Section 2.3: Calculating limits using the limits laws

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$ $\Rightarrow \lim_{x\to a} [f(x)]^2 = \left(\lim_{x\to a} f(x)\right)^2$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a} c = c$
6. $\lim_{x\to a} x = a$ $\quad \lim x^n \quad \left(\lim x\right)^n$
7. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer.
8. $\lim_{x\to a} x^n = a^n$, where n is a positive integer.
9. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer and if n is even, then we assume that $\lim_{x\to a} f(x) > 0$.
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x}$ where n is a positive integer and if n is even, then we assume that $a > 0$.

REMARK 1. Note that *all these properties also hold for the one-sided limits.*

REMARK 2. The analogues of the *laws 1-3 also hold when f and g are vector functions* (the product in Law 3 should be interpreted as a dot product).

EXAMPLE 3. *Compute the limit:*

$$\lim_{x\to -1}(7x^5+2x^3-8x^2+3) \overset{1}{=} \lim_{x\to -1}(7x^5) + \lim_{x\to -1}(2x^3) - \lim_{x\to -1}(8x^2) + \lim_{x\to -1} 3$$

$$\overset{2,5}{=} 7\lim_{x\to -1} x^5 + 2\lim_{x\to -1} x^3 - 8\lim_{x\to -1} x^2 + 3$$

$$\overset{8}{=} 7\cdot(-1)^5 + 2(-1)^3 - 8(-1)^2 + 3 = -14$$

REMARK 4. If we had defined $f(x) = 7x^5 + 2x^3 - 8x^2 + 3$ then Example 3 would have been,

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}(7x^5+2x^3-8x^2+3) = 7(-1)^5 + 2(-1)^3 - 8(-1)^2 + 3 = -14 = f(-1)$$

EXAMPLE 5. *Compute the limit:*

$$\lim_{x\to -2}\frac{x^2+x+1}{x^3-10} = \frac{(-2)^2+(-2)+1}{(-2)^3-10} = \frac{4-2+1}{-8-10} = \frac{3}{-18} = -\frac{1}{6}$$

REMARK 6. The function from Example 5 also satisfies "direct substitution property":

$$\lim_{x\to a} f(x) = f(a).$$

Later we will say that such functions are *continuous*. Note that in both examples it was important that a in the domain of f.

EXAMPLE 7. *Compute the limit:*

$$\lim_{x\to 3}\frac{x-3}{x^2-9} = \lim_{x\to 3}\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)} = \frac{1}{3+3} = \frac{1}{6}$$

EXAMPLE 8. *Compute the limit:*

$$\lim_{x\to 1}\frac{x-1}{x^2-4x+3} = \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x-3)} = \frac{1}{1-3} = -\frac{1}{2}$$

EXAMPLE 9. *Given*

$$g(x) = \begin{cases} x^2 + 4, & \text{if} \quad x \le -1 \\ 2 - 3x & \text{if} \quad x > -1 \end{cases}$$

Compute the limits:

(a) $\lim_{x \to 4} g(x) = \lim_{x \to 4} (2 - 3x) = 2 - 3 \cdot 4 = -10$

(b) $\lim_{x \to -1} g(x) = 5$, because

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (x^2 + 4) = (-1)^2 + 4 = 5$$

$x < -1$

$$\|$$

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (2 - 3x) = 2 - 3(-1) = 5$$

$x > -1$

EXAMPLE 10. *Evaluate these limits.*

(a) $\lim_{x \to 4} \frac{x^{-1} - 0.25}{x - 4}$ $= \lim_{x \to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x-4} = \lim_{x \to 4} \frac{4-x}{4x(x-4)}$

$= \lim_{x \to 4} \frac{-\cancel{(x-4)}}{4x\cancel{(x-4)}} = \lim_{x \to 4} \left(-\frac{1}{4x} \right) = -\frac{1}{4 \cdot 4} = -\frac{1}{16}$

(b) $\lim_{x \to 0} \frac{(x+5)^2 - 25}{x}$ $= \lim_{x \to 0} \frac{x^2 + 10x + \cancel{25} - \cancel{25}}{x} = \lim_{x \to 0} \frac{x(x+10)}{x}$

$= \lim_{x \to 0} (x+10) = 10$

$$|a| = \begin{cases} a, & a \geq 0 \\ -a & a \leq 0 \end{cases}$$

(c) $\lim_{x\to -1} \frac{|x+1|}{x+1}$ DNE, because

$$|x+1| = \begin{cases} x+1, & x+1 \geq 0 \\ -(x+1), & x+1 \leq 0 \end{cases} = \begin{cases} x+1, & x \geq -1 & \text{right hand limit} \\ -(x+1), & x \leq -1 & \text{left-hand limit} \end{cases}$$

$$\lim_{x\to -1^-} \frac{|x+1|}{x+1} = \lim_{\substack{x\to -1^- \\ x<-1}} \frac{-(x+1)}{x+1} = -1$$

$$\lim_{x\to -1^+} \frac{|x+1|}{x+1} = \lim_{x\to -1^+} \frac{x+1}{x+1} = 1$$

$\neq$

(d) $\lim_{x\to -1} \frac{x^2+x}{|x+1|}$ DNE, because

$$\lim_{x\to -1^-} \frac{x^2+x}{|x+1|} = \lim_{x\to -1^-} \frac{x(x+1)}{-(x+1)} = \lim_{x\to -1} (-x) = 1$$

$$\lim_{x\to -1^+} \frac{x^2+x}{|x+1|} = \lim_{x\to -1^+} \frac{x(x+1)}{x+1} = \lim_{x\to -1} x = -1$$

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(e) $\displaystyle\lim_{\underbrace{x\to 0^-}_{x<0}} \left\{\frac{1}{x} - \frac{1}{|x|}\right\} = \lim_{x\to 0^-} \left(\frac{1}{x} - \left(-\frac{1}{x}\right)\right) = \lim_{x\to 0^-} \frac{2}{x}$ DNE as a finite limit

$\displaystyle\frac{1}{|x|} = -\frac{1}{x}$ if $x<0$

but as an infinite one it is $-\infty$.

(f) $\displaystyle\lim_{x\to 0} \frac{\sqrt{6-x}-\sqrt{6}}{x} = \lim_{x\to 0} \frac{\sqrt{6-x}-\sqrt{6}}{x} \cdot \frac{\sqrt{6-x}+\sqrt{6}}{\sqrt{6-x}+\sqrt{6}}$

$$= \lim_{x\to 0} \frac{(\sqrt{6-x})^2-(\sqrt{6})^2}{x(\sqrt{6-x}+\sqrt{6})} = \lim_{x\to 0} \frac{\cancel{6}-x-\cancel{6}}{x(\sqrt{6-x}+\sqrt{6})}$$

$$= \lim_{x\to 0} -\frac{1}{\sqrt{6-x}+\sqrt{6}} = -\frac{1}{\sqrt{6-0}+\sqrt{6}} = -\frac{1}{2\sqrt{6}}$$

Conclusion from the above examples:

To calculate the limit of $f(x)$ as $x \to a$:

* PLUG IN $x = a$ if a is in the domain of f. Then $\lim_{x \to a} f(x) = f(a)$

Otherwise "FACTOR" or "MULTIPLY BY CONJUGATE" and then plug in.

Consider one sided limits if necessary.

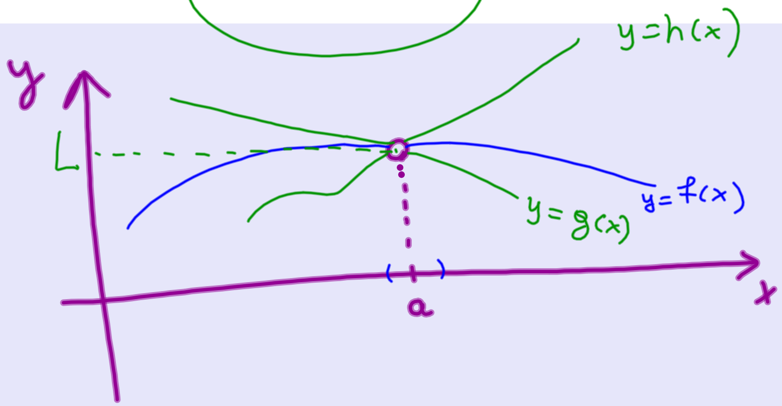
Also, use the following Theorem

Squeeze Theorem. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$g(x) \le f(x) \le h(x)$$

and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$. Then

$$\lim_{x\to a} f(x) = L.$$



Corollary. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$|f(x)| \le h(x) \quad (equivalently, \quad -h(x) \le f(x) \le h(x))$$

and $\lim_{x\to a} h(x) = 0$. Then

$$\lim_{x\to a} f(x) = 0.$$

EXAMPLE 11. *Given* $3x \le f(x) \le x^3 + 2$ *for* $0 \le x \le 2$. *Find* $\boxed{\lim_{x\to 1} f(x) = 3}$

$0 \le 1 \le 2$

$3x \xrightarrow{x\to 1} 3$

$f(x) \xrightarrow{x\to 1} 3$

$x^3+2 \xrightarrow{x\to 1} 3$, 1^3+2

EXAMPLE 12. *Evaluate:*

(a) $\lim_{x \to 0} x \sin \frac{1}{x}$ $= 0$, because

$$\left| x \sin \frac{1}{x} \right| = |x| \cdot \left| \sin \frac{1}{x} \right| \le |x| \cdot 1 = |x| \xrightarrow[x \to 0]{\text{by S.T.}} 0$$

(b) $\lim_{t \to 0} (t^5) \cos^3(\frac{1}{t^2})$ $= 0$, because

$$\left| t^5 \cos^3\left(\frac{1}{t^2}\right) \right| \le |t|^5 \cdot \left| \cos \frac{1}{t^2} \right|^3 \le |t|^5 \xrightarrow[t \to 0]{\text{by S.T.}} 0$$

EXAMPLE 13. *Is there a number c such that*

$$L = \lim_{x \to -2} \frac{3x^2 + cx + c + 3}{x^2 + x - 2} = \lim_{x \to -2} \frac{\overbrace{3x^2+cx+c+3}^{=f(x)}}{(x+2)(x-1)}$$

exists? If so, find the value c and the value of the limit.

It is sufficient that $x=-2$ is zero of the numerator, i.e. if $f(x) = 3x^2+cx+c+3$ then $f(-2)=0$, or

$$3\cdot(-2)^2 + c\cdot(-2) + c + 3 = 0$$

$$12 - 2c + c + 3 = 0$$

$$\boxed{c = 15}$$

In this case $f(x) = 3x^2 + 15x + 15 + 3 = 3(x^2+5x+6)$

$$= 3(x+2)(x+3).$$

So, $L = \lim_{x\to -2} \frac{f(x)}{(x+2)(x-1)} = \lim_{x \to -2} \frac{3\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-1)}$

$$= \frac{3(-2+3)}{-2-1} = \boxed{-1}$$