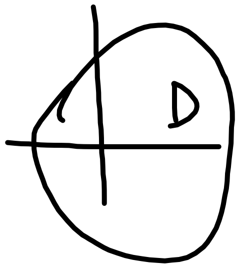
14.6: Parametric surfaces and their areas

Consider a continuous vector valued function of two variables

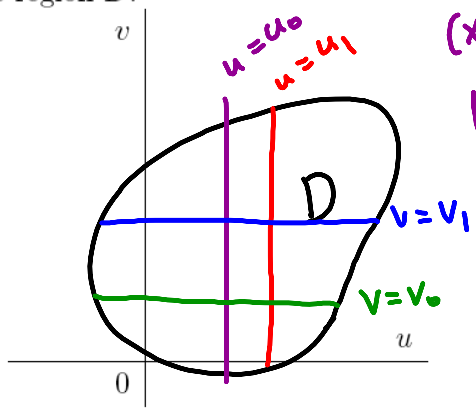
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D.$$

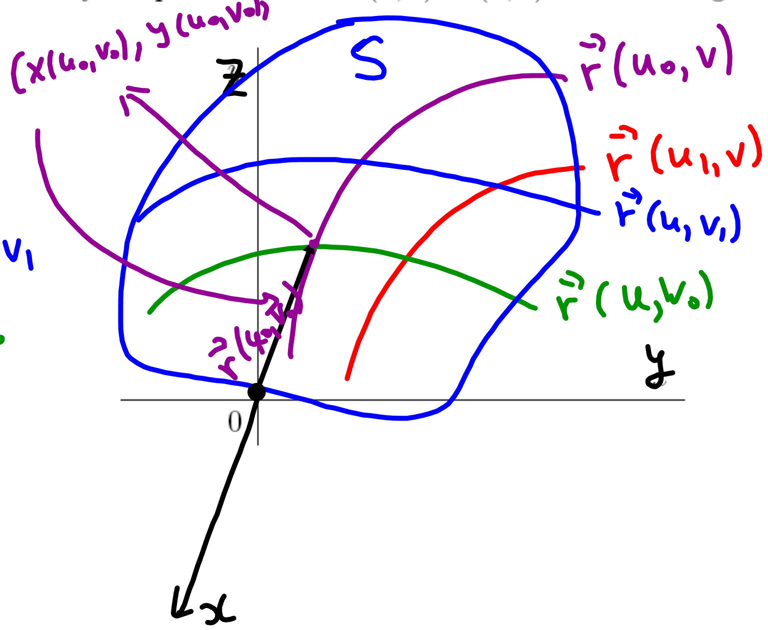


Parametric surface:

$$S: \;\; x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D.





EXAMPLE 1. *Determine the surface given by the parametric representation*

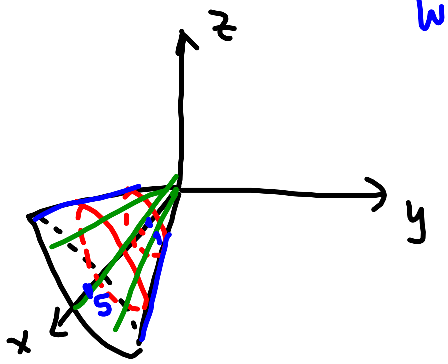
$$\mathbf{r}(u,v) = \langle \underbrace{u}_{x}, \underbrace{u\cos v}_{y}, \underbrace{u\sin v}_{z} \rangle\,, \quad \underbrace{1 \le u \le 5, \quad 0 \le v \le 2\pi}_{D}$$

$x = u$

$\left.\begin{aligned} y &= u\cos v \\ z &= u\sin v \end{aligned}\right\} \Rightarrow y^2 + z^2 = u^2$

$\boxed{y^2 + z^2 = x^2}$ cone

where $1 \le x \le 5$



EXAMPLE 2. *Give parametric or vector representations for each of the following surfaces (indicate the domain of the parameters):*

(a) *cylinder:* $x^2 + y^2 = 9, \quad 1 \le z \le 5.$

Use cylindrical coordinates

$$\boxed{\begin{gathered} x = r\cos\theta, \quad y = r\sin\theta \\ x^2+y^2=9 \end{gathered}}, \quad z = z$$

$$\Downarrow$$

$$r = 3$$

$$x = 3\cos\theta, \quad y = 3\sin\theta, \quad z = z, \quad D = \{(\theta, z) \mid 0 \le \theta \le 2\pi, 1 \le z \le 5\}$$

(b) *upper half-sphere:* $z = \sqrt{100 - x^2 - y^2}$.

projection onto the xy-plane

Way I $\vec{r}(x,y) = \langle x, y, \sqrt{100-x^2-y^2} \rangle$, $D = \{(x,y) \mid x^2+y^2 \le 100\}$

Spherical coordinates (ρ, θ, φ) $\quad \rho = 10$

Way II $\vec{r}(\theta, \varphi) = \langle 10\sin\varphi\cos\theta, 10\sin\varphi\sin\theta, \underbrace{10\cos\varphi} \rangle$

$$D = \{(\theta,\varphi) \mid 0 \le \theta \le 2\pi, 0 \le \varphi \le \tfrac{\pi}{2}\}$$

(c) *elliptic paraboloid:* $x = 3y^2 + z^2 + 1$, *where* $1 \le x \le 2$.

$\vec{r}(y,z) = \langle 3y^2 + z^2 + 1,\ y,\ z \rangle$

answer

$D = \{ (y,z) \mid 1 \le 3y^2 + z^2 + 1 \le 2 \}$

$D = \{ (y,z) \mid 0 \le 3y^2 + z^2 \le 1 \}$

$D = \{ (y,z) \mid 3y^2 + z^2 \le 1 \}$

(d) *elliptic paraboloid* $y = x^2 + 4z^2$, *where* $0 \le y \le 1$

Way I $\quad \vec{r}(x,z) = \langle x,\ x^2 + 4z^2,\ z \rangle$

$D = \{ (x,z) \mid x^2 + 4z^2 \le 1 \}$

Way II Use cylindrical coordinates as follows:

$x = r\cos\theta, \quad y = y, \quad z = \frac{1}{2} r \sin\theta$

(r, θ, y)
3 parameters.

Eliminate y!

$y = x^2 + 4z^2 = r^2\cos^2\theta + 4 \cdot \frac{r^2\sin^2\theta}{4}$

$0 \le y = r^2 \le 1 \Rightarrow 0 \le r \le 1$

$$\boxed{\begin{array}{l} \vec{R}(r,\theta) = \langle r\cos\theta,\ r^2,\ \frac{1}{2} r\sin\theta \rangle \\ D = \{ (r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1 \} \end{array}}$$

CONCLUSION: To parametrize surface we may use polar, cylindrical or spherical coordinates, or

	Parameter domain D
• $z = f(x,y) \longrightarrow \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$	projection onto the xy-plane
• $y = f(x,z) \longrightarrow \mathbf{r}(x,z) = x\mathbf{i} + f(x,z)\mathbf{j} + z\mathbf{k}$	— " — xz-plane
• $x = f(y,z) \longrightarrow \mathbf{r}(y,z) = f(y,z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	— " — yz-plane

Sometimeis we use generalized ~~of~~ polar, cylindrical or spherical coordinates

$$x = a r\cos\theta \qquad y = b r\sin\theta$$

f.ex. $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$

$$x = a r\cos\theta \qquad y = b r\sin\theta \qquad z = z$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$z = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

$$x = a\rho\sin\varphi\cos\theta \qquad y = b\rho\sin\varphi\sin\theta \qquad z = c\rho\cos\varphi$$

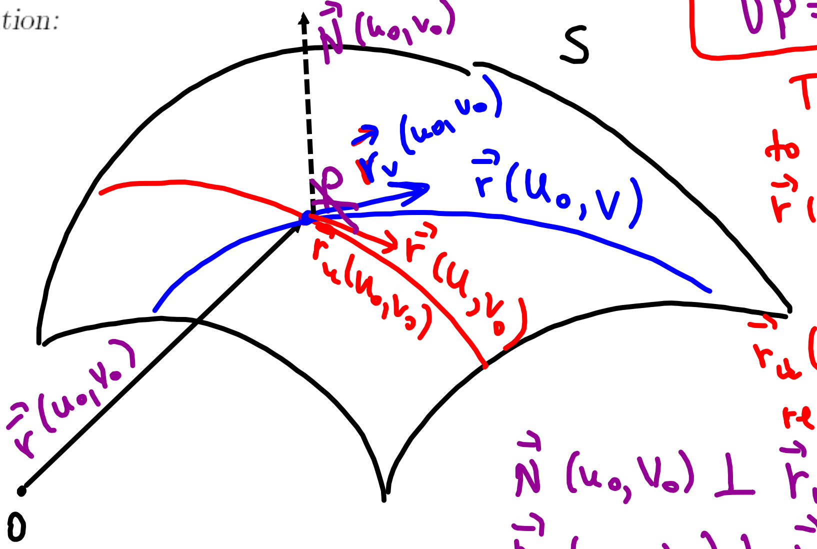
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- **Tangent planes:**

PROBLEM: Find the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u,v)$ *at a point* P_0 *with position vector* $\mathbf{r}(u_0,v_0)$.

Solution:

$$\boxed{\vec{OP} = \vec{r}(u_0,v_0)}$$



Tangent vectors to the curves $\vec{r}(u,v_0)$ and $\vec{r}(u_0,v)$ are $\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$ respectively

$\vec{N}(u_0,v_0) \perp \vec{r}_u(u_0,v_0)$

$\vec{N}(u_0,v_0) \perp \vec{r}_v(u_0,v_0)$

The **normal vector**

$$\boxed{\mathrm{N} = \mathrm{N}(u_0,v_0) = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)}$$

If the normal vector is not $\mathbf{0}$ then the surface S is called **smooth** (it has no "corner").

Special Case: a surface S given by a *graph* $z = f(x, y)$. Then one can choose the following parametrization of S:

$$\mathbf{r}(x, y) = \langle x, y, f(x,y) \rangle$$

and the then the normal vector is

$$\mathrm{N} = \langle f_x, f_y, -1 \rangle$$

Indeed, by above

$$\vec{N}(x,y) = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

Similarly, if S is given by $y = g(x,z)$ then

$$\vec{N}(x,z) = \langle f_x, -1, f_z \rangle$$

If S is given by $x = h(y,z)$ then

$$\vec{N}(y,z) = \langle 1, f_y, f_z \rangle$$

EXAMPLE 3. *Find the tangent plane to the surface with parametric equations* $x = uv + 1, y = ue^v, z = ve^u$ *at the point* $(1, 0, 0)$ $= P$

$$\vec{r}(u,v) = \langle uv+1,\ ue^v,\ ve^u \rangle$$

$$\vec{OP} = \vec{r}(u_0, v_0)$$

$$\langle 1,0,0 \rangle = \langle u_0 v_0 + 1,\ u_0 e^{v_0},\ v_0 e^{u_0} \rangle$$

$$1 = u_0 v_0 + 1$$

$$0 = u_0 e^{v_0} \Rightarrow u_0 = 0$$

$$0 = v_0 e^{u_0} \Rightarrow v_0 = 0$$

$$(u_0, v_0) = (0, 0)$$

Find normal at $(0,0)$.

$$\vec{r}_u = \langle v,\ e^v,\ ve^u \rangle \Rightarrow \vec{r}_u(0,0) = \langle 0,1,0 \rangle$$

$$\vec{r}_v = \langle u,\ ue^v,\ e^u \rangle \Rightarrow \vec{r}_v(0,0) = \langle 0,0,1 \rangle$$

$$\vec{N}(0,0) = \vec{r}_u(0,0) \times \vec{r}_v(0,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 1,0,0 \rangle$$

Tangent plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$1(x-1) + 0(y-0) + 0(z-0) = 0$$

$$\boxed{x = 1}$$

- **Surface Area:**

Consider a smooth surface S given by

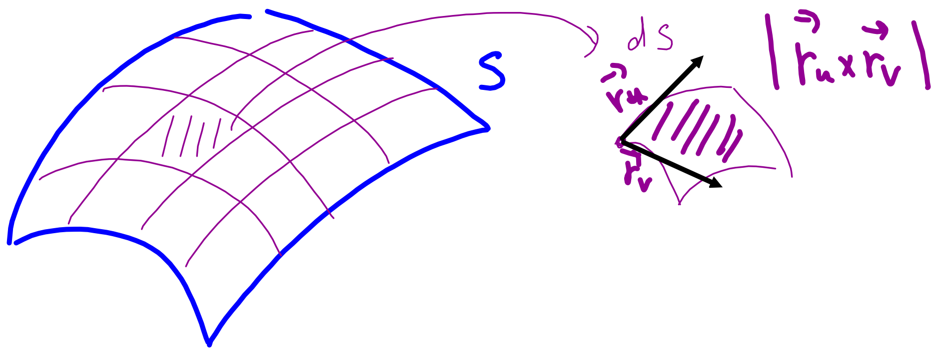
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D,$$

then

$$\mathrm{d}S = |N(u,v)|\mathrm{d}u\mathrm{d}v = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

and the **surface area**

$$A(S) = \iint_S \mathrm{d}S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}A_{uv}$$



REMARK 4. *Special Case*: a surface S given by a *graph* $z = f(x,y)$ we have

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

and

$$\mathrm{d}S = |\mathbf{N}(x,y)|\,\mathrm{d}A = |\langle f_x, f_y, -1\rangle|\, dA_{xy}$$

$$= \sqrt{f_x^2 + f_y^2 + 1}\, dA_{xy}$$

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u + v, \quad z = u - v, \qquad u^2 + v^2 \le 1.$$

$$S: \ \vec{r}(u,v) = \langle uv,\ u+v,\ u-v \rangle, \quad D = \{(u,v) \mid u^2+v^2 \le 1\}$$

$$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = \langle -2,\ v+u,\ v-u \rangle$$

$$|\vec{N}(u,v)| = \sqrt{4 + (v+u)^2 + (v-u)^2}$$

$$= \sqrt{4 + v^2 + u^2 + 2vu + v^2 + u^2 - 2vu} = \sqrt{4 + 2(u^2+v^2)}$$

$$= \sqrt{2}\sqrt{2 + u^2 + v^2}$$

double

$$SA = \iint_S dS = \iint_D |\vec{N}(u,v)|\, du\, dv = \sqrt{2} \iint_{u^2+v^2 \le 1} \sqrt{2+u^2+v^2}\, du\, dv$$

Use polar coordinates:

$u = r\cos\theta$

$v = r\sin\theta$

$D^* = \{0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 \sqrt{2+r^2}\, r\, dr\, d\theta$$

$$= 2\sqrt{2}\pi \underbrace{\int_0^1 \sqrt{2+r^2}\, r\, dr}_{u\text{-sub.}} = \ldots$$

EXAMPLE 6. *Find the surface area of the part paraboloid $z = x^2 + y^2$ between two planes: $z = 0$ and $z = 4$.* $0 \le z \le 4$

Parametrization I

$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

$D = \{(x,y) \mid \underbrace{x^2+y^2 \le 4}_{\text{because } 0 \le z \le 4}\}$

$\vec{N}(x,y) = \langle z_x, z_y, -1 \rangle = \langle 2x, 2y, -1 \rangle$

$|\vec{N}(x,y)| = \sqrt{4(x^2+y^2)+1}$

$$SA = \iint_D |\vec{N}(x,y)| \, dA$$

$$= \iint_D \sqrt{4(x^2+y^2)+1} \, dA$$

use polar

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1} \, r \, dr \, d\theta$$

Parametrization 2

Use cylindrical coordinates

$\vec{R}(\theta, r) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$

$D = \{(\theta, r) \mid 0 \le r \le 2, \ 0 \le \theta \le 2\pi\}$

because $0 \le z = r^2 \le 4$

$$\vec{N}(\theta, r) = \vec{R}_\theta \times \vec{R}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix}$$

$$= \langle 2r^2\cos\theta, -2r^2\sin\theta, -r \rangle$$

$|\vec{N}(\theta, r)| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$

$$SA = \iint_D |\vec{N}(\theta, r)| \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r\sqrt{4r^2+1} \, dr \, d\theta$$