14.1: Vector Fields

A vector function

$$\vec{r}(t) = \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is an example of a function whose domain is a set of real numbers and whose range is a set of vectors in $\mathbb{R}^3$:

$$\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3.$$

Consider a type of functions (**vector fields**) whose domain is $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$):

Vector field over $\mathbb{R}^2$.

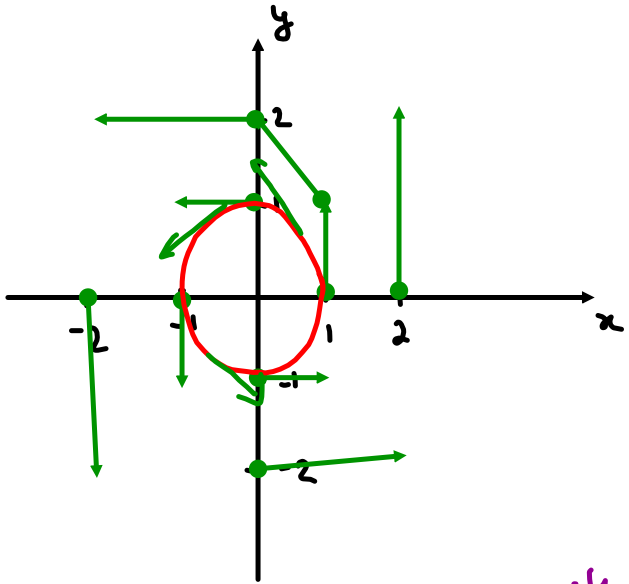
$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = \langle P(x,y), Q(x,y)\rangle$$

Vector field over $\mathbb{R}^3$:

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

EXAMPLE 1. *Describe the vector field* $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ *by sketching.*

(x,y)	$\vec{F}(x,y) = \langle -y, x\rangle$
$(0,0)$	$\vec{0}$
$(1,0)$	$\langle 0, 1\rangle$
$(0,1)$	$\langle -1, 0\rangle$
$(-1,0)$	$\langle 0, -1\rangle$
$(0,-1)$	$\langle 1, 0\rangle$
$(1,1)$	$\langle -1, 1\rangle$
$(2,0)$	$\langle 0, 2\rangle$



$|\vec{F}(x,y)| = |\langle -y, x\rangle| = \sqrt{x^2+y^2} = |\langle x, y\rangle|$ magnitude of position vector

$\vec{F}(x,y) \cdot \langle x,y\rangle = \langle -y, x\rangle \cdot \langle x, y\rangle = 0$

$\vec{F}(x,y) \perp \langle x,y\rangle$

Function $u = f(x, y, z)$ is also called a **scalar field**. Its gradient is also called **gradient vector field**:

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

EXAMPLE 2. *Find the gradient vector field of $f(x, y, z) = xyz$.*

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle.$$