Section 2.2: The Limit of a function

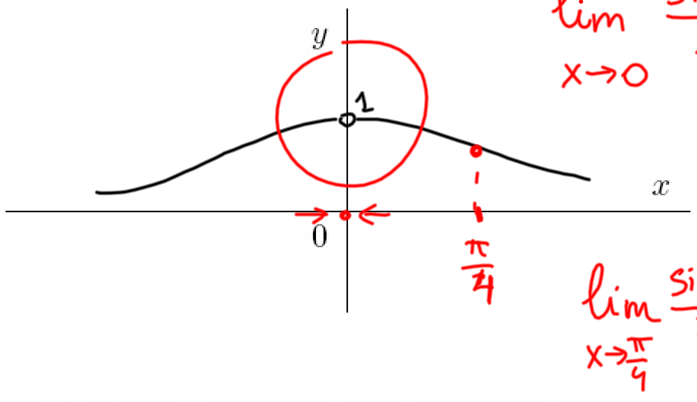
A limit is a way to discuss how the values of a function $f(x)$ behave when x approaches a number a, whether or not $f(a)$ is defined.

Let's consider the following function:

$$f(x) = \frac{\sin x}{x} \qquad (x \text{ in radians}).$$

Note that $f(0) = \dfrac{\sin 0}{0}$ is undefined. However, one can compute the values of $f(x)$ for values of x close to 0.

x	$f(x)$
$\pm$ 0.1	0.99833417
$\pm$ 0.05	0.99958339
$\pm$ 0.01	0.99998333
$\pm$ 0.005	0.99999583
$\pm$ 0.001	0.99999983



$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to \frac{\pi}{4}} \frac{\sin x}{x} = \frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}}$$

The table allows us to guess (correctly) that that our function gets closer and closer to 1 as x approaches 0 through positive and negative values. In limit notation it can be written as

$$\underbrace{\lim_{x\to 0^-} \frac{\sin x}{x}}_{\text{left-hand limit}} = \underbrace{\lim_{x\to 0^+} \frac{\sin x}{x}}_{\text{right-hand limit}} = 1$$

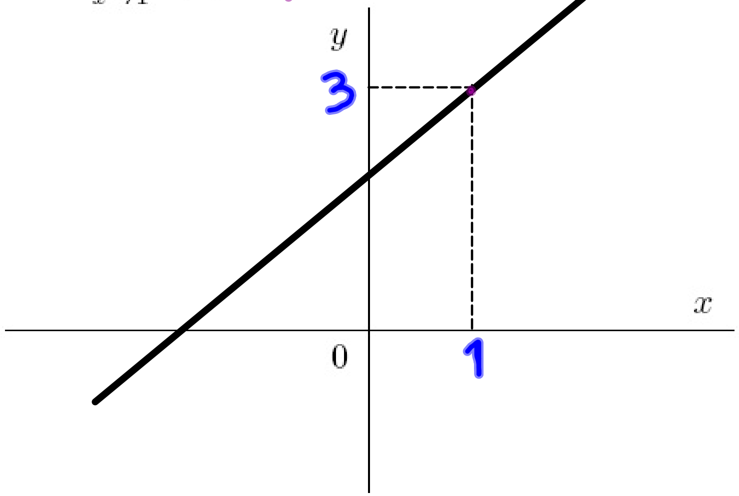
which implies that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

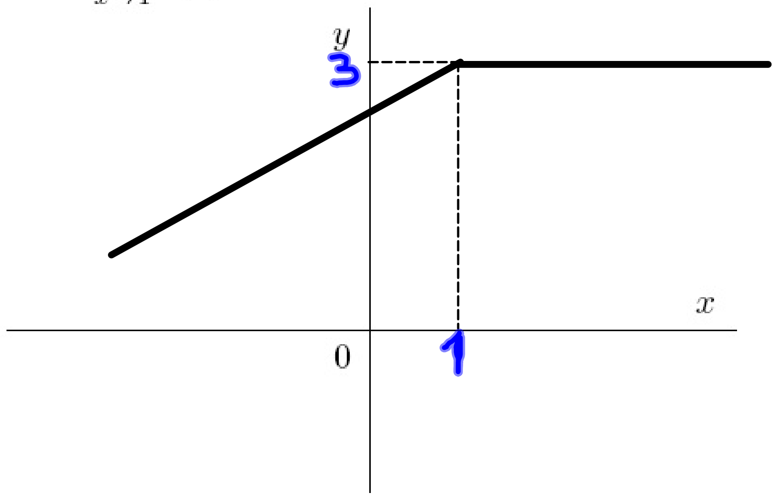
DEFINITION 1.

- *If* $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$ *then* $\lim_{x \to a} f(x)$ *exists and* $\lim_{x \to a} f(x) = L$*;*
- *If* $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) = L$ *then* $\lim_{x \to a} f(x)$ *does not exist.*

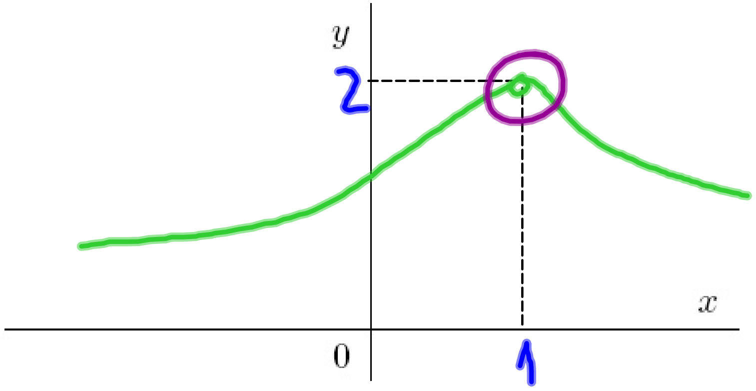
$\lim_{x \to 1} f(x) = f(1) = 3$



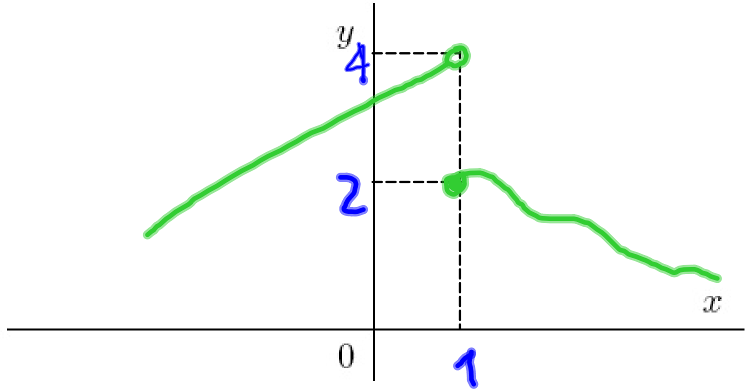
$\lim_{x \to 1} f(x) = f(1) = 3$



$\lim_{x \to 1} f(x) = 2$

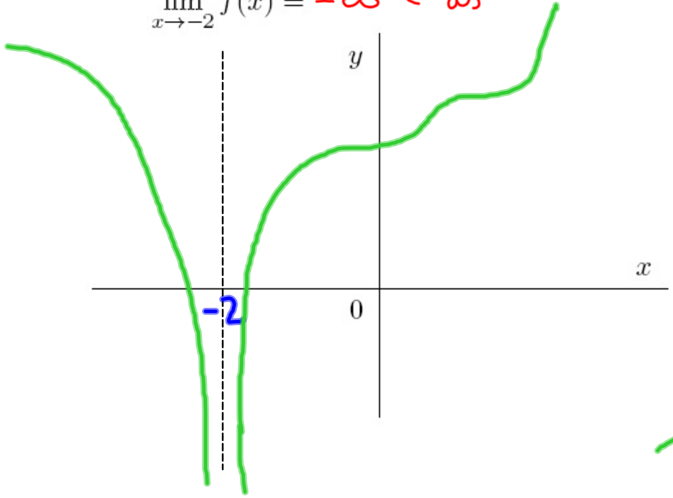


$\lim_{x \to 1} f(x) = \text{DNE}$

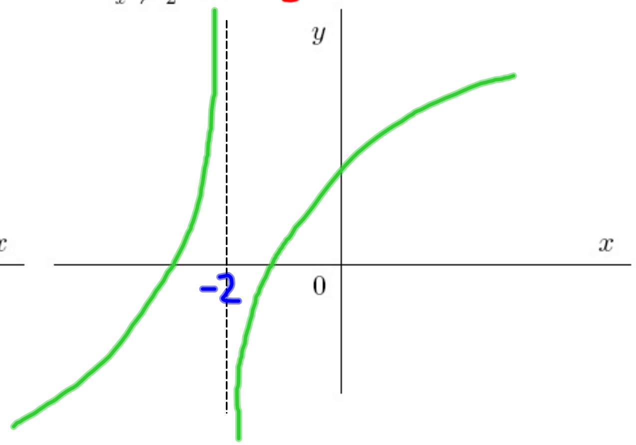


$$\lim_{x \to 1^-} f(x) = 4 \neq \lim_{x \to 1^+} f(x) = 2$$

$\lim_{x \to -2} f(x) = -\infty$ (DNE as a finite limit)



$\lim_{x \to -2} f(x) =$ DNE



$\lim_{x \to -2^-} f(x) = \infty$

$\lim_{x \to -2^+} f(x) = -\infty$

Limits of piecewise defined function.

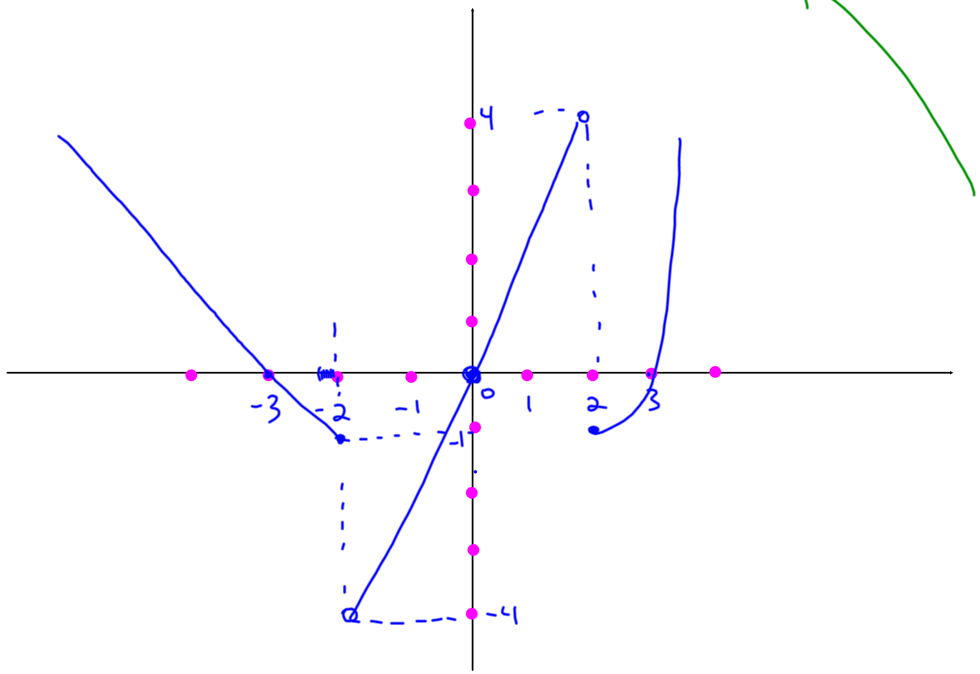
EXAMPLE 2. *Plot the graph of the function*

$$f(x) = \begin{cases} -3 - x & \text{if} \quad x \le -2 \\ 2x & \text{if} \quad -2 < x < 2 \\ x^2 - 4x + 3 & \text{if} \quad x \ge 2 \end{cases}$$

x	y
-2	-1
-3	0

x	y
-2	-4
2	4

x	y
2	-1
3	0



Find the limits (using the graph above):

$f(0) = 0$

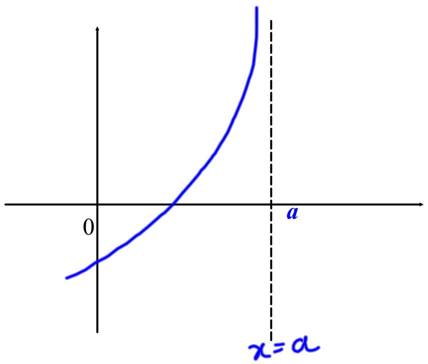
$\lim\limits_{x\to 0^-} f(x) = 0$ $=$ $\lim\limits_{x\to 0^+} f(x) = 0$ $\lim\limits_{x\to 0} f(x) = 0$

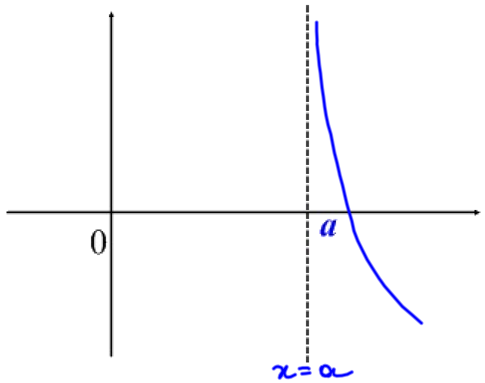
$\lim\limits_{x\to -2^-} f(x) = -1$ $\neq$ $\lim\limits_{x\to -2^+} f(x) = -4$ $\lim\limits_{x\to -2} f(x) =$ DNE

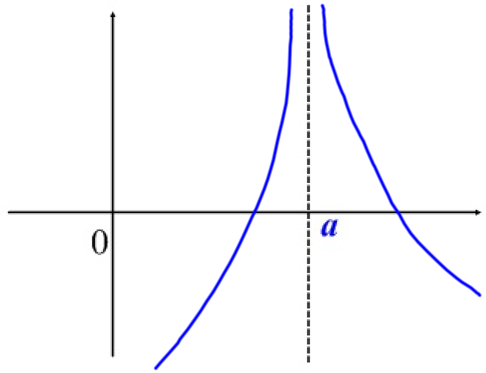
$\lim\limits_{x\to 2^-} f(x) = 4$ $\neq$ $\lim\limits_{x\to 2^+} f(x) = -1$ $\lim\limits_{x\to 2} f(x) =$ DNE

DEFINITION 3. *The line $x = a$ is said to be a* **vertical asymptote** *of the curve $y = f(x)$ if at least one of the following six statements is true:*

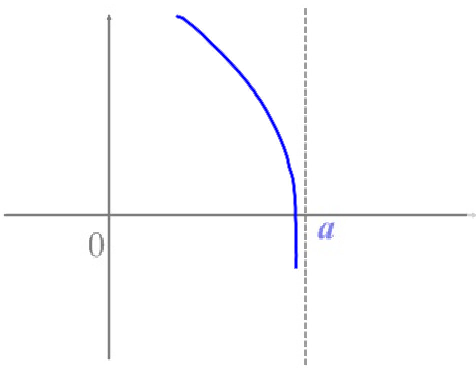
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a} f(x) = \infty$$

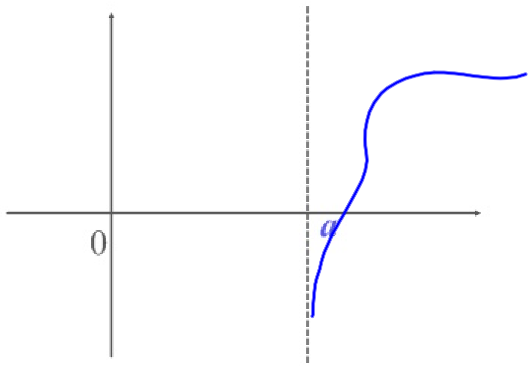


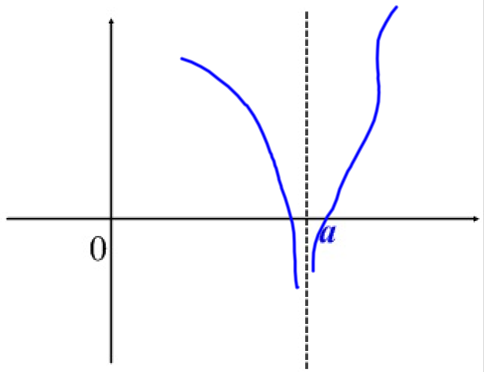




$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a} f(x) = -\infty$$







REMARK 4. The vertical asymptotes of a rational function come from the zeroes of the denominator.

EXAMPLE 5. *Determine the infinite limit:*

(a) $\lim_{x\to 4^-} \frac{7}{x-4} = -\infty$ $\qquad$ $x=4$ is vertical asymptote

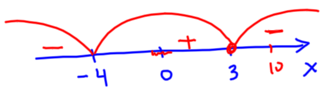
$x<4 \Rightarrow x-4<0 \Rightarrow \frac{7}{x-4}<0$

(b) $\lim_{x\to 4^+} \frac{7}{x-4} = \infty$

$x>4 \Rightarrow x-4>0 \Rightarrow \frac{7}{x-4}>0$

(c) $\lim_{x\to 4} \frac{7}{x-4} =$ DNE (compare (a) & (b))

(d) $\lim_{x\to 0^-} \frac{3-x}{x^4(x+4)} = +\infty$



$\lim_{x\to -4} \frac{3-x}{x^4(x+4)}$ DNE

because

$\lim_{x\to -4^-} \frac{3-x}{x^4(x+4)} = -\infty$, but

$\lim_{x\to 4^+} \frac{3-x}{x^4(x+4)} = +\infty$

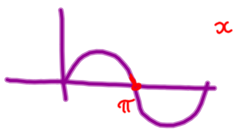
(e) $\lim_{x\to 0^+} \frac{3-x}{x^4(x+4)} = +\infty$

(f) $\lim_{x\to 0} \frac{3-x}{x^4(x+4)} = +\infty$

(g) $\lim_{x\to \pi^-} \csc x = \lim_{x\to\pi^-} \frac{1}{\sin x} = \infty$ $\qquad$ $x=\pi$ is vertical asymptote

$\sin \pi = 0$

$x<\pi$ (but near π) then $\sin x > 0$



EXAMPLE 6. *Given:* $f(x) = \dfrac{x-4}{x^2-5x+4}$.

(a) *What are the vertical asymptotes of $f(x)$?*

$$f(x) = \frac{x-4}{x^2-5x+4} = \frac{\cancel{x-4}}{(x-1)\cancel{(x-4)}} = \frac{1}{x-1} \quad (x \neq 4)$$

$$\boxed{x = 1}$$

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{1}{x-1} = \frac{1}{4-1} = \frac{1}{3}$$

(b) *How does $f(x)$ behave near the asymptotes?*

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

$$x<1 \Rightarrow x-1<0 \Rightarrow \frac{1}{x-1}<0$$

$$\lim_{x\to 1^+} f(x) = +\infty$$