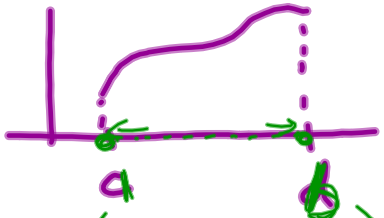
Section 2.5:Continuity

direct subst rule

DEFINITION 1. *A function $f(x)$ is* **continuous** *at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. More implicitly: if f is continuous at a then*

1. *$f(a)$ is defined (i.e. a is in the domain of f);*
2. *$\lim_{x\to a} f(x)$ exists.*
3. *$\lim_{x\to a} f(x) = f(a)$.*

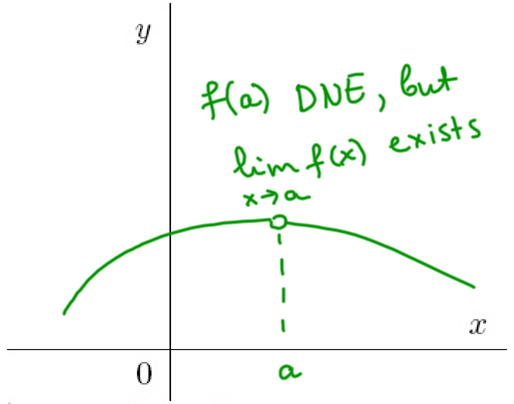


A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

$\lim_{x\to a^+} f(x) = f(a)$ $\lim_{x\to b^-} f(x) = f(b)$

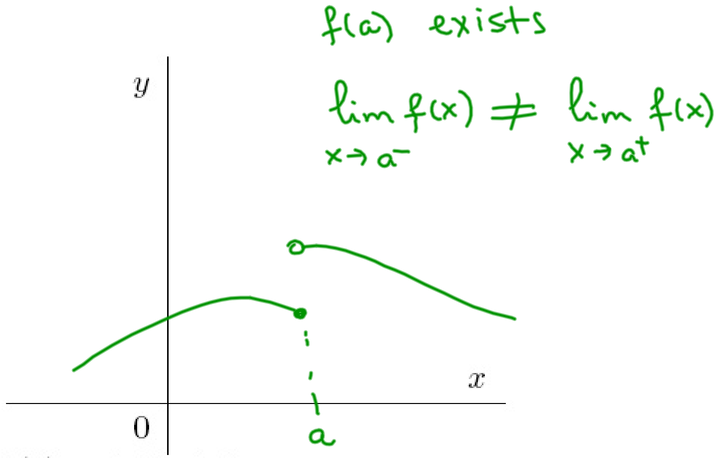
Geometrically, if f is continuous at any point in an interval then its graph has no break in it (i.e. can be drawn without removing your pen from the paper).

REASONS FOR BEING DISCONTINUOUS:



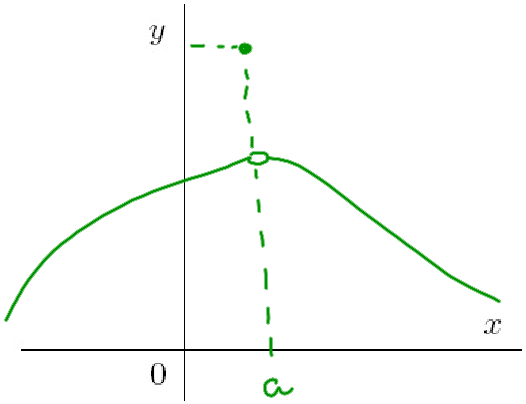
(a) is not defined

(i.e. a is not in the domain of f)

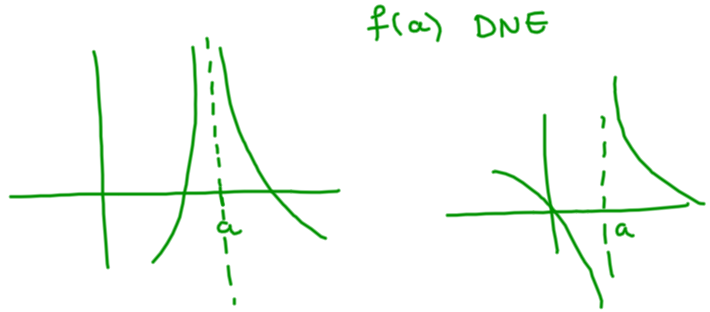


$f(a)$ is defined, but

the limit as $x \to a$ DNE

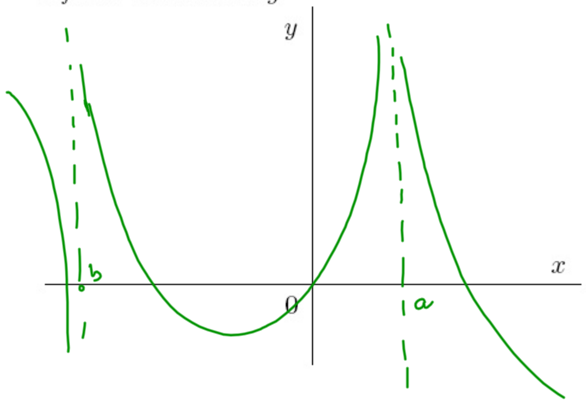


$f(a)$ is defined and $\lim_{x \to a} f(x)$ exists,
but $\lim_{x \to a} f(x) \neq f(a)$



Classification of discontinuities:

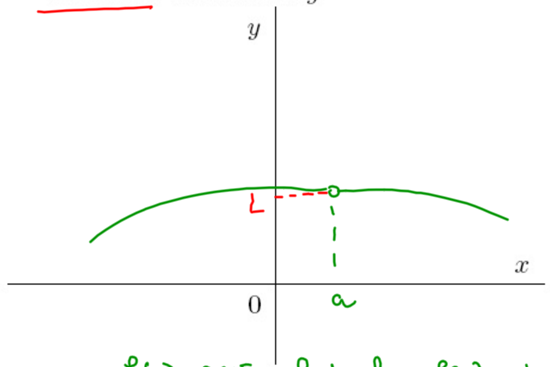
infinite discontinuity



$f(a)$ is not cont. at a and b.

($x=a$ and $x=b$ are vertical asymptotes)

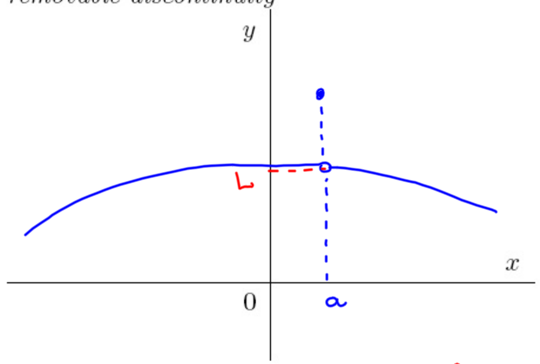
removable discontinuity



$f(a)$ DNE, but $\lim_{x\to a} f(x) = L$

To fix such discontinuity, set $f(a)=L$

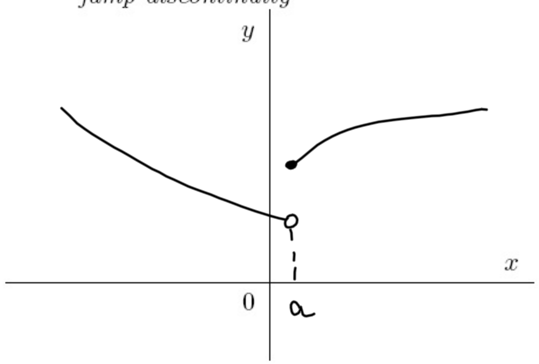
removable discontinuity



$f(a)$ exists, but $f(a) \neq L = \lim_{x\to a} f(x)$

To fix: set $f(a)=L$ to make $f(x)$ continuous.

jump discontinuity



$\lim_{x\to a} f(x)$ DNE, because both one-sided limits exists, but do not coincide.

EXAMPLE 2. *Explain why each function is discontinuous at the given point:*

(a) $f(x) = \dfrac{2x}{x-3}, \quad x = 3$

$f(3)$ DNE (i.e. $x=3$ does not belong to the domain of f.)

(b) $f(x) = \begin{cases} \dfrac{x^2-2x+1}{x-1} & \text{if } \; x \neq 1 \\ 5 & \text{if } \; x = 1, \end{cases}$, $x = 1$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-2x+1}{x-1} = \lim_{x\to 1} \frac{(x-1)^2}{x-1} = \lim_{x\to 1} (x-1) = 0 \neq f(1) = 5$$

DEFINITION 3. *A function f is* **continuous from the right at** $x = a$ *if*

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a *if*

$$\lim_{x \to a^-} f(x) = f(a).$$

REMARK 4. *Functions* **continuous on an interval** *if it is continuous at every number in the interval.* At the end point of the interval we understand continuous to mean continuous from the right or continuous from the left.

EXAMPLE 5. *Find the interval(s) where $f(x) = \sqrt{9 - x^2}$ is continuous.*

Find domain of f : $9 - x^2 \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow |x| \leq 3$, or $-3 \leq x \leq 3$.

$[-3, 3]$

$\lim\limits_{x \to a} f(x) = f(a) = \sqrt{9 - a^2}$ for all $-3 < a < 3$

$\lim\limits_{x \to -3^+} f(x) = f(-3)$

$\lim\limits_{x \to 3^-} f(x) = f(3)$.

f is continuous on $[-3, 3]$

EXAMPLE 6. *Find the constant c that makes g continuous on $(-\infty, \infty)$:*

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx^2 - 1 & \text{if } x \geq 4 \end{cases}$$

polynomials

Since polynomials are continuous for all x, it is sufficient to find c such that

$$x^2 - c^2 = cx^2 - 1 \quad \text{at } x = 4$$

$$4^2 - c^2 = c \cdot 4^2 - 1$$

$$c^2 + 16c - 17 = 0$$

$$(c-1)(c+17) = 0$$

Answer: $c = 1$ or $c = -17$.

EXAMPLE 7. *For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function. If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.*

(a) $f(x) = \dfrac{x^2-9}{x^4-81} = \dfrac{\cancel{x^2-9}}{\cancel{(x^2-9)}(x^2+9)} = \dfrac{1}{x^2+9} \quad (x \neq \pm 3)$

$f(3)$ and $f(-3)$ is undefined. However

$$\lim_{x \to \pm 3} f(x) = \lim_{x \to \pm 3} \frac{1}{x^2+9} = \frac{1}{(\pm 3)^2+9} = \frac{1}{18}$$

So, $f(x)$ has removable discontinuity at $x = \pm 3$

To fix discontinuity:

$$g(x) = \begin{cases} f(x), & x \neq \pm 3 \\ \frac{1}{18}, & x = \pm 3 \end{cases}$$

(b) $f(x) = \dfrac{7}{x+12}$

$x+12=0 \Rightarrow x=-12$, i.e. $f(-12)$ DNE

$$\lim_{x \to -12^+} \frac{7}{x+12} = +\infty \Rightarrow \text{infinite discontinuity at } x=-12$$

$x > -12 \Rightarrow x+12 > 0$

(c) $f(x) = \begin{cases} x^2 + x & \text{if } x < 2 \\ 8 - x & \text{if } x > 2 \\ 4 & \text{if } x = 2 \end{cases}$

Since $f_1(x) = x^2 + x$, $f_2(x) = 8 - x$ are polynomials, they are continuous for all x. Thus f is continuous for all $x \neq 2$. Investigate separately $x = 2$.

$f(2) = 4$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + x) = 2^2 + 2 = 6 \neq 4 = f(2)$$

$x < 2$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (8 - x) = 8 - 2 = 6$$

$x > 2$

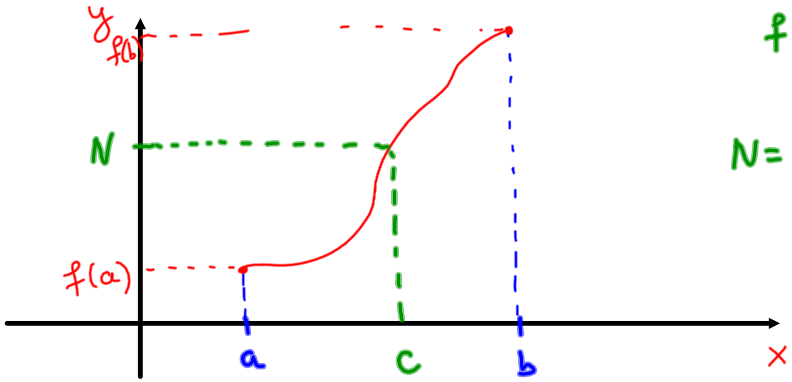
So $\lim_{x \to 2} f(x) = 6 \neq 4 = f(2)$

removable discont. at $x = 2$

To fix that define

$$g(x) = \begin{cases} f(x), & x \neq 2 \\ 6, & x = 2 \end{cases}$$

Intermediate Value Theorem: If $f(x)$ is continuous on the closed interval $[a,b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $a<c<b$, so that $f(c)=N$.



$f(a) < N < f(b)$

$N = f(c)$

EXAMPLE 8. *If $f(x)=x^5-2x^3+x^2+2$, show there a number c so that $f(c)=1$.*

Reasoning

$f(c)=1 \qquad c^5-2c^3+c^2+2=1$

$g(c) = c^5 - 2c^3 + c^2 + 1 = 0$

c^5 is a dominant term as $c \to +\infty$ and as $c \to -\infty$

So, $\lim\limits_{c\to\infty} g(c) = \infty$ and $\lim\limits_{c\to-\infty} g(c) = -\infty$,

So $g(x)$ changes sign on $(-\infty, \infty)$, i.e. there exists c such that $g(c)=0$, or $f(c)=1$.

x	0	1	-1	-2
$f(x)$	2	2	4	-10

Note that

$-10 = f(-2) < \underset{f(c)}{\overset{}{1}} < f(-1) = 4$

Since $f(x)$ is continuous (as a polynomial), By IVT we conclude that there exists c on $(-1,-2)$ such that $f(c)=1$.

EXAMPLE 9. *Show that following equation has a solution (a root) between* 1 *and* 2*:*

$$f(x) = 3x^3 - 2x^2 - 2x - 5 = 0.$$

In other words show that there exists a number c such that $1 < c < 2$ and $f(c) = 0$.

Apply IVT. $f(x)$ is a polynomial, so $f(x)$ is continuous on $[1, 2]$. We have

$$f(1) = 3 - 2 - 2 - 5 = -6 < 0$$

$$f(2) = 3 \cdot 8 - 2 \cdot 4 - 4 - 5 = 7 > 0$$

So, $f(x)$ has at least one root on $(1, 2)$