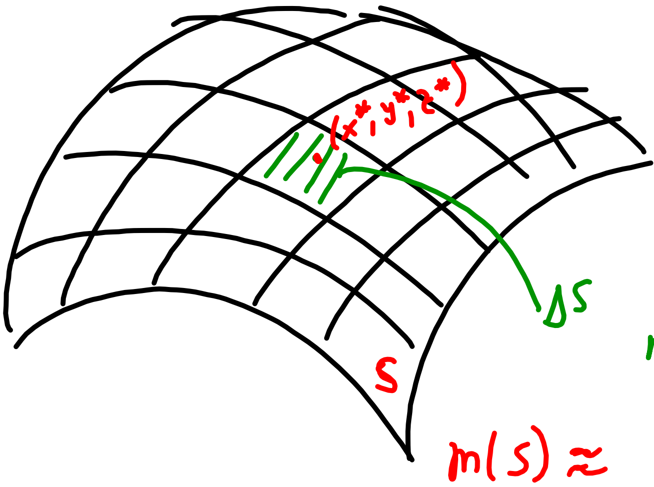
14.7: Surface Integrals

Problem: Find the **mass** of a thin sheet (say, of aluminum foil) which has a shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$.

Solution:



$\rho = \text{const}$

$\Downarrow$

$$m = \rho \cdot SA = \rho \iint_S dS$$

$$m(\Delta S) = \rho(x^*, y^*, z^*)\,\Delta S$$

$$m(S) \approx \sum m(\Delta S) = \sum \rho\,\Delta S$$

$$m(S) = \iint_S \rho(x, y, z)\,dS$$

If S is given by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$, then the **surface integral of f over the surface S** is:

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))|\mathbf{N}(u,v)|\,\mathrm{d}A = \iint_D f(r(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$$dS = |\vec{N}(u,v)|\,du\,dv$$

$$dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

EXAMPLE 1. *Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$ inside the cylinder $x^2+y^2 \le 2x$, if its density is a function $\rho(x,y,z) = x^2+y^2+z^2$.*

$$m = \iint_S \rho(x,y,z)\, dS \quad \text{, where}$$

$$S = \{(x,y,z) \mid z = \sqrt{x^2+y^2},\ x^2+y^2 \le 2x\}$$

Parameterize S

Way I

$\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$

$D = \{(x,y) \mid x^2+y^2 \le 2x\}$

$\vec{N}(x,y) = \langle z_x, z_y, -1 \rangle$

$= \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$

$$|\vec{N}(x,y)| = \sqrt{\underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{1} + (-1)^2}$$

$|\vec{N}(x,y)| = \sqrt{2}$

$$m = \iint_D \rho(x,y,\sqrt{x^2+y^2})\, |\vec{N}(x,y)|\, dA_{x,y}$$

$$= \iint_D 2(x^2+y^2)\sqrt{2}\, dA$$

Use polar coordinates

$D^* = \{(r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$$m = 2\sqrt{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$= 2\sqrt{2}\int_{-\pi/2}^{\pi/2} \left.\frac{r^4}{4}\right|_{r=0}^{2\cos\theta} d\theta = 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\pi/2} 4\cos^4\theta\, d\theta = \ldots$$

Way II Use cylindrical

$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$

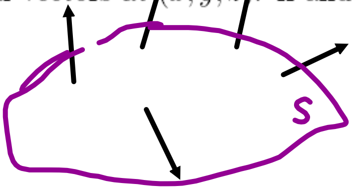
$D = \{(r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$\vec{N}(r,\theta) = \vec{R}_r \times \vec{r}_\theta$

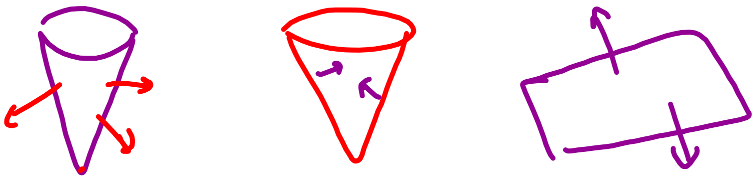
and so on...

• **Oriented surfaces.** We consider only two-sided surfaces.

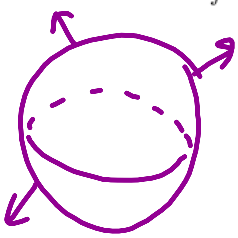
Let a surface S has a tangent plane at every point (except at any boundary points). There are two unit normal vectors at (x, y, z): $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$.



If it is possible to choose a unit normal vector $\hat{\mathbf{n}}$ at every point (x, y, z) of a surface S so that $\hat{\mathbf{n}}$ varies continuously over S, then S is called **oriented surface** and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:



`Convention:` *For closed surfaces the positive orientation is outward.*



• **Surface integrals of vector fields.**

DEFINITION 2. *If* **F** *is a continuous vector field defined on an oriented surface* S *with unit normal vector* $\hat{\mathbf{n}}$, *then the* **surface integral of F over** S *is*

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \underbrace{\mathbf{F} \cdot \hat{\mathbf{n}}}\, dS. \quad , \quad d\vec{S} = \hat{n}\, dS$$

This integral is also called the **flux** *of* **F** *across* S.

Note that if S is given by $\mathbf{r}(u, v)$, $(u, v) \in D$, then

$$\text{unit} \leftarrow \hat{\mathrm{n}} = \frac{\mathrm{n}}{|\mathrm{n}|} = \frac{\vec{n}(u,v)}{|\vec{n}(u,v)|} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and

$$d\vec{\mathrm{S}} = \hat{n}\, dS = \frac{\vec{n}(u,v)}{\cancel{|\vec{n}(u,v)|}} \cdot \cancel{|\vec{n}(u,v)|}\, du\, dv$$

$$= \vec{n}(u,v)\, du\, dv$$

Finally,

$$\iint_S \mathrm{F} \cdot d\mathrm{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v)\, du\, dv$$

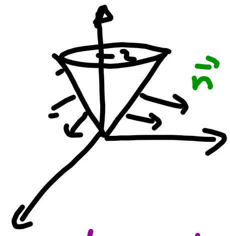
where $\vec{n}(u,v) = \pm \vec{r}_u \times \vec{r}_v$

EXAMPLE 3. *Find the flux of the vector field*

$$\mathbf{F} = \langle x^2, y^2, z^2 \rangle$$

across the surface

$$S = \{z^2 = x^2 + y^2, 0 \le z \le 2\}.$$



Note: Direction of $\vec{n}$ is not indicated, so we have to choose $\vec{n}$ pointed outward.

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

Parameterize S: Way 1

$\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$.

$D = \{(x,y) \mid 0 \le x^2+y^2 \le 4\}$

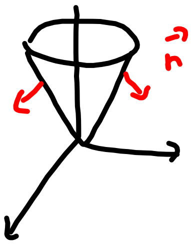
and so on

See also Example 1

We use here parameterization using cylindrical coordinates:

$$\vec{R}(\theta, r) = \langle \underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y}, \underbrace{r}_{z} \rangle, \quad D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$\vec{n}(\theta, r) = \pm R_\theta \times R_r = \pm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \pm \langle r\cos\theta, r\sin\theta, -r \rangle$$



$\vec{n}$ is pointed down $\Rightarrow$ its last component must be negative

$$\vec{n}(\theta, r) = \langle r\cos\theta, r\sin\theta, -r \rangle$$

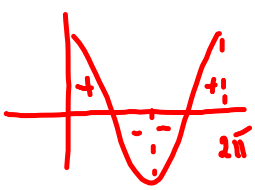
$$\text{Flux} = \iint_D \vec{F}(\vec{R}(\theta, r)) \cdot \vec{n}(r,\theta)\, dA_{r,\theta}$$

$$= \iint_D \langle r^2\cos^2\theta,\ r^2\sin^2\theta,\ r^2 \rangle \cdot \langle r\cos\theta,\ r\sin\theta,\ -r \rangle\, dA_{r,\theta}$$

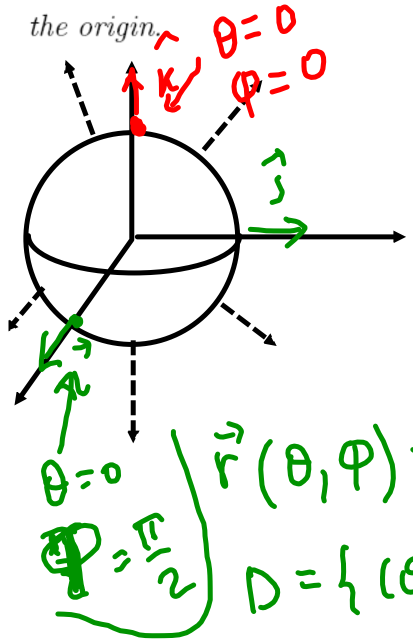
$$= \int_0^{2\pi}\int_0^{2} \left(r^3\cos^3\theta + r^3\sin^3\theta - r^3\right) dr\, d\theta$$

$$\underbrace{\int_0^{2\pi}\left(\cos^3\theta + \sin^3\theta - 1\right) d\theta}_{-2\pi} \int_0^2 r^3\, dr = -2\pi \left.\frac{r^4}{4}\right|_0^2 = -\frac{\pi}{2}\cdot 16 = \boxed{-8\pi}$$

because $\int_0^{2\pi} \cos^3\theta\, d\theta = \int_0^{2\pi} \sin^3\theta\, d\theta = 0$



EXAMPLE 4. *Evaluate* $I = \oiint_S \mathbf{F} \cdot d\mathbf{S}$ *where* $\mathbf{F} = \langle z, y, x \rangle$ *and* S *is the unit sphere centered at the origin.*



S is closed $\Rightarrow \vec{n}$ is pointed outward.

$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$

Use spherical coordinates to parameterize S: $\rho = 1$

$\vec{r}(\theta, \varphi) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$; S

$D = \{(\theta, \varphi) \mid 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

$$\vec{n}(\theta, \varphi) = \pm \vec{r}_\theta \times r_\varphi = \pm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix} =$$

$$\pm \langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi \rangle =$$

$$= \pm \sin\varphi \langle -\sin\varphi\cos\theta, -\sin\varphi\sin\theta, -\cos\varphi \rangle$$

$$\vec{F}(\vec{r}(\theta,\varphi))\cdot\vec{n}(\theta,\varphi)$$

$$= \langle \cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta \rangle$$

$$\cdot\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle \sin\varphi$$

$$= \sin^2\varphi(\cos\varphi\cos\theta + \sin\varphi\sin^2\theta + \cos\theta\cos\varphi)$$

$$= \sin^2\varphi(2\cos\varphi\cos\theta + \sin\varphi\sin^2\theta)$$

$$\oiint_S \vec{F}\cdot d\vec{S} = \iint_D (2\cos\varphi\cos\theta\sin^2\varphi + \sin^3\varphi\sin^2\theta)\,dA_{\theta,\varphi}$$

$$= \underbrace{2\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{=0}\int_0^{\pi}\cos\varphi\sin^2\varphi\,d\varphi}_{=0} + \int_0^{2\pi}\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\,d\theta\int_0^{\pi}\underbrace{\sin^3\varphi\,d\varphi}_{(1-\cos^2\varphi)\sin\varphi\,d\varphi}$$

$$= \left(\int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\int_0^{2\pi}\cos 2\theta\,d\theta\right)\left(\int_0^{\pi}\sin\varphi\,d\varphi - \int_0^{\pi}\cos^2\varphi\sin\varphi\,d\varphi\right)$$

$$(\pi - 0)\left(-\cos\varphi\Big|_0^{\pi} + \frac{\cos^3\varphi}{3}\Big|_0^{\pi}\right) = \pi\left(2+\left(-\frac{1-1}{3}\right)\right)$$

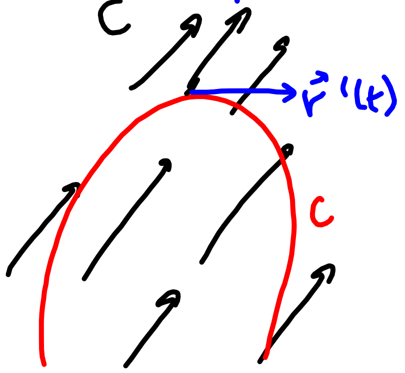
$$= \boxed{\frac{4\pi}{3}}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \underbrace{\vec{r}\,'(t)}_{\text{tangent to } C}\, dt$$

$\vec{F} = \rho \vec{V}$ — $\vec{V}$ velocity, ρ density of fluid

$$\left[\frac{Kg}{m^3}\right]\cdot\left[\frac{m}{c}\right] = \left[\frac{Kg}{m^2\cdot c}\right]$$

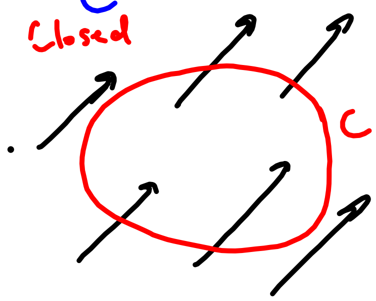
amount of fluid per unit area per unit time.



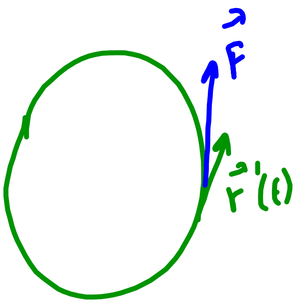
$\int_C \vec{F}\cdot d\vec{r}$ measures how much vector field is aligned with the curve C.

$\oint_C \vec{F}\cdot d\vec{r}$ measures how much vector field tends to circulate around the C.

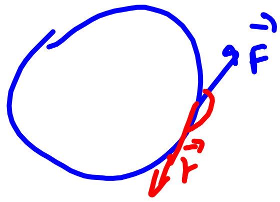
Closed



$\oint_C \vec{F}\cdot d\vec{r}$ Macroscopic circulation of $\vec{F}$



$\vec{F}\cdot\vec{r}\,' > 0$ positive circulation

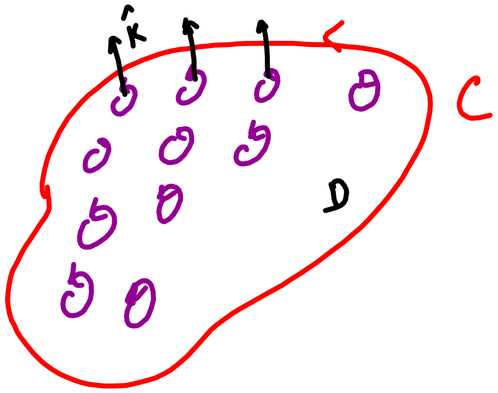


negative circulation

$$\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{macroscopic circulation}} = \iint_D \underbrace{\text{curl}\,\vec{F}}_{\text{microscopic circulation of } \vec{F}}\cdot\hat{k}\, dA$$

Green's Theorem.

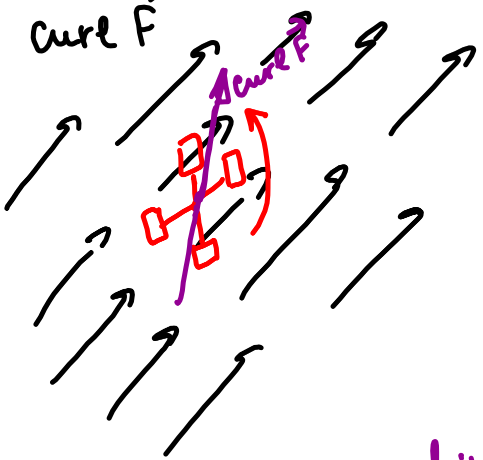
$\hat{k} \perp D$



Stokes' Theorem: $$\oint_C \vec{F}\cdot d\vec{r} = \underbrace{\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS}_{\text{Flux of curl } \vec{F}}$$

$C = \partial S$

curl $\vec{F}$



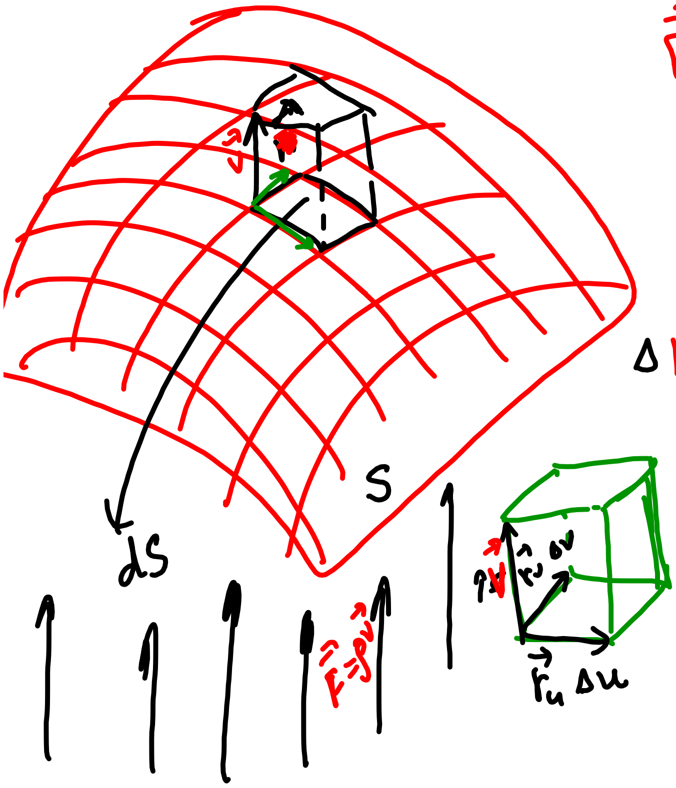
curl $\vec{F}$ shows how fluid may rotate.

Curl $\vec{F}$ measures the tendency of the vector field to curl around the given point in the direction given by the right hand rule.

$|\text{curl } \vec{F}| = \text{angular velocity}.$

Flux of $\vec{F}$ across S $\quad \iint_S \vec{F}\cdot\hat{n}\, d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot\vec{n}(u,v)\, du\, dv$

$\vec{F} = \rho\vec{v}$ (= mass of fluid per unit area and per unit time.)



$$\Delta m = \rho\cdot dV = \rho\underbrace{(\vec{r}_u \times \vec{r}_v)}_{\vec{n}}\cdot\vec{v}\, du\, dv$$

$$= \underbrace{\rho\vec{v}}_{\vec{F}}\cdot\hat{n}\,\Delta u\,\Delta v$$

$$\text{Totall mass} = \iint_S dm$$

$$= \iint_S \vec{F}\cdot\hat{n}\, dS$$