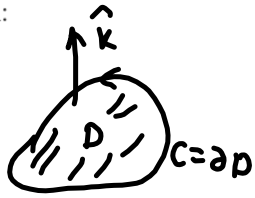
14.8: STOKES' THEOREM

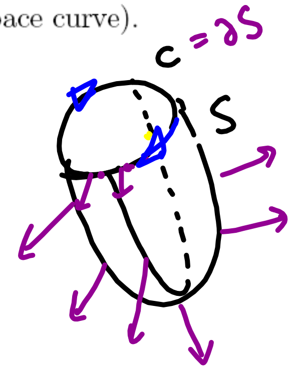
Stokes' Theorem can be regarded as a 3-dimensional version of Green's Theorem:

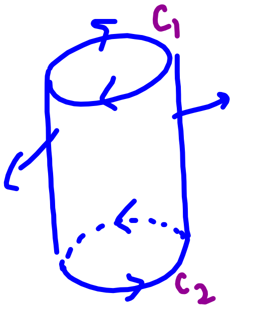
$$\oint_{C} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A = \iint_D \mathrm{curl}\mathbf{F}\cdot\mathbf{k}\,\mathrm{d}A.$$

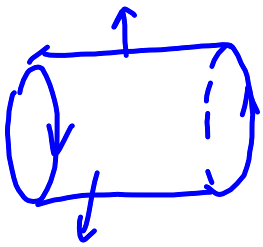
∂D



Let S be an oriented surface with unit normal vector $\hat{\mathbf{n}}$ and with the boundary curve C (which is a space curve).







$\partial S = C_1 \cup C_2$

The orientation on S induces the **positive orientation of the boundary curve** C: if you walk in the positive direction around C with your head pointing in the direction of $\hat{\mathbf{n}}$, then the surface will always be on your left.

The positively oriented boundary curve of an oriented surface S is often written as ∂S.

Stokes' Theorem: *Let S be an oriented piece-wise-smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

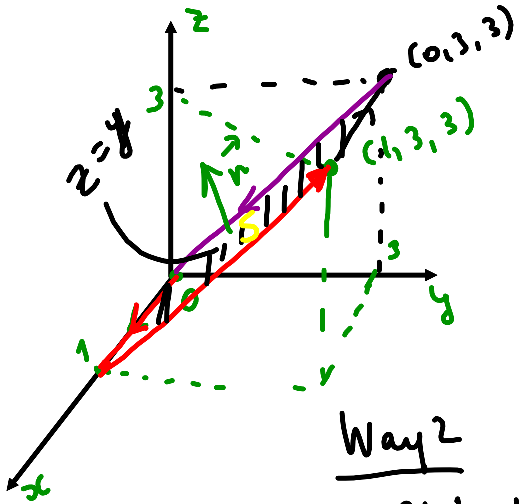
$$\oint_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F} \cdot \,\mathrm{d}\mathbf{S},$$

or

$$\iint_S \mathrm{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \,\mathrm{d}\mathbf{r}.$$

EXAMPLE 1. *Find the work performed by the forced field* $\mathbf{F}(x,y,z) = \langle 3x^8, 4xy^3, y^2x\rangle$ *on a particle that traverses the curve* C *in the plane* $z = y$ *consisting of 4 line segments from* $(0,0,0)$ *to* $(1,0,0)$*, from* $(1,0,0)$ *to* $(1,3,3)$*, from* $(1,3,3)$ *to* $(0,3,3)$*, and from* $(0,3,3)$ *to* $(0,0,0)$.

S



$$W = \oint_C \vec{F}\cdot d\vec{r}$$

Way 1 Parameterize C and evaluate 4 integrals

Way 2 Since C is closed, we use Stokes' Theorem:

$$\partial S = C$$

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS$$

We choose as S the part of the plane $z=y$ bounded by the given curve C.

$$W = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS$$

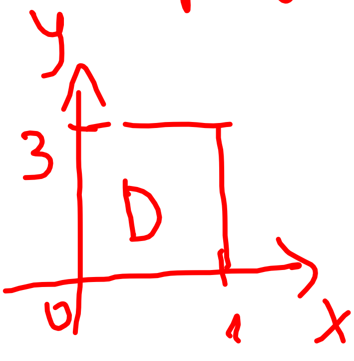
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^8 & 4xy^3 & y^2 x \end{vmatrix}$$

$$= \langle 2yx, -y^2, 4y^3 \rangle$$

Parameterize S:

$x = x,\ y = y,\ z = y$ (or $\vec{r}(x,y) = \langle x, y, y\rangle$)

D is projection of S onto the xy-plane



$\vec{n} = \pm\langle 0, 1, -1\rangle = \langle 0, -1, 1\rangle$

(z-component should be positive).

$$W = \iint_S \langle 2yx, -y^2, 4y^3 \rangle \cdot \hat{n} \, dS$$

$$= \iint_D \langle 2yx, -y^2, 4y^3 \rangle \cdot \langle 0, -1, 1 \rangle \, dA_{xy}$$

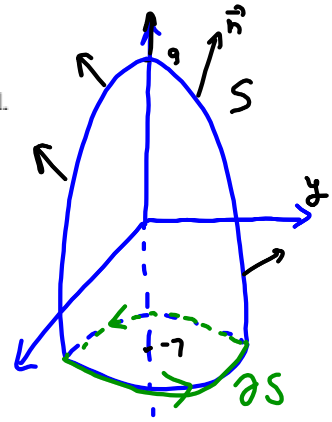
$$= \int_0^1 \left(\int_0^3 \left(y^2 + 4y^3 \right) dy \right) dx$$

$$= \left(\frac{y^3}{3} + y^4 \right) \Big|_0^3 = 9 + 81 = \boxed{90}$$

EXAMPLE 2. *Verify Stokes' Theorem* $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{r}$ *for the vector field* $\vec{F} = \langle 3y, 4z, -6x\rangle$ *and the paraboloid* $z = 9 - x^2 - y^2$ *that lies above the plane* $z = -7$ *and oriented upward. Be sure to check and explain the orientations.*

Solution: Use the following steps:

- Parametrize the boundary circle ∂S and compute the line integral.



$$S = \{(x,y,z) \mid z = 9 - x^2 - y^2,\ z \geq -7\}$$
$$\partial S = \{(x,y,z) \mid x^2+y^2=16,\ z=-7\}$$

$$\begin{cases} z = 9-x^2-y^2 \\ z=-7 \end{cases}$$
$$-7 = 9-x^2-y^2$$
$$x^2+y^2=16$$

Use polar coordinates

∂S: $\vec{r}(t) = \langle 4\cos t, 4\sin t, -7\rangle$

$\vec{r}\,'(t) = \langle -4\sin t, 4\cos t, 0\rangle$

$0 \leq t \leq 2\pi$

$$\vec{F}(\vec{r}(t)) = \langle 3\cdot 4\sin t, 4\cdot(-7), -6\cdot 4\cos t\rangle$$
$$\vec{F}(\vec{r}(t)) = 4\langle 3\sin t, -7, -6\cos t\rangle$$
$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 4\langle 3\sin t, -7, -6\cos t\rangle\cdot\langle -4\sin t, 4\cos t, 0\rangle$$
$$= 4\cdot\left[(-12\sin^2 t) - 28\cos t\right] = -16\left(3\sin^2 t + 7\cos t\right)$$

$$\int_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi}\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = -16\int_0^{2\pi}(3\sin^2 t + 7\cos t)\,dt$$
$$= -16\left[3\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt + 7\underbrace{\int_0^{2\pi}\cos t\,dt}_{=0}\right]$$
$$= -16\left[\frac{3}{2}\left(\int_0^{2\pi}dt - \underbrace{\int_0^{2\pi}\cos 2t\,dt}_{=0}\right) + 7\cdot 0\right]$$
$$= -\frac{48}{2}\cdot 2\pi = \boxed{-48\pi}$$

•Parametrize the surface of the paraboloid and compute the surface integral:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 4z & -6x \end{vmatrix} = \langle 0-4, -(-6-0), 0-3 \rangle$$
$$= \langle -4, 6, -3 \rangle$$

Parameterize S :

$$S = \{ (x, y, z) \mid z = 9 - x^2 - y^2, \ z \geq -7 \}$$

$$\vec{r}(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle$$

$$D = \{ (x, y) \mid x^2 + y^2 \leq 16 \}$$

$$\vec{n}(x,y) = \pm \langle z_x, z_y, -1 \rangle = \pm \langle -2x, -2y, -1 \rangle = + \langle 2x, 2y, 1 \rangle$$

$$\text{curl}\,\vec{F}(\vec{r}(x,y)) \cdot \vec{n}(x,y) = \langle -4, 6, -3 \rangle \cdot \langle 2x, 2y, 1 \rangle = -8x + 12y - 3$$

Compute the surface integral

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, dS$$

$$= \iint_D \text{curl}\,\vec{F}(\vec{r}(x,y)) \cdot \vec{n}(x,y)\, dA_{xy}$$

$$= \iint_D (-8x + 12y - 3)\, dA \overset{\text{use polar}}{=}$$

$$= \int_0^{2\pi} \int_0^4 (-8r\cos\theta + 12r\sin\theta)\, r\,dr\,d\theta - 3\iint_D dA$$

$$= -8 \underbrace{\int_0^{2\pi} \cos\theta}_{=0} \int_0^4 r^2\,dr + 12 \underbrace{\int_0^{2\pi} \sin\theta}_{=0} \int_0^4 r^2\,dr - 3\underbrace{A(D)}_{\pi 4^2} = \boxed{-48\pi}$$

THEOREM 3. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\operatorname{curl}\mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is a conservative vector field.*

SUMMARY: Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ *be a continuous vector field in* $\mathbb{R}^3$.

A → B

There exists f s.t. $\nabla f = \mathbf{F}$

potential function

$\displaystyle\int_{\overset{\frown}{AB}} \mathbf{F}\cdot d\mathbf{r}$ is independent of path $= f(B) - f(A)$ FTLI

$\Updownarrow$

$\mathbf{F}$ is conservative in $\mathbb{R}^3$

$\Updownarrow$

$\operatorname{curl}\mathbf{F} = \mathbf{0}$

$\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C

Now we proof Theorem 7 from Section 14.5:

THEOREM 3. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\mathrm{curl}\mathbf{F} = 0$*, then* $\mathbf{F}$ *is a conservative vector field.*

Proof:

SUMMARY: Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ *be a continuous vector field in* $\mathbb{R}^3$*.*

There exists f s.t. $\nabla f = \mathbf{F}$

$\int_{\breve{AB}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path

$\mathbf{F}$ is conservative in $\mathbb{R}^3$

$\text{curl}\mathbf{F} = \mathbf{0}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C