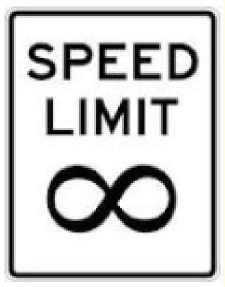
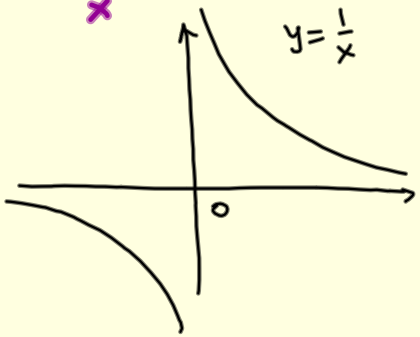


Section 2.6: Limits at infinity: horizontal asymptotes

The end behavior of a function is computed by $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

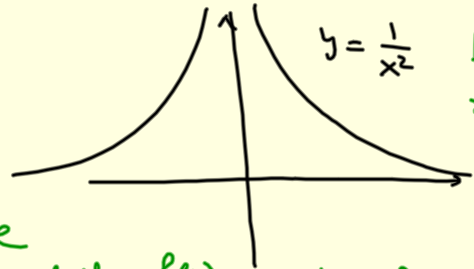
$f(x) = \frac{1}{x}$

$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$



$y = 0$ is horizontal asymptote for both $f(x)$ and $g(x)$

$g(x) = \frac{1}{x^2}$



$\lim_{x\to\infty} g(x) = 0$

$\lim_{x\to-\infty} g(x) = 0$

THEOREM 1. *If $r > 0$ is a rational number such that x^r is defined for all x, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \textit{and} \quad \lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

$y=0$ is horizontal asymptote for $y = \frac{1}{x^r}$

EXAMPLE 2. *Examine the limits at infinity for the following rational functions (the quotient of polynomials):*

$$f(x) = \frac{7x^2 + 4x - 3}{2x^2 - 12x + 16}, \quad g(x) = \frac{9x - 4}{7x^2 + 4x - 3}, \quad h(x) = \frac{2x^4 - 12x^3 + 16}{x^2 - 12x + 1}$$

Solution: Let's divide the numerator and denominator by the highest power of x that appears and then apply Theorem 1.

- $\lim_{\substack{\cancel{x\to\infty} \\ x\to\pm\infty}} f(x) = \lim_{x\to\infty} \frac{(7x^2 + 4x - 3)/x^2}{(2x^2 - 12x + 16)/x^2} = \lim_{x\to\pm\infty} \frac{\frac{7x^2}{x^2} + \frac{4x}{x^2} - \frac{3}{x^2}}{\frac{2x^2}{x^2} - \frac{12x}{x^2} + \frac{16}{x^2}}$

$$= \lim_{x\to\pm\infty} \frac{7 + \frac{4}{x} - \frac{3}{x^2}}{2 - \frac{12}{x} + \frac{16}{x^2}} = \frac{7 + 0 - 0}{2 - 0 + 0} = \frac{7}{2}$$

ratio of leading coefficients of $f(x)$

$y = \frac{7}{2}$ is horizontal asymptote for $f(x)$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{7x^2 + 4x - 3}{2x^2 - 12x + 16} =$$

- $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{(9x-4)/x^2}{(7x^2+4x-3)/x^2} =$ $\lim_{x\to\infty} \frac{\frac{9}{x}-\frac{4}{x^2}}{7+\frac{4}{x}-\frac{3}{x^2}} = \frac{0-0}{7+0-0} = \frac{0}{7} = 0$

y=0 is horizontal asymptote

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{9x-4}{7x^2+4x-3} = 0$$

- $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{(2x^4 - 12x^3 + 16)/x^4}{(x^2 - 12x + 1)/x^4} =$

$$\lim_{x\to\infty} \frac{2 - \frac{12}{x} + \frac{16}{x^4}}{\frac{1}{x^2} - \frac{12}{x^3} + \frac{1}{x^4}}$$

DNE as a finite limit

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{2x^4 - 12x^3 + 16}{x^2 - 12x + 1} =$$

no horizontal asymptote.

DEFINITION 3. *If* $\lim_{x\to\infty} f(x) = L$, *or* $\lim_{x\to-\infty} f(x) = L$, *then* $y = L$ *is called a* **horizontal asymptote** *of* $f(x)$.

EXAMPLE 4. *Analyze the rational functions*

$$f(x) = \frac{7x^2 + 4x - 3}{2x^2 - 12x + 16}, \quad g(x) = \frac{9x - 4}{7x^2 + 4x - 3}, \quad h(x) = \frac{2x^4 - 12x^3 + 16}{x^2 - 12x + 1}$$

from Example 2 for horizontal asymptotes.

function	$f(x)$	$g(x)$	$h(x)$
horizontal asymptotes	$y = \frac{7}{2}$	$y = 0$	—

RULES FOR HORIZONTAL ASYMPTOTES of rational functions:

1. If the degree (highest power) of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote (cf. $h(x)$ from Examples 2&4).

2. If the degree of the numerator is smaller than the degree of the denominator, then the horizontal asymptote is at $y = 0$ (or the $x$axis) ((cf. $g(x)$ from Examples 2,4)).

3. If the degree of the numerator is equal to the degree of the denominator, then you must compare the coefficients in front of the terms with the highest power. The horizontal asymptote is the coefficient of the highest power of the numerator divided by the coefficient of the highest power of the denominator. (cf. $f(x)$ from Examples 2,4).

EXAMPLE 5. *Find the equations for all vertical and horizontal asymptotes for*

rational function

(a) $f(x) = \dfrac{3x^2 + 2x - 3}{2(x-1)(x+2)}$

$3 \cdot 1^2 + 2 \cdot 1 - 3 \neq 0$

$x = 1$ is vertical asymptote

$3 \cdot (-2)^2 + 2 \cdot (-2) - 3$
$= 12 - 4 - 3 \neq 0$

$x = -2$ is vertical asymptote.

$$\lim_{x \to \pm\infty} f(x) = \frac{3}{2}$$

$y = \frac{3}{2}$ is horizontal asymptote.

Reminder
$\sqrt{a^2} = |a|$

not a rational function

(b) $f(x) = \dfrac{2x}{\sqrt{x^2+5}} = \begin{cases} \sqrt{\dfrac{4x^2}{x^2+5}}, & x \geq 0 \\ -\sqrt{\dfrac{4x^2}{x^2+5}}, & x \leq 0. \end{cases}$

Note $x^2 + 5 > 0$, i.e. $x^2 + 5 \neq 0$. So, no vertical asymptote

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \sqrt{\frac{4x^2}{x^2+5}} = \sqrt{\lim_{x \to \infty} \frac{4x^2}{x^2+5}} = \sqrt{4} = 2$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -\sqrt{\frac{4x^2}{x^2+5}} = -\sqrt{4} = -2$$

$y = 2$ and $y = -2$ are horizontal asymptotes

Remark Another method to find lim at $x \to -\infty$:

replace x by $-x$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x \to \infty} \frac{2(-x)}{\sqrt{(-x)^2+5}}$$

$$= -\underbrace{\lim_{x \to \infty} \frac{2x}{\sqrt{x^2+5}}}_{=2} = -2$$

EXAMPLE 6. *Compute these limits:* multiply by conjugate both numerator and denominator

$(a-b)(a+b)=a^2-b^2$

(a) $\lim_{x\to\infty}(\sqrt{x^2+3x}-x) =$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{\sqrt{x^2+3x}+x}$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+3x})^2 - x^2}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x}$$

divide both num. and denom. by highest power of x

$$= \lim_{x\to\infty} \frac{(3x)/x}{(\sqrt{x^2+3x}+x)/x} = \lim_{\underbrace{x\to\infty}_{x>0}} \frac{3}{\frac{\sqrt{x^2+3x}}{\sqrt{x^2}}+1}$$

$x>0$

$x=\sqrt{x^2}$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{3}{x}}+1} = \frac{3}{\sqrt{1+0}+1} = \boxed{\frac{3}{2}}$$

(b) $\lim_{x\to\infty} \frac{\sqrt{3x^4+2x^2+1}}{x(x-1)} = \lim_{x\to\infty} \sqrt{\frac{3x^4+2x^2+1}{x^2(x-1)^2}}$

$x>0$, $x-1>0 \Rightarrow x(x-1)=\sqrt{(x(x-1))^2}$

rational function, also it is ratio of two polynomials of degree 4.

$$= \sqrt{\lim_{x\to\infty} \frac{3x^4+2x^2+1}{1x^2(x-1)^2}} = \sqrt{\frac{3}{1}} = \sqrt{3}$$

(c) $\lim_{x\to\infty}(\sqrt{x^2+4x+5}+x) = \lim_{x\to\infty} \dfrac{(\sqrt{x^2+4x+5}+x)(\sqrt{x^2+4x+5}-x)}{\sqrt{x^2+4x+5}-x}$

conjugate

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+4x+5})^2 - x^2}{\sqrt{x^2+4x+5}-x} = \lim_{x\to\infty} \frac{(\cancel{x^2}+4x+5-\cancel{x^2})/x}{(\sqrt{x^2+4x+5}-x)/x}$$

$$= \lim_{x\to\infty} \frac{4+\frac{5}{x}}{\frac{\sqrt{x^2+4x+5}}{\sqrt{x^2}} - 1} = \lim_{x\to\infty} \frac{4+\frac{5}{x}}{\sqrt{\frac{x^2+4x+5}{x^2}}-1}$$

$x>0$
$x=\sqrt{x^2}$

$$= \lim_{x\to\infty} \frac{4+\frac{5}{x}}{\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}-1}$$

DNE (behaives as $\frac{4}{0}$)

(d) $\lim_{x\to-\infty}(\sqrt{x^2+4x+5}+x) = \lim_{x\to\infty} \sqrt{(-x)^2+4(-x)+5} - x$

$= \lim_{x\to\infty} \sqrt{x^2-4x+5} - x \quad = \ldots\ldots \quad = -2$