$$

$$0 \le \rho \le \sqrt{18} = 3\sqrt{2}$$

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3\sqrt{2}} \rho^2\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \sin\varphi\, d\varphi \int_0^{3\sqrt{2}} \rho^4\, d\rho$$

$$= \frac{\pi}{2}\left(-\cos\varphi\right)\Big|_0^{\pi/4}\ \frac{\rho^5}{5}\Big|_0^{3\sqrt{2}} = \ldots$$