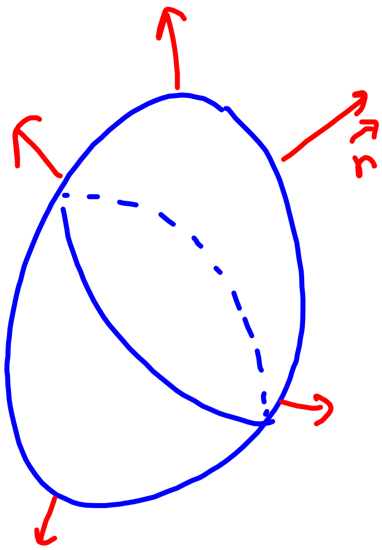
14.9: The Divergence Theorem

Let E be a simple solid region with the boundary surface S (which is a closed surface.) Let S be positively oriented (i.e.the orientation on S is outward that is, the unit normal vector $\hat{\mathbf{n}}$ is directed outward from E).



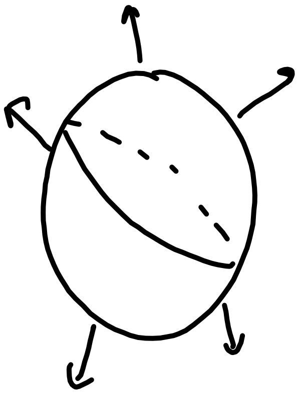
	object		Boundary
Green	plain region	D	∂D is a closed plane curve.
Stokes	surface	S	∂S is a closed curve (edge)
Divergence	solid	E	∂E is a closed surface

Gauss

The Divergence Theorem: Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a continuous vector field on an open region that contains E. Then

$$\oiint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \underbrace{\text{div}\mathbf{F}}_{\text{a scalar function}}\, dV.$$

$S = \partial E$



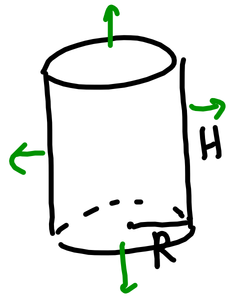
$$= \iint_S \vec{F}\cdot\hat{n}\, dS =$$

Flux through a closed surface S

F.ex. if S is oriented in then

$$\oiint_S \vec{F}\cdot d\vec{S} = -\iiint_E \text{div}\vec{F}\, dV$$

EXAMPLE 1. *Let* $E = \{(x, y, z) : x^2 + y^2 \leq R^2, 0 \leq z \leq H\}$. *Find the flux of the vector field* $\mathbf{F} = \langle 1 + x, 3 + y, z - 10\rangle$ *over* ∂E.



Flux $\displaystyle\oiint_{\partial E} \vec{F} \cdot d\,S \overset{\text{Divergence Theorem}}{=}$

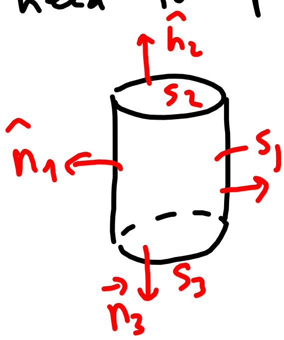
$$= \iiint_E \operatorname{div} \vec{F}\, dV = 3 \iiint_E dV$$

$$= 3V(E)$$

$$= \boxed{3\pi R^2 H}$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(1+x) + \frac{\partial}{\partial y}(3+y) + \frac{\partial}{\partial z}(z-10) = 3$$

Remark To calculate this integral directly, we need to parametrize 3 surfaces



REMARK 2. If $\mathbf{F} = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$ then $\operatorname{div} \vec{F} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

then flux over a closed surface S which is boundary of a solid E is

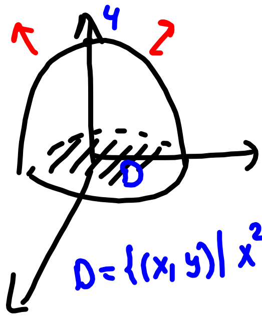
$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div} \vec{F}}_{1} \, dV = \iiint_E dV = V(E).$$

$$\boxed{V(E) = \oiint_S \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle \cdot d\vec{S}}$$

EXAMPLE 3. *Let E be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Evaluate* $I = \iint_S \langle x^3, 2xz^2, 3y^2z\rangle \cdot \mathrm{d}\mathbf{S}$ *if*

(a) *S is the boundary of the solid E.*

$S = \partial E \Rightarrow S$ is closed



$D = \{(x,y) \mid x^2+y^2 \le 4\}$

$$I = \oiint_S \langle x^3, 2xz^2, 3y^2z\rangle \cdot d\vec{S}$$

$$= \oiint_{\partial E} \langle x^3, 2xz^2, 3y^2z\rangle\, d\vec{S} \;\underline{\underline{DT}}$$

$$= \iiint_E \operatorname{div}\langle x^3, 2xz^2, 3y^2z\rangle\, dV$$

$E = \{(x,y,z) \mid 0 \le z \le 4 - x^2 - y^2\}$

$$= \iiint_E (3x^2 + 0 + 3y^2)\, dV = \iint_D \left[\int_0^{4-x^2-y^2} 3(x^2+y^2)\, dz\right] dA_{xy}$$

polar coordinates

$$= 3\iint_D (x^2+y^2)(4-x^2-y^2)\, dA = \int_0^{2\pi}\int_0^2 r^2(4-r^2)\, r\, dr\, d\theta$$

$$= 2\pi \cdot 3 \int_0^2 (4r^3 - r^5)\, dr = 6\pi\left(r^4 - \frac{r^6}{6}\right)\Big|_0^2$$

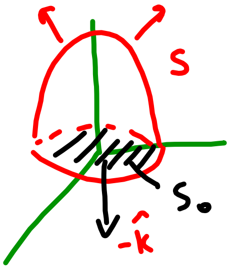
$$= 6\pi r^4\left(1 - \frac{r^2}{6}\right)\Big|_0^2$$

$$= \pi r^4 (6 - r^2)\Big|_0^2 = 16\pi(6-4) = \boxed{32\pi}$$

(B) *S is the part of the paraboloid $z = 4 - x^2 - y^2$ between the planes $z = 0$ and $z = 4$.*

S is not closed here. Close S to use Divergense Theorem.

$S_0 = \{(x,y,z) \mid x^2+y^2 \le 4,\ z=0\}$



$\partial E = S \cup S_0$

$$\underbrace{\underset{\partial E}{\oiint} \vec{F}\cdot d\vec{S}}_{32\pi \text{ (see part A)}} = \underbrace{\iint_S \vec{F}\, d\vec{S}}_{I} + \iint_{S_0} \vec{F}\, d\vec{S}$$

$I = 32\pi - \iint_{S_0} \vec{F}\, d\vec{S}$

Parameterize S_0: $\vec{r}(x,y) = \langle x, y, 0\rangle$
$\vec{n}(x,y) = \langle 0,0,-1\rangle$

$$\vec{F}(\vec{r}(x,y))\cdot \vec{n}(x,y) = \langle x^3, 0, 0\rangle \cdot \langle 0,0,-1\rangle = 0$$

$$\iint_{S_0} \vec{F}\, d\vec{S} = 0$$

Then $I = 32\pi - 0 = 32\pi$.

EXAMPLE 4. *Evaluate* $I = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S}$ *if* S *is the boundary of*

(a) *ellipsoid* $E = \underbrace{\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}}_{\partial E = S \text{ is closed}}$ *and* $\mathbf{F} = \langle$ 🙂 , ❄️ , 🌼 $\rangle$

By DT,

$$\oiint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iiint_E \text{div}(\text{curl}\,\vec{F})\,dV = \iiint_E 0\,dV = 0.$$

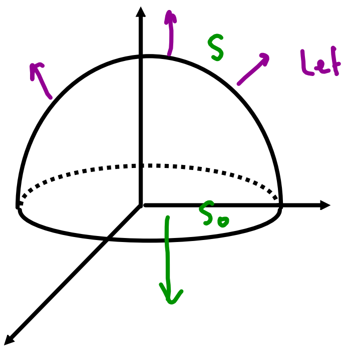
(b) *an arbitrary simple solid region* E *and* F *is an arbitrary continuous vector field.*

$$\oiint_{\partial E} \text{curl}\,\vec{F}\cdot d\vec{S} = 0 \quad \forall E, \vec{F}$$

6. Apply the Divergence Theorem to compute $I = \iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = \langle xz^2 + \cos(y+z), \frac{1}{3}y^3 + e^z, x^2 z + y + x^3 \rangle$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ with outward normal.



Note that S is not closed.

Let $S_0 = \{(x,y,z) \mid x^2+y^2 \le 1,\ z=0\}$.

Then $S \cup S_0 = \partial E$,

where E is the solid hemisphere:

$E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ z \ge 0\}$

By DT

$$\oint\!\!\oint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}} + \iint_{S_0} \vec{F} \cdot d\vec{S} = \iiint_E (z^2 + y^2 + x^2)\, dV$$

Flux across S:

$$\iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iiint_E (x^2+y^2+z^2)\, dV} - \underbrace{\iint_{S_0} \vec{F} \cdot d\vec{S}} = \frac{2\pi}{5} + 0 = \boxed{\frac{2\pi}{5}}$$ Final answer

Use spherical coordinates:

$E^* = \{(\rho, \theta, \varphi) : 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{2}\}$

$$\iiint_E (x^2+y^2+z^2)\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin\varphi\, d\varphi \int_0^1 \rho^4\, d\rho$$

$$= 2\pi (-\cos\varphi)\Big|_0^{\pi/2} \frac{\rho^5}{5}\Big|_0^1 = \frac{2\pi}{5}$$

Parametrize S_0: $D = \{x^2+y^2 \le 1\}$

S_0: $\vec{r}(x,y) = \langle x, y, 0 \rangle$,

$\vec{n}(x,y) = \langle 0, 0, -1 \rangle$

$\vec{F}(\vec{r}(x,y)) \cdot \vec{n}(x,y) =$

$= \langle ☺, ☆, y+x^3 \rangle \cdot \langle 0, 0, -1 \rangle$

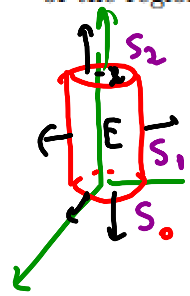
$= -(y+x^3)$

$$-\iint_{S_0} \vec{F} \cdot d\vec{S} = -\iint_D -(y+x^3)\, dA_{xy}$$

$$= \iint_D (y+x^3)\, dA = \int_0^{2\pi}\int_0^1 (r\sin\theta + r^3\cos^3\theta)\, r\, dr\, d\theta$$

$$= \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{0} \int_0^1 r^2\, dr + \underbrace{\int_0^{2\pi} \cos^3\theta\, d\theta}_{0} \int_0^1 r^4\, dr = 0$$

1. Verify the Divergence Theorem for the vector field $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S, which is the surface of the region enclosed by $x^2 + y^2 = 1$ and the planes $z = 0$, $z = 2$.



DT $\underbrace{\oiint_{\partial E} \vec{F}\cdot d\vec{S}}_{LHS} = \underbrace{\iiint_E \operatorname{div}\vec{F}\, dV}_{RHS}$

$\boxed{LHS}$ $\partial E = S_1 \cup S_2 \cup S_0 \Rightarrow \oiint_{\partial E} = \iint_{S_1} + \iint_{S_2} + \iint_{S_0}$

Parameterize S_1, S_2, S_0

$S_1: \vec{r}_1(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle$, $D_1 = \{(\theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le z \le 2\}$

$$\vec{n}_1 = \pm (\vec{r}_1)_\theta \times (\vec{r}_1)_z = \pm \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \pm \langle +\cos\theta, +\sin\theta, 0\rangle = \langle \cos\theta, \sin\theta, 0\rangle$$

$$\vec{F}(\vec{r}_1(\theta,z)) \cdot \vec{n}_1(\theta, z) = \langle \cos^3\theta, \sin^3\theta, z^3\rangle \langle \cos\theta, \sin\theta, 0\rangle$$

$$= \cos^4\theta + \sin^4\theta$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D_1} (\cos^4\theta + \sin^4\theta)\, dA_{\theta,z} = \int_0^2 \left(\int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\, d\theta \right) dz$$

$$= 2 \cdot 2 \int_0^{2\pi} \cos^4\theta\, d\theta = \cancel{4} \int_0^{2\pi} \left(\frac{1 + \cos 2\theta}{\cancel{2}} \right)^2 d\theta$$

$$= \int_0^{2\pi} 1^2\, d\theta + 2 \underbrace{\int_0^{2\pi} \cos 2\theta\, d\theta}_{0} + \int_0^{2\pi} \cos^2 2\theta\, d\theta$$

$$= 2\pi + \int_0^{2\pi} \frac{1 + \cos 4\theta}{2}\, d\theta = 2\pi + \underbrace{\int_0^{2\pi} \frac{d\theta}{2}}_{\pi} + \underbrace{\int_0^{2\pi} \frac{\cos 4\theta\, d\theta}{2}}_{0}$$

$$= 3\pi$$

S_2: $\vec{r}_2(x,y) = \langle x, y, 2 \rangle$

$D_2 = \{ x^2 + y^2 \leq 1 \}$

$\vec{n}_2(x,y) = \langle 0, 0, 1 \rangle$

$\vec{F}(\vec{r}_2(x,y)) \cdot \vec{n}_2(x,y) =$

$= \langle x^3, y^3, 8 \rangle \langle 0, 0, 1 \rangle = 8$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{D_2} 8 \, dA = 8 A(D_2) = 8\pi$$

S_0: $\vec{r}_3(x,y) = \langle x, y, 0 \rangle$

$D_3 = D_2$, $\vec{n}_3 = \langle 0, 0, -1 \rangle$

$\vec{F}(\vec{r}_3(x,y)) \cdot \vec{n}_3(x,y) = 0$

$$\iint_{S_0} \vec{F} \, d\vec{S} = 0$$

$$\text{LHS} = 3\pi + 8\pi + 0 = \boxed{11\pi}$$

Remind

$$\vec{F} = \langle x^3, y^3, z^3 \rangle$$

$$\operatorname{div}\vec{F} = 3(x^2+y^2+z^2)$$

$$\iiint_E \operatorname{div}\vec{F}\, dV = 3\iiint_E (x^2+y^2+z^2)\,dV$$

$$= 3\int_0^2\int_0^{2\pi}\int_0^1 (r^2+z^2)\, r\,dr\,d\theta\,dz$$

$$= 3\cdot 2\pi \int_0^2\int_0^1 (r^3+rz^2)\,dr\,dz$$

$$= 6\pi\int_0^2 \left(\frac{r^4}{4}+\frac{r^2z^2}{2}\right)\Big|_0^1 dz = 6\pi\int_0^2\left(\frac{1}{4}+\frac{z^2}{2}\right)dz$$

$$= 6\pi\left(\frac{z}{4}+\frac{z^3}{6}\right)\Big|_0^2 = \pi\left(\frac{3z}{2}+z^3\right)\Big|_0^2$$

$$= \pi(3+8) = \boxed{11\pi}$$