Section 3.10: Related rates

In this section, we have two or more quantities that are changing with respect to time t. We will apply the following strategy:

1. Read the problem carefully and draw a diagram if possible. (units)
2. Express the given information and the required rates in terms of derivatives and state your "find" and "when".
3. Find a formula (equation) that relates the quantities in the problem. (If necessary, use Geometry[1] of the situation to eliminate one of the variables by substitution.) **Don't substitute the given numerical information at this step!!!**
4. Use the Chain Rule to differentiate both sides of the equation with respect to t.
5. Substitute the given numerical information in the resulting equation and solve for the desired rate of change.

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[1] Useful formulas:

- Triangle: $A = \frac{1}{2}bh$
  - Equilateral Triangle: $h = \dfrac{\sqrt{3}}{2}s$; $A = \dfrac{\sqrt{3}s^2}{4}$
  - Right Triangle: Pythagorean Theorem $c^2 = a^2 + b^2$
- Trapezoid: $A = \dfrac{h}{2}(b_1 + b_2)$
- Parallelogram: $A = bh$
- Circle: $A = \pi r^2$; $C = 2\pi r$
- Sector of Circle: $A = \frac{1}{2}r^2\theta$; $s = r\theta$
- Sphere: $V = \dfrac{4}{3}\pi r^3$; $A = 4\pi r^2$
- Cylinder: $V = \pi r^2 h$
- Cone: $V = \dfrac{1}{3}\pi r^2 h$

EXAMPLE 1. *A spherical balloon is inflated with gas at a rate of* $25\text{ft}^3/\text{min}$. *How fast is the radius changing when the radius is* 2ft*?*

Quantities that are changing : V , r

Given $\frac{dV}{dt} = 25 \ \text{ft}^3/\text{min}$

Find $\left.\frac{dr}{dt}\right|_{r=2\text{ft}}$ – ?

$$V = \frac{4}{3}\pi r^3 \quad (\text{don't substitute } r = 2)$$

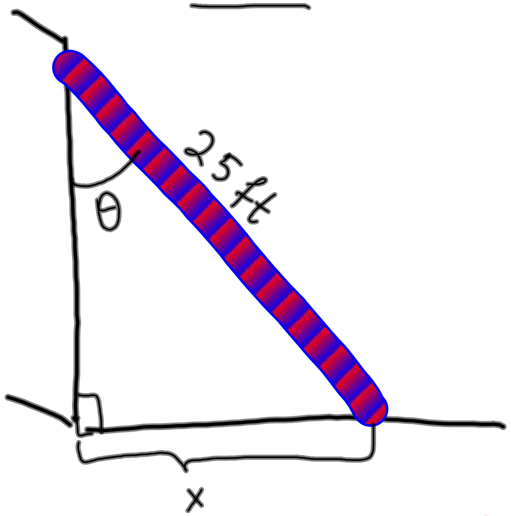
$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

$$\frac{d}{dt}\left[V(t)\right] = \frac{4}{3}\pi \frac{d}{dt}\left[r(t)\right]^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \,\cancel{3}\, r^2 \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=2} = \left.\frac{1}{4\pi r^2}\frac{dV}{dt}\right|_{r=2} = \frac{1}{4\pi\cdot 2^2}\cdot 25 = \boxed{\frac{25}{16\pi}\,\frac{\text{ft}}{\text{min}}}$$

EXAMPLE 2. *A ladder* 25 *feet long and leaning against a vertical wall. The bottom of the ladder slides away from the wall at speed* 3 feet/sec. *Determine how fast the angle between the top of the ladder and the wall is changing when the angle is* $\frac{\pi}{4}$ *radians.*



Quantities x, θ

$$\frac{dx}{dt} = 3$$

Find $\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{4}} = ?$

$$\frac{d}{dt}\left(\sin\theta(t)\right) = \frac{d}{dt}\left(\frac{x(t)}{25}\right)$$

$$\cos\theta \, \frac{d\theta}{dt} = \frac{1}{25}\frac{dx}{dt}$$

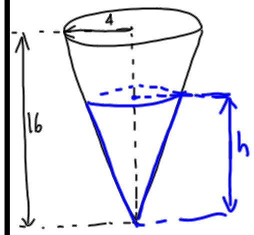
$$\theta = \frac{\pi}{4} \Rightarrow \underbrace{\cos\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} \frac{d\theta}{dt} = \frac{1}{25}\cdot 3$$

$$\frac{d\theta}{dt} = \boxed{\frac{3\sqrt{2}}{25} \text{ rad/sec}}$$

EXAMPLE 3. *A water tank has the shape of an inverted right circular cone with height 16m and base radius 4m. Water is pouring into the tank at* $3\text{m}^3/\text{min}$.

(a) *How fast is the water level rising when the water in the tank is 5 meters deep?*

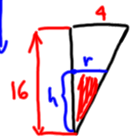
Quantities: V, h



$$\frac{dV}{dt} = 3$$

$$\left.\frac{dh}{dt}\right|_{h=5}$$

$$V = \frac{1}{3}\pi r^2 h$$



$$\frac{16}{h} = \frac{4}{r}$$

$$16r = 4h$$

$$4r = h \Rightarrow \boxed{r = \frac{h}{4}}$$

use in (b)

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{48}$$

$$\frac{d}{dt}\left(V(t)\right) = \frac{\pi}{48}\frac{d}{dt}\left(\left(h(t)\right)^3\right)$$

$$\frac{dV}{dt} = \frac{\pi}{48}3h^2\frac{dh}{dt}$$

$$h=5 \Rightarrow \cancel{3} = \frac{\pi}{48}\cdot\cancel{3}\cdot 25\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \boxed{\frac{48}{25\pi}\ \text{m/min}}$$

(b) *At what rate is the radius of the top of the water in the tank changing when the depth of the water is 5 meters?*

Quantities: V, r

$$\frac{dV}{dt} = 3 \qquad \text{Find } \left.\frac{dr}{dt}\right|_{h=5}$$

$$V = \frac{1}{3}\pi r^2 h$$

$h = 4r$ (see above)

$$\Rightarrow V = \frac{1}{3}\pi r^2 4r$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi\,\cancel{3}r^2\frac{dr}{dt}$$

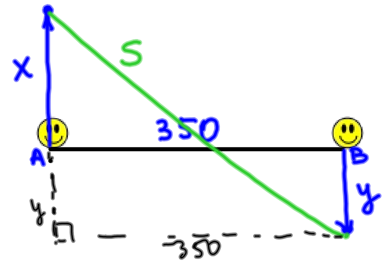
$$3 = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \left.\frac{3}{4\pi r^2}\right|_{h=5}$$

when $h=5$ we have $r = \frac{h}{4} = \frac{5}{4}$

$$\frac{dr}{dt} = \frac{3}{\cancel{4}\pi\cdot\frac{5^2}{4^{\cancel{2}}}} = \frac{12}{25\pi}\ \text{m/min}$$

EXAMPLE 4. *Two people are separated by 350 meters. Person A starts walking north at a rate of 0.6 m/sec and 7 minutes later Person B starts walking south at 0.5 m/sec. At what rate is the distance separating the two people changing 25 minutes after Person A starts walking?*

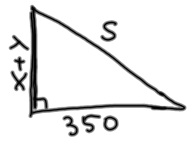


$$\frac{dx}{dt} = 0.6\ m/s$$

$$\frac{dy}{dt} = 0.5\ m/s$$

$$\left.\frac{ds}{dt}\right|_{t=1500}$$

Quantities
$x,\ y,\ s$



$$s^2 = 350^2 + (x+y)^2$$

$$25\,\text{min} = 25 \cdot 60\,s = 1500\,s$$

$$\frac{d}{dt}\left[(s(t))^2\right] = \frac{d}{dt}\left[350^2 + [x(t)+y(t)]^2\right]$$

$$\cancel{2}\, s(t) \left.\frac{ds}{dt}\right|_{t=1500} = \cancel{2}\,(x(t)+y(t))\left.\left(\frac{dx}{dt} + \frac{dy}{dt}\right)\right|_{t=1500}$$

$$x(1500) = (\text{speed}_A)(\text{time } A) = \frac{dx}{dt} \cdot 1500 = 0.6 \cdot 1500 = 900\ m$$

$$y(1500) = (\text{speed } B)(\text{time } B) = 0.5 \cdot (25-7) \cdot 60 = 540\ m$$

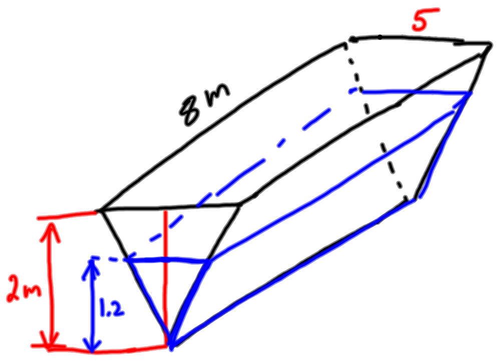
$$s(1500) = \left.\sqrt{350^2 + (x+y)^2}\right|_{t=1500} = \sqrt{350^2 + (900+540)^2} \approx 1482\ m$$

$$1482 \frac{ds}{dt} \approx (900+540) \cdot (0.6+0.5)$$

$$\boxed{\frac{ds}{dt} \approx 1.07\ m/s}$$

EXAMPLE 5. *A trough of water is 8 meters long and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If the trough is filled with water at a constant rate of* $6\text{m}^3/\text{s}$*, how fast the water level (the height of the water) changing when the water is* 120cm *deep?*

$= 1.2\,\text{m}$



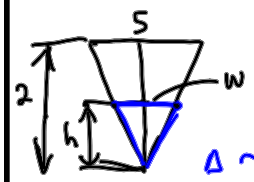
Quantities: V, h

Given $\frac{dV}{dt} = 6 \; m^3/s$

Find $\left.\frac{dh}{dt}\right|_{h=1.2\,m} = ?$

$$V = (\text{Area of end}) \cdot \text{length}$$

$$\text{Area of end} = \frac{\text{Base} \cdot \text{height}}{2}, \qquad \text{length} = 8$$



$\Delta \sim \Delta$

$$V = \frac{w\,h}{2} \cdot 8 = 4wh$$

$$\frac{5}{w} = \frac{2}{h} \Rightarrow 5h = 2w \Rightarrow w = \frac{5h}{2}$$

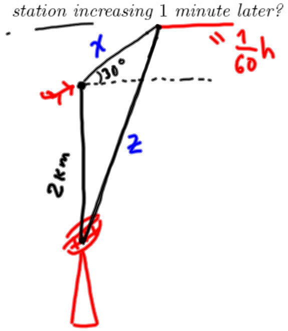
$$\Rightarrow V = 4\,\frac{5h}{2} \cdot h = 10h^2$$

$$\frac{d}{dt} V(t) = \frac{d}{dt} 10\left[h(t)\right]^2$$

$$\frac{dV}{dt} = 10 \cdot 2h \frac{dh}{dt}$$

$h = 1.2 \Rightarrow 6 = 20 \cdot 1.2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = 0.25 \; m/s$

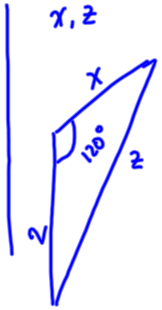
EXAMPLE 6. *A plane flying with a constant speed of 360km/hour passes over a radar station at an altitude of 2km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing 1 minute later?*



$\frac{dx}{dt} = 360\ km/h$

$\left.\frac{dz}{dt}\right|_{t=\frac{1}{60}h} = ?$

x, z



Use the Law of Cosines

$z^2 = x^2 + 2^2 - 2\cdot 2\cdot x\cos 120°$

$z^2 = x^2 + 4 - 4x\left(-\frac{1}{2}\right)$

$z^2 = x^2 + 4 + 2x$

Differentiate

$\frac{d}{dt}[z(t)]^2 = \frac{d}{dt}\left[[x(t)]^2 + 4 + 2x(t)\right]$

$2z(t)\frac{dz}{dt} = 2x(t)\frac{dx}{dt} + 0 + 2\frac{dx}{dt}$

$t = \frac{1}{60}$

$x\left(\frac{1}{60}\right) = (\text{speed})(\text{time}) = 360\cdot\frac{1}{60} = 6\ km$

$z\left(\frac{1}{60}\right) = \sqrt{\left[x\left(\frac{1}{60}\right)\right]^2 + 4 + 2x\left(\frac{1}{60}\right)}$

$= \sqrt{6^2 + 4 + 2\cdot 6} = 2\sqrt{13}\ km$

$\cancel{2}\cdot\cancel{2}\sqrt{13}\frac{dz}{dt} = \cancel{2}\cdot 6\cdot \overset{180}{\cancel{360}} + \cancel{2}\cdot\overset{180}{\cancel{360}}$

$\frac{dz}{dt} = \frac{7\cdot 180}{\sqrt{13}} \approx 350\ km/h$