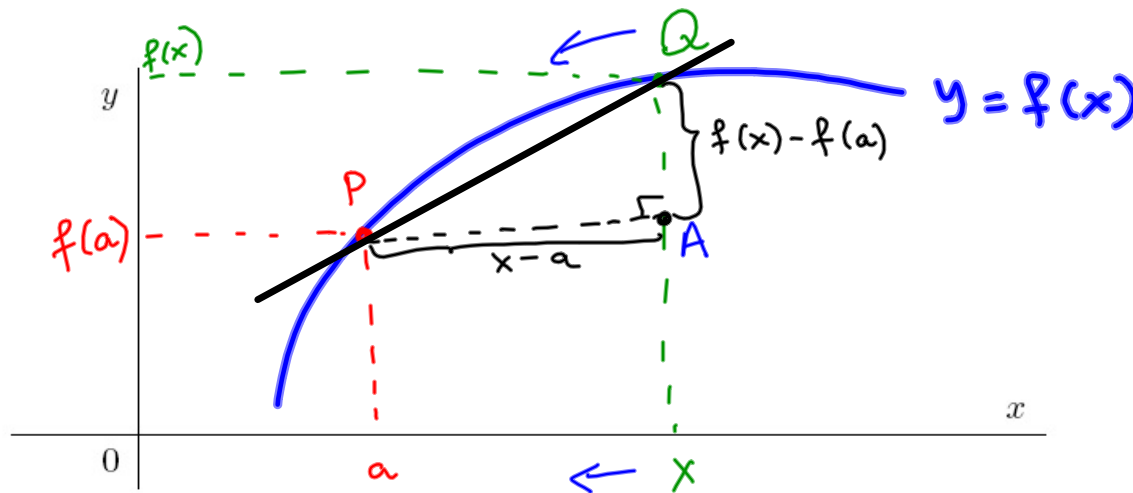


## Section 2.7: Tangents, velocities, and other rates of change.

Consider a curve with equation  $y = f(x)$  and points  $P(a, f(a))$  and  $Q(x, f(x))$  on it. The slope of the secant line  $PQ$  (also known as average rate or average velocity) is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} = \frac{QA}{PA}$$



DEFINITION 1. The tangent line to the curve  $y = f(x)$  at the point  $P(a, f(a))$  is the line through  $P$  with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{Q \rightarrow P} m_{PQ} \quad (1)$$

provided that this limit exists.

The slope (1) of tangent line also known as instantaneous rate of change or instantaneous velocity.

EXAMPLE 2. (a) Find the slope of the tangent line to the graph of  $f(x) = x^2 - 3x - 4$  at  $(5, 6)$ .

$$m = \lim_{x \rightarrow 5} \frac{f(x) - 6}{x - 5} = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 4 - 6}{x - 5} = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} \quad \begin{matrix} a \\ f(a) \end{matrix}$$

$$= \lim_{x \rightarrow 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = 5 + 2 = 7.$$

(b) Find the equation of the tangent line to the graph of  $f(x)$  at  $x = 5$ . (Recall that point-slope form for a line through the point  $(x_1, y_1)$  with slope  $m$  is:  $y - y_1 = m(x - x_1)$ )

$(5, 6)$

$$y - 6 = 7(x - 5)$$

**Velocities.** Denote by  $f(t)$  the position of an object at time  $t$ .

The **Average Velocity** of the object from  $t = a$  to  $t = b$  is

$$\frac{f(b) - f(a)}{b - a} = V_{ave}$$

The **Instantaneous Velocity** of the object at time  $t = a$  is

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

(Compare this formula with (1) by substitution  $h = x - a$ .)  
 $x = a + h$

EXAMPLE 3. The position (in meters) of an object moving in a straight path is given by

$$s(t) = t^2 - 2t + 8,$$

where  $t$  is measured in seconds.

(a) Find the average velocity over the time interval  $[4, 5]$ .

$$V_{\text{ave}} = \frac{s(5) - s(4)}{5 - 4} = \frac{25 - 10 + 8 - (16 - 8 + 8)}{1} = 7 \text{ m/s}$$

*Note: In the original image, a green arrow points from  $s(5)$  to  $25 - 10 + 8$ , and another green arrow points from  $s(4)$  to  $16 - 8 + 8$ . The value  $16$  is also labeled as  $s(4) = 16$ .*

(b) Find the instantaneous velocity at time  $t = 4$ .

$$\begin{aligned} \lim_{t \rightarrow 4} \frac{s(t) - s(4)}{t - 4} &= \lim_{t \rightarrow 4} \frac{t^2 - 2t + 8 - 16}{t - 4} = \lim_{t \rightarrow 4} \frac{t^2 - 2t - 8}{t - 4} \\ &= \lim_{t \rightarrow 4} \frac{(t - 4)(t + 2)}{t - 4} = 4 + 2 = 6 \text{ m/s} \end{aligned}$$

**Other Rates of Change:**

The **Average Rate** of change of function  $f(x)$  from  $x = a$  to  $x = b$  is

$$\frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Rate of Change** of  $f(x)$  at  $x = a$  is

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$