Line integral w.r.t. arc length

over plane curve $\mathbb{R}^2$

over space curve $\mathbb{R}^3$

Line integral over vector field

over plane curve, $\mathbb{R}^2$

over space curve $\mathbb{R}^3$

14.2: Line Integrals

$\mathbb{R}^2$

Line integrals on plane: Let C be a plane curve with parametric equations:

$$x = x(t), y = y(t), \quad \underbrace{a \leq t \leq b}, \text{ parameter domain}$$

or we can write the parametrization of the curve as a vector function:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad a \leq t \leq b.$$

DEFINITION 1. *The line integral of $f(x, y)$ with respect to arc length, or the* **line integral of f along C** *is*

$$\int_C f(x, y)\, \mathrm{d}s$$

Recall that the *arc length* of a curve given by parametric equations $x = x(t), y = y(t), \quad a \leq t \leq b$ can be found as

$$\int_C ds = L = \int_a^b \mathrm{d}s, = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

where

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

The line integral is then

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathrm{r}'(t) = \langle x'(t), y'(t) \rangle$$

and then

$$\boxed{\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t = |\vec{r}\,'(t)|\, dt}$$

$a < b$

Using this notation the line integral becomes,

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t),y(t))|\mathrm{r}'(t)|\,\mathrm{d}t.$$

REMARK 2. The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as t increases from a to b.*

Let us emphasize that $\quad \mathrm{d}s = |r'(t)|\,\mathrm{d}t = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t.$

EXAMPLE 3. *Evaluate the line integral* $\int_C y\,ds$, *where* $C: x = t^3, y = t^2, 0 \le t \le 1.$

$$ds = \sqrt{(x')^2 + (y')^2}\,dt = \sqrt{(3t^2)^2 + (2t)^2}\,dt$$

$$= \sqrt{9t^4 + 4t^2}\,dt = t\sqrt{9t^2+4}\,dt$$

$$\int_C y\,ds = \int_0^1 \underbrace{t^2}_{\frac{u-4}{9}}\, t\sqrt{9t^2+4}\,dt = \int_4^{13} \frac{u-4}{9}\sqrt{u}\,\frac{du}{18}$$

$u = 9t^2 + 4$

$du = 18t\,dt$

$4 \le u \le 13$

$$= \int_4^{13} \frac{u\sqrt{u} - 4\sqrt{u}}{9 \cdot 18}\,du = \ldots$$

Line integrals in space: Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

The **line integral of f along C** is

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(x(t),y(t),z(t))|r'(t)|\,\mathrm{d}t.$$

Here

$$\boxed{\boxed{\mathrm{d}s = |r'(t)|\,\mathrm{d}t} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t.}$$

EXAMPLE 4. *Evaluate the line integral* $\int_C (x+y+z)\,\mathrm{d}s$*, where* C *is the line segment joining* $(-1,1,2)$ *and* $(2,3,1)$.

Parameterize C:

$\vec{v} = \langle 2,3,1\rangle - \langle -1,1,2\rangle = \langle 3,2,-1\rangle$

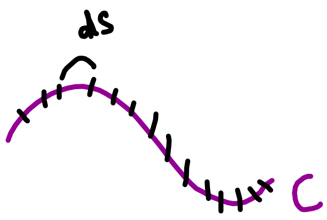
$\vec{r}(t) = \langle \underbrace{2+3t}_{x}, \underbrace{3+2t}_{y}, \underbrace{1-t}_{z} \rangle, \quad -1 \le t \le 0$

$\vec{r}\,'(t) = \langle 3,2,-1\rangle \implies ds = |\vec{r}\,'(t)|\,dt = \sqrt{3^2+2^2+1^2}\,dt = \sqrt{14}\,dt$

$$\int_C (x+y+z)\,ds = \int_{-1}^{0} (2+3t + 3+2t+1-t)\sqrt{14}\,dt$$

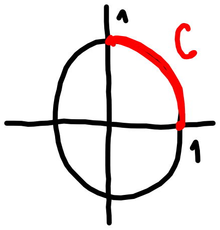
$$= \sqrt{14}\int_{-1}^{0} (6+4t)\,dt = \ldots$$

Physical interpretation of a line integral: Let $\rho(x, y, z)$ represents the linear density at a point (x, y, z) of a thin wire shaped like a curve C. Then the **mass** m of the wire is:



$$\boxed{m = \int_C \rho(x, y, z)\, ds.}$$

EXAMPLE 5. *A thin wire with the linear density $\rho(x, y) = x^2 + 2y^2$ takes the shape of the curve C which consists of the arc of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$. Find the mass of the wire.* in counterclockwise direction



$C: \vec{r}(t) = \langle \cos t, \sin t \rangle, 0 \le t \le \frac{\pi}{2}$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$m(C) = \int_C \rho(x,y)\, ds = \int_C (x^2 + 2y^2)\, ds$$

$$= \int_0^{\pi/2} (\overbrace{\cos^2 t + 2\sin^2 t}^{\overbrace{\sin^2 t + \sin^2 t}^{1}})\, |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{\pi/2} (1 + \sin^2 t) \cdot 1\, dt = \frac{\pi}{2} + \int_0^{\pi/2} \sin^2 t\, dt$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \frac{1 - \cos 2t}{2}\, dt = \ldots$$

Line integrals with respect to x, y, and z. Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \leq t \leq b,$$

The **line integral of f with respect to** x is,

$$\int_C f(x, y, z)\,\mathrm{d}x = \int_a^b f(x(t), y(t), z(t))x'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** y is,

$$\int_C f(x, y, z)\,\mathrm{d}y = \int_a^b f(x(t), y(t), z(t))y'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** z is,

$$\int_C f(x, y, z)\,\mathrm{d}z = \int_a^b f(x(t), y(t), z(t))\, z'(t)\,dt$$

These two integral often appear together by the following notation:

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

or

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

EXAMPLE 6. *Compute*

$$= \int_C \vec{F}\cdot d\vec{r} \quad \text{where } \vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

$$I = \int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int -y\,dx + x\,dy$$

where C is the circle $x^2 + y^2 = 1$ oriented in the counterclockwise direction.

Parameterize C :

C : $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

$dx = -\sin t\,dt \;, \quad dy = \cos t\,dt$

$$I = \int_0^{2\pi} -\sin t(-\sin t)\,dt + \cos t\cos t\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

Line integrals of vector fields.

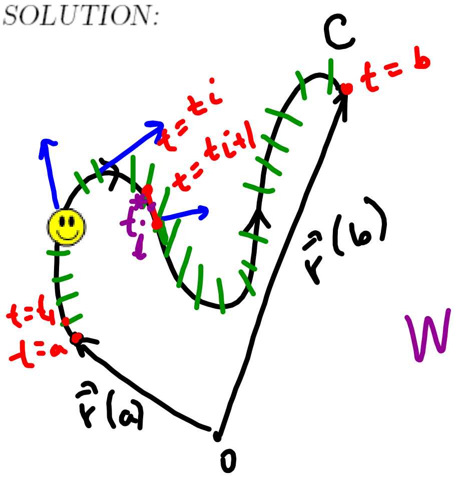
PROBLEM: Given a continuous force field,

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

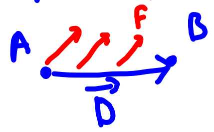
such as a gravitational field. Find the work done by the force **F** in moving a particle along a curve

$$C : \vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b.$$

SOLUTION:



Recall



$[t_i, t_{i+1}]$

$W = \vec{F} \cdot \vec{D}$

$W_i \approx \vec{F}(\vec{r}(t_i^*)) \cdot \Delta\vec{r}(t_i)$

where $\Delta\vec{r}(t_i) \approx \vec{r}(t_{i+1}) - \vec{r}(t_i)$

$\|P\| = \max_i |\Delta\vec{r}(t_i)|$

$$W = \lim_{\|P\|\to 0} \sum_i W_i$$

$$W = \lim_{\|P\|\to 0} \sum_i \vec{F}(\vec{r}(t_i^*)) \cdot \Delta\vec{r}(t_i) = \int_C \vec{F} \cdot d\vec{r}$$

Line integral of vector field

DEFINITION 7. *Let* **F** *be a continuous vector field defined on a curve* C *given by a vector function* $\mathbf{r}(t)$, $a \le t \le b$. *Then the* **line integral of F along** C *is*

$$W = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}(t) = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,\mathrm{d}t.$$

REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F}\cdot \mathrm{d}\mathbf{r}(t) = -\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}(t)$$

EXAMPLE 9. *Find the work done by the force field* $\mathbf{F}(x,y,z) = \langle \underbrace{xy}_{P}, \underbrace{yz}_{Q}, \underbrace{xz}_{R}\rangle$ *in moving a particle along the curve* $C : \mathbf{r}(t) = \langle t, t^2, t^3\rangle$, $0 \le t \le 1$.

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$\int_C xy\,dx + yz\,dy + xz\,dz$$

$$= \int_0^1 \langle t\cdot t^2, t^2\cdot t^3, t\cdot t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle\,dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\,dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

Relationship between line integrals of vector fields and line integrals with respect to x, y, and z.

$\int_C \mathbf{F} \cdot d\mathbf{r}$ [redacted] $= \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle$

$$= \int_C P\,dx + Q\,dy + R\,dz$$