14.5: Curl and Divergence

Introduce the vector differential operator ∇ as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the **curl of F** is the *vector field* on $\mathbb{R}^3$ defined by

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\hat{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\hat{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\hat{k}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

EXAMPLE 1. *Find the curl of the vector field*

$$\mathbf{F}(x,y,z) = \langle xy, x^2, yz \rangle.$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & yz \end{vmatrix} = \langle z-0, -(0-0), 2x-x \rangle$$

$$= \langle z, 0, x \rangle$$

Question *What is the curl of a two-dimensional vector field*

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \ ?$$

Answer: Then $R(x, y, z) = 0$ and $\frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0$.

Thus $\text{curl}\,\vec{F} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$

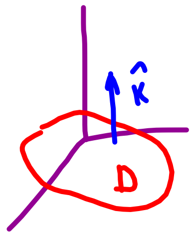
ok $= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$

CONCLUSION: Green's Theorem in vector form:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \underline{\hat{k} \cdot \hat{k} = 1}$$

$$= \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}}_{\text{curl}\,\vec{F}} \cdot \hat{k} \, dA = \iint_D \text{curl}\,\vec{F} \cdot \hat{k} \, dA$$

$$\boxed{\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\,\vec{F} \cdot \hat{k} \, dA}$$



$\hat{k} \perp D$ (to the plane containing D).

THEOREM 2. *If a function $f(x, y, z)$ has continuous partial derivatives of second order then*

$$\underbrace{\mathrm{curl}}_{}(\nabla f) = \vec{0}.$$

gradient vector field

Proof.

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \underbrace{(\nabla \times \nabla)}_{\substack{\parallel \\ 0}} f = \vec{0}$$

COROLLARY 3. *If* **F** *is conservative, then* curl**F** $= \vec{0}$.

$\Updownarrow$

there exists a potential function, say f, s.t.

$\vec{F} = \nabla f$

Then $\quad \text{curl } \vec{F} = \text{curl}(\nabla f) = \vec{0}$.

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. *If* $\mathbf{F}$ *is a vector field defined on* R^3 *whose component functions have continuous partial derivatives and* $\mathrm{curl}\mathbf{F} = 0$, *then* $\mathbf{F}$ *is a conservative vector field.*

EXAMPLE 5. *Let* $\mathbf{F}(x, y, z) = \langle x^9, y^9, z^9 \rangle$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^9 & y^9 & z^9 \end{vmatrix} = \vec{0}$$

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$.

Guess $f(x,y,z) = \frac{x^{10}}{10} + \frac{y^{10}}{10} + \frac{z^{10}}{10} + \text{Const}$

(check!)

(c) *Evaluate* $\int_{(1,0,1)}^{(-1,-1,-1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(1,0,1)}^{(-1,-1,-1)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=}$

$$= f(-1,-1,-1) - f(1,0,1)$$

$$= \frac{3}{10} - \frac{2}{10} = \frac{1}{10} = \boxed{0.1}$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on R^3 and the partial derivatives P_x, Q_y, R_z exist, then the **divergence of F** is the *scalar field* on defined by

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle \quad .$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ ← scalar field

EXAMPLE 6. *Find the divergence of the vector field*

$$\mathbf{F}(x,y,z) = \langle \sin(xyz), x^2, yz \rangle .$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(\sin(xyz)) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(yz)$$

$$= yz\cos(xyz) + y$$

THEOREM 7. *If the components of a vector field* $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ *has continuous partial derivatives of second order then*

$$\text{div}\ \underbrace{\text{curl}\ \mathbf{F}} = 0.$$

Proof.

$$\text{div}\left(\text{curl}\ \vec{F}\right) = \nabla \cdot \left(\nabla \times \vec{F}\right) = 0$$

$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ because $\vec{a}, \vec{b}$ are alway coplanar

EXAMPLE 8. *Is there a vector field* $\mathbf{G}$ *on* R^3 *s.t.* curl $G = \langle yz, xyz, zy\rangle$*?*

We know that $\text{div}\left(\text{curl}\ \vec{G}\right) = 0$

If such a field does exist then

$$\text{div}\left(\text{curl}\ \vec{G}\right) = \text{div}\langle yz, xyz, zy\rangle$$

$$= 0 + xz + y = 0$$

for all x, y, z

a contradiction.

Such field $\vec{G}$ DNE.