14.3: The fundamental Theorem for Line Integrals

14.4: Green's Theorem

• Conservative vector field. (see also Section 14.1 in textbook)

DEFINITION 1. *A vector field* **F** *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* **F**.

$$\vec{F} = \langle P, Q, R \rangle = \nabla f = \langle f_x, f_y, f_z \rangle \Rightarrow \begin{array}{l} f_x = P \\ f_y = Q \\ f_z = R \end{array}$$

Example if $\vec{F} = \langle x, y^2, z^3 \rangle$ conservative

if $\underbrace{f}_{\text{a potential function}} = \frac{x^2}{2} + \frac{y^3}{3} + \frac{z^4}{4} + C$ then $\nabla f = \vec{F}$

REMARK 2. Not all vector fields are conservative, but such fields do arise frequently in Physics.

`Illustration:` **Gravitational Field:** By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m and M is The gravitational force acting on the object at (x, y, z) is

$$|\mathbf{F}| = G\frac{mM}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the objects and G is the gravitational constant.

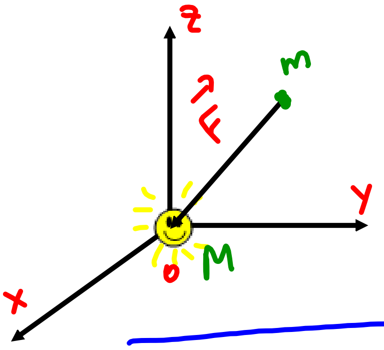
Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. ~~Then~~

Then the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\mathbf{F}(x, y, z) = |\vec{F}| \cdot \hat{F} = |\vec{F}| \left(\frac{-\langle x, y, z\rangle}{r} \right)$$

$$= -\frac{|\vec{F}|}{r}\langle x, y, z\rangle = -\frac{GmM}{r^2 \cdot r}\langle x, y, z\rangle$$

$$= -\frac{GmM}{r^3}\langle x, y, z\rangle$$



$$\boxed{\vec{F}(x,y,z) = -\left\langle \frac{GmMx}{(x^2+y^2+z^2)^{3/2}}, \frac{GmMy}{(x^2+y^2+z^2)^{3/2}}, \frac{GmMz}{(x^2+y^2+z^2)^{3/2}} \right\rangle}$$

EXAMPLE 3. *Let*

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}.$$

Find its gradient and answer the questions:

$$\left(\square^{-\frac{1}{2}}\right)' = -\frac{1}{2}\square^{\cdot 3/2}\cdot\square'$$

(a) *Is the gravitational field conservative?* Yes, because $\exists f \Rightarrow \nabla f = \vec{F}$

(b) *What is a potential function of the gravitational field?*

$$f_x = -\frac{GmM\cdot\cancel{2}x}{\cancel{2}(x^2+y^2+z^2)^{3/2}}$$

$$\nabla f = -\left\langle \frac{GmMx}{(\quad)^{3/2}}, \frac{GmMy}{(\quad)^{3/2}}, \frac{GmMz}{(\quad)^{3/2}} \right\rangle$$

$\parallel$ $\vec{F}$, where $\vec{F}$ is gravitational field.

• **The fundamental Theorem for Line Integrals:** Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

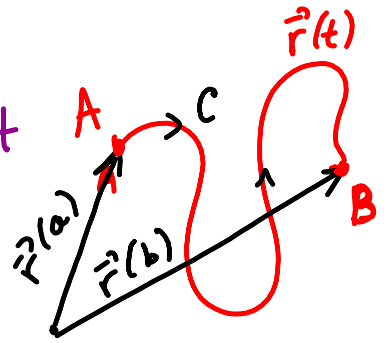
where F' is continuous on $[a,b]$.

parametrization

Let C be a smooth curve given by $\mathbf{r}(t),\ a \le t \le b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \mathrm{dr} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$$

Proof.



$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \langle f_x(\vec{r}(t)), f_y(\vec{r}(t)), f_z(\vec{r}(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\, dt$$

$$= \int_a^b \left(f_x(\vec{r}(t))x'(t) + f_y(\vec{r}(t))\, y'(t) + f_z(\vec{r}(t))\, z'(t) \right) dt \overset{\text{by Chain Rule}}{=}$$

$$= \int_a^b \frac{d f(\vec{r}(t))}{dt}\, dt \overset{FT}{=} f(\vec{r}(b)) - f(\vec{r}(a)) \qquad \square$$

REMARK 4. If C is a closed curve then $\vec{r}(a) = \vec{r}(b)$ and then

$$\oint_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

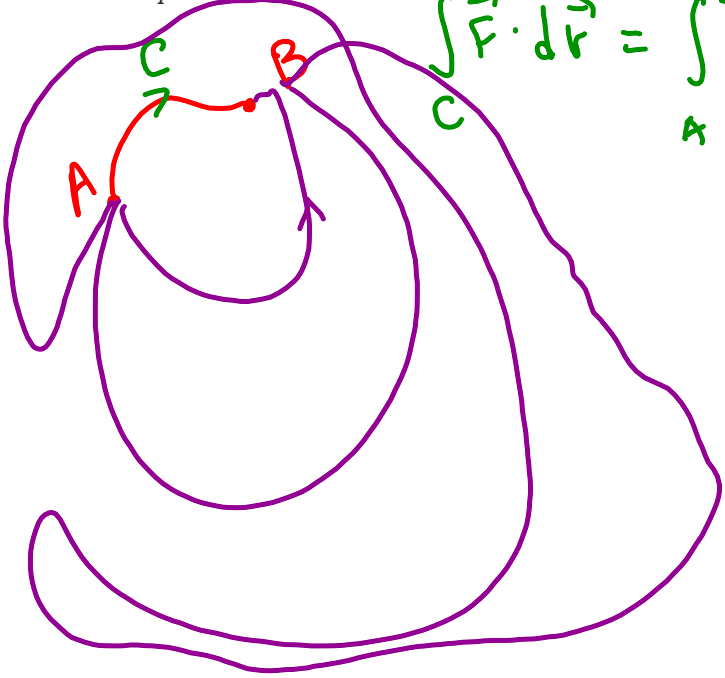
$$\boxed{\oint_C \nabla f \cdot d\vec{r} = 0}$$

If $\vec{F}$ is conservative then $\oint_C \vec{F} \cdot d\vec{r} = 0$

COROLLARY 5. *If F is a conservative vector field and C is a curve with initial point A and terminal point B then:*

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

independenly of path of integration



EXAMPLE 6. *Find the work done by the gravitational field*

$$\mathbf{F}(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

in moving a particle with mass m *from the point* $(1,2,2)$ *to the point* $(3,4,12)$ *along a piecewise-smooth curve* C.

By Example 3, $\vec{F}$ is conservative and its potential is

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}.$$

Thus, $\vec{F} = \nabla f$.

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_{(1,2,2)}^{(3,4,12)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=\!=} f(3,4,12) - f(1,2,2)$$

$$= \frac{GmM}{\sqrt{12^2+4^2+3^2}} - \frac{GmM}{\sqrt{1^2+2^2+2^2}} = GmM\left(\frac{1}{13}-\frac{1}{3}\right)$$

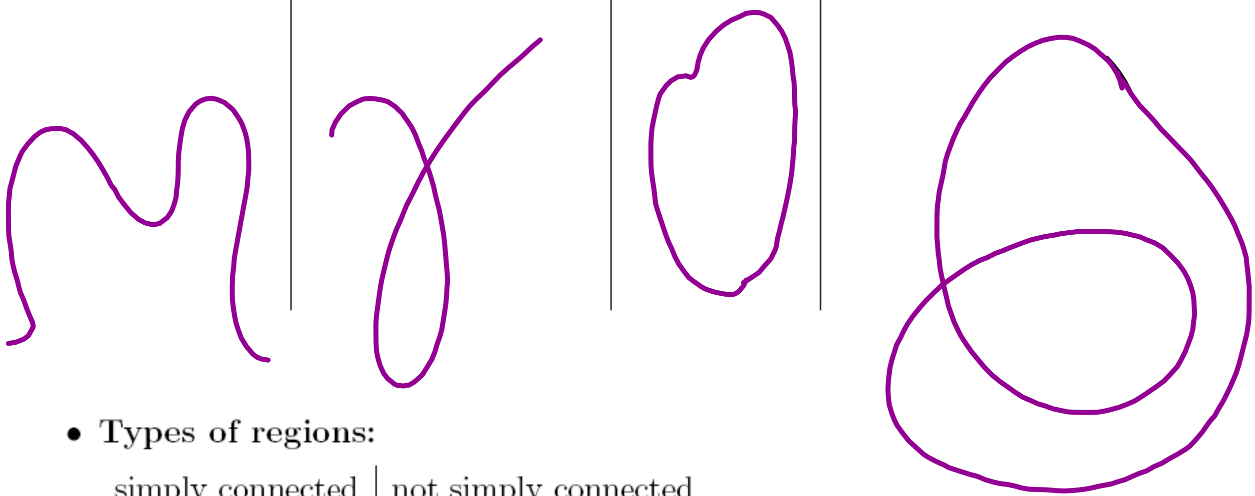
$$= -\frac{10}{39}GmM$$

Notations And Definitions:

DEFINITION 7. *A piecewise-smooth curve is called a* **path**.

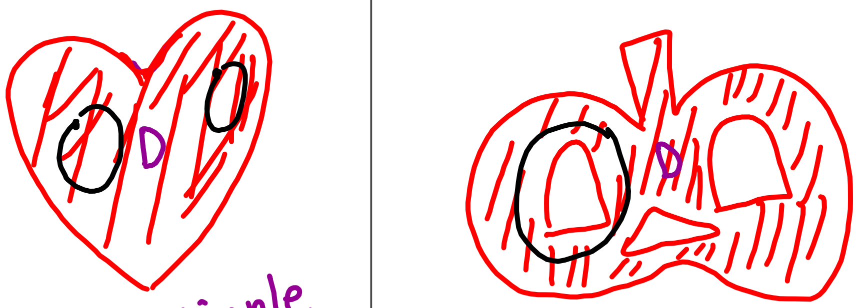
- **Types of curves:**

simple not closed	not simple not closed	simple closed	not simple, closed



- **Types of regions:**

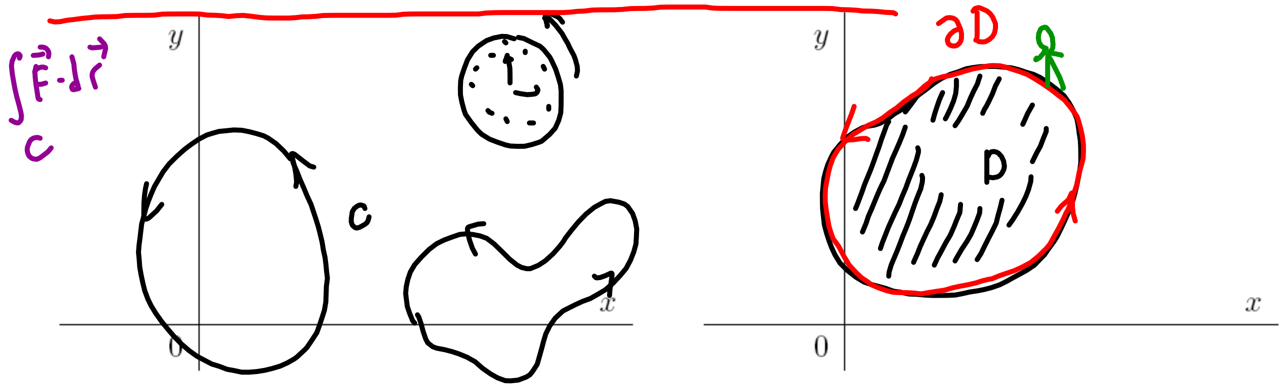
simply connected	not simply connected



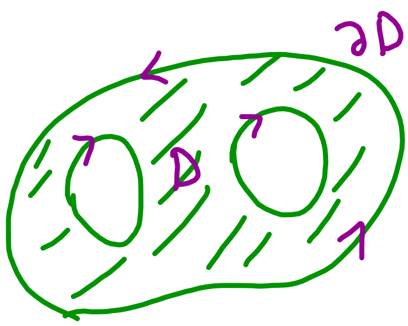
∂D is a simple closed curve

∂D is a union of simple closed curves

- **Convention:** The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.

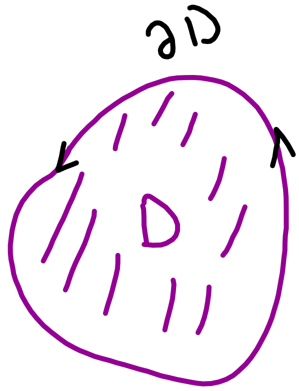


- The positively oriented boundary curve of D is denoted by ∂D.



•**GREEN's THEOREM:** *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then*

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A.$$



$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$

Important: C is a <u>closed</u> curve

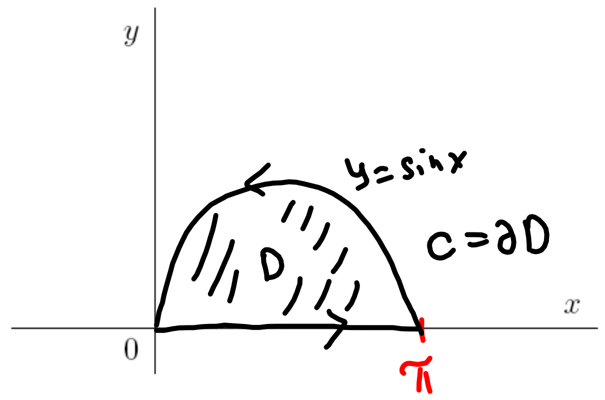
If C is oriented clockwise then

$$\oint_C \vec{F}\cdot d\vec{r} = -\oint_{\partial D} \vec{F}\cdot d\vec{r} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

EXAMPLE 8. *Evaluate:*

$$I = \oint_C \overbrace{e^x(1-\cos y)}^{P}\,\mathrm{d}x \overbrace{- e^x(1-\sin y)}^{Q}\,\mathrm{d}y$$

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



Way 1

Parameterize C

Way 2 C is a closed simple path.

Use Green's Theorem

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(-e^x(1-\sin y)\right) - \frac{\partial}{\partial y}\left(e^x(1-\cos y)\right)$$

$$= -e^x(1-\sin y) - e^x \sin y$$

$$= -e^x + e^x \cancel{\sin y} - e^x \cancel{\sin y} = -e^x$$

$$I = \iint_D (-e^x)\,dA = -\int_0^{\pi}\int_0^{\sin x} e^x\,dy\,dx$$

$$= -\int_0^{\pi} e^x \sin x\,dx$$

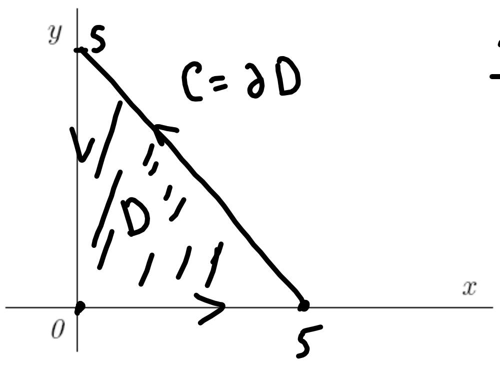
loop Integral

(Integrate by parts twice)

EXAMPLE 9. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to $(5,0)$, from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integrals:*

(a) $I_1 = \oint_C \underbrace{(x^2y + \frac{1}{2}y^2)}_{P}\,\mathrm{d}x + \underbrace{(xy + \frac{1}{3}x^3 + 3x)}_{Q}\,\mathrm{d}y \overset{GT}{=} \iint_D 3\,dA = 3\iint_D dA = 3A(D) = 3\cdot\frac{5\cdot 5}{2} = \boxed{\frac{75}{2}}$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y + x^2 + 3 - (x^2 + y) = 3$$



(b) $I_2 = \oint_C \underbrace{(x^2y + \frac{1}{2}y^2 + e^{x\sin x})}_{P}\,\mathrm{d}x + \underbrace{(xy + \frac{1}{3}x^3 + x - 4\arctan(e^y))}_{Q}\,\mathrm{d}y = \frac{I_1}{3} = A(D) = \boxed{\frac{25}{2}}$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

(c) $I_3 = \oint_C (x^2y + \frac{1}{2}y^2 - 55\arcsin(\sec x))\,\mathrm{d}x + (12y^5\cos y^3 + xy + \frac{1}{3}x^3 + x)\,\mathrm{d}y = A(D) = \frac{25}{2}$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

•Application: Computing areas. Choose P & Q s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

$\boxed{P = 0, \quad Q = x}$ $\quad \boxed{P = -y, \ Q = 0}$ $\quad \boxed{P = -\frac{y}{2}, \quad Q = \frac{x}{2}}$

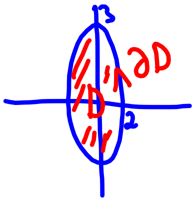
$$\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1}$$

$\oint_{\partial P} P \cancel{dx} + Q\, dy$ ($P = 0$)

$$\boxed{A(D) = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx.}$$

EXAMPLE 10. *Find the area enclosed by the ellipse* $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1.$

$D = \{(x,y) \mid \frac{x^2}{4} + \frac{y^2}{9} \le 1\}$



line integral

$$A(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$$

where ∂D: $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Parameterize ∂D: $x = 2\cos t$, $y = 3\sin t$

$0 \le t \le 2\pi$

$dx = -2\sin t\, dt$ $\qquad dy = 3\cos t\, dt$

$$A(D) = \frac{1}{2}\int_0^{2\pi} \left(2\cos t \cdot 3\cos t - 3\sin t(-2\sin t)\right) dt$$

$$= \frac{1}{2}\cdot 6 \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = 3 \cdot \int_0^{2\pi} dt = 6\pi$$

Note: The area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal $\boxed{\pi ab}$

$\vec{F}$ in $\mathbb{R}^2$

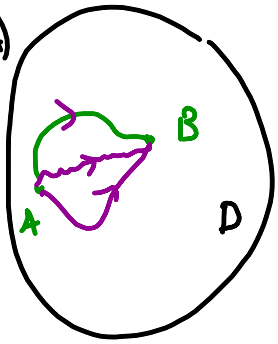
SUMMARY: Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *be a vector field on ~~an open~~ simply connected domain* D*. Suppose that* P *and* Q *have continuous partial derivatives through* D*. Then the facts below are equivalent.*

The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	There exists $\underbrace{f}_{\text{potential}}$ s.t. $\nabla f = \mathbf{F}$

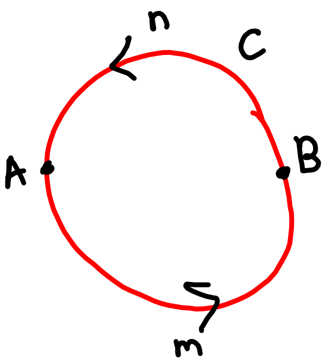
$\Downarrow$

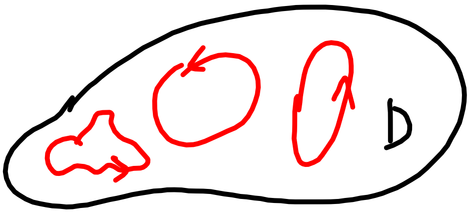
$$\int_{AB} \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	$\int_{\overset{\frown}{AB}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path in D



The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for every closed curve C in D





$$\left.\begin{array}{l} \oint_C \vec{F} \cdot d\vec{r} = \int_{AmB} + \int_{BnA} \\ \int_{AmB} = -\int_{BnA} \end{array}\right\} \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$

The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	$\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout D

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Uparrow$

$$0 = \oint_{\partial D} \vec{F} \cdot d\vec{r} \overset{GT}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

EXAMPLE 11. *Determine whether or not the vector field is conservative:*

(a) $\mathbf{F}(x, y) = \langle \underbrace{x^2 + y^2}_{P}, \underbrace{2xy}_{Q} \rangle$.

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \text{ is conservative}$$

(b) $\mathbf{F}(x, y) = \langle \underbrace{x^2 + 3y^2 + 2}_{P}, \underbrace{3x + ye^y}_{Q} \rangle$

$\vec{F}$ is not conservative

$$\frac{\partial P}{\partial y} = 6y \neq \frac{\partial Q}{\partial x} = 3$$

EXAMPLE 12. *Given* $\mathbf{F}(x,y) = \underbrace{\sin y}_{P}\mathbf{i} + \underbrace{(x\cos y + \sin y)}_{Q}\mathbf{j}$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$$\frac{\partial P}{\partial y} = \cos y = \frac{\partial Q}{\partial x}$$

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$

$$\langle f_x, f_y \rangle = \langle P, Q \rangle$$

$$\begin{cases} f_x = \sin y \\ f_y = x\cos y + \sin y \end{cases}$$

$\Rightarrow$ $f(x,y) = \int \sin y \, dx = x\sin y + C(y)$

$\Rightarrow$ $\frac{\partial}{\partial y}(x\sin y + C(y)) = x\cos y + \sin y$

$\cancel{x\cos y} + C'(y) = \cancel{x\cos y} + \sin y$

$C'(y) = \sin y \Rightarrow C(y) = -\cos y + C$

$$\boxed{f(x,y) = x\sin y - \cos y + C}$$

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3,0)$ *to the point* $(0,\pi/2)$.

$$W = \int_{(3,0)}^{(0,\pi/2)} \vec{F}\cdot d\vec{r} = \int_{(3,0)}^{(0,\pi/2)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(0,\tfrac{\pi}{2}) - f(3,0)$$

$$= 0\cdot \sin\tfrac{\pi}{2} - \cos\tfrac{\pi}{2} - (3\sin 0 - \cos 0)$$

$$= \boxed{1}$$

(d) *Evaluate* $\oint_C \mathbf{F}\,\mathrm{d}\mathbf{r}$ *where* C *is an arbitrary path in* $\mathbb{R}^2$.

$\vec{F}$ is conservative $\qquad \oint_C \vec{F}\cdot d\vec{r} = 0 \quad \forall C$ in $\mathbb{R}^2$ closed

EXAMPLE 13. *Given*

$$\mathbf{F} = \langle \overbrace{2xy^3 + z^2}^{P}, \overbrace{3x^2y^2 + 2yz}^{Q}, \overbrace{y^2 + 2xz}^{R} \rangle . = \vec{F}(x,y,z)$$

Find a function f s.t. $\nabla f = \mathbf{F}$

$\langle f_x, f_y, f_z \rangle = \langle P, Q, R \rangle$

$f_x = 2xy^3 + z^2 \Rightarrow f = \int (2xy^3 + z^2)\,dx$

$$\left.\begin{array}{l} f = x^2y^3 + xz^2 + C(y,z) \\ f_y = 3x^2y^2 + 2yz \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial y}\left(x^2y^3 + xz^2 + C(y,z)\right) = 3x^2y^2 + 2yz$$

$$\cancel{3x^2y^2} + 0 + \frac{\partial C(y,z)}{\partial y} = \cancel{3x^2y^2} + 2yz$$

$$C(y,z) = \int 2yz\,dy$$

$$C(y,z) = y^2z + K(z)$$

$$\left.\begin{array}{l} f = x^2y^3 + xz^2 + y^2z + K(z) \\ f_z = y^2 + 2xz \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial z}\left(x^2y^3 + xz^2 + y^2z + K(z)\right) = y^2 + 2xz$$

$$0 + \cancel{2xz} + \cancel{y^2} + K'(z) = \cancel{y^2} + \cancel{2xz}$$

$$K'(z) = 0 \Rightarrow \underline{K(z) = Const}$$

$$\boxed{f(x,y,z) = x^2y^3 + xz^2 + y^2z + Const}$$