Section 3.1: Derivative

DEFINITION 1. *The* **Derivative** *of a function $f(x)$ at $x = a$ is*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Other common notations for the derivative of $y = f(x)$ are f', $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$.

It follows from the definition that the derivative $f'(a)$ measures:

- The slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$;
- The instantaneous rate of change of $f(x)$ at $x = a$;
- The instantaneous velocity of the object at time at $t = a$ (if $f(t)$ is the position of an object at time t).

EXAMPLE 2. *Given* $f(x) = \dfrac{3}{x+5}$.

(a) *Find the derivative of* $f(x)$ *at* $x = -3$.

$$f'(-3) = \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h\to 0} \frac{\frac{3}{-3+h+5} - \frac{3}{-3+5}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{h+2} - \frac{3}{2}}{h} = \lim_{h\to 0} \frac{6 - 3(h+2)}{2\cdot(h+2)\,h} = \lim_{h\to 0} \frac{\cancel{6} - 3\cancel{h} - \cancel{6}}{2(h+2)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-3}{2(h+2)} = -\frac{3}{2(0+2)} = \boxed{-\frac{3}{4}}$$

also slope of the tangent line of $y=f(x)$ at $x=-3$

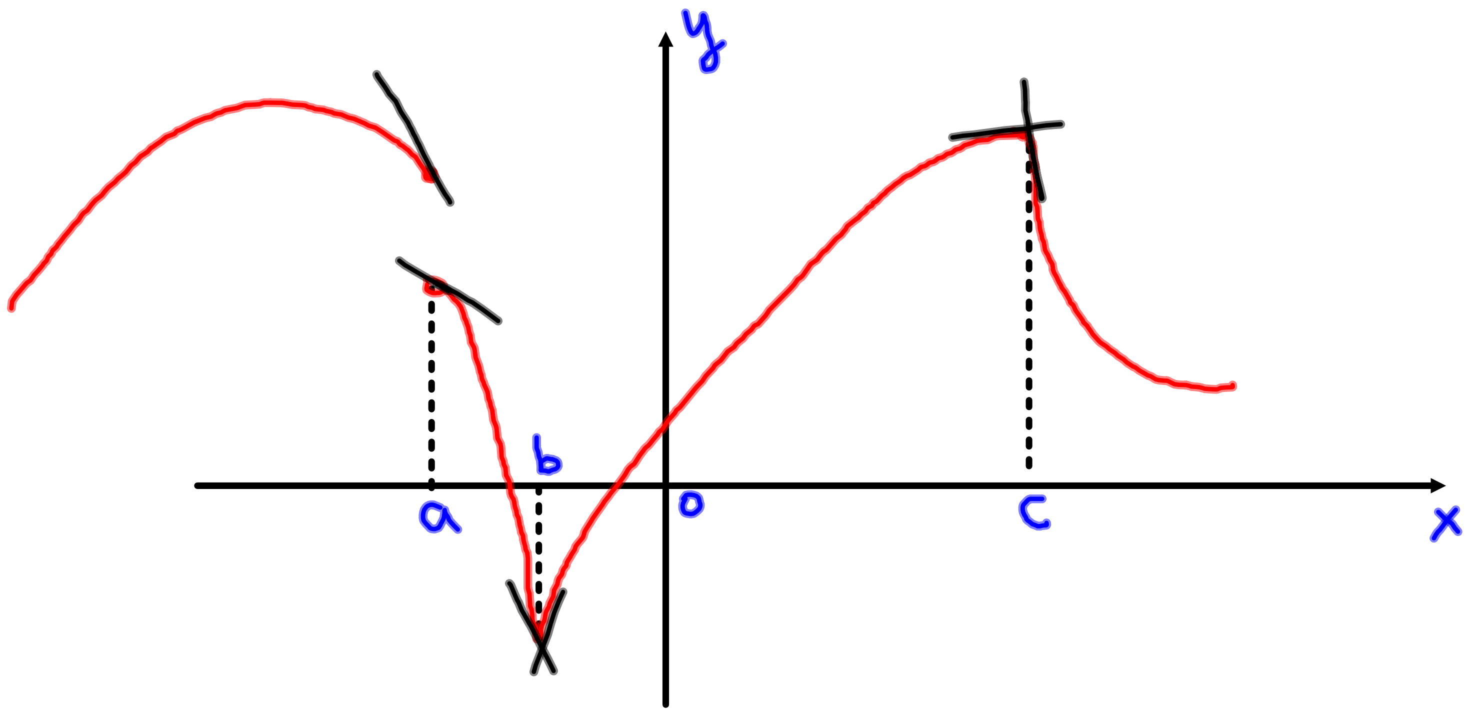
(b) *Find the equation of the tangent line of* $y = f(x)$ *at* $x = -3$.

$$y - y_1 = m(x - x_1)$$

$$f(-3) = \frac{3}{-3+5} = \frac{3}{2}$$

$$\boxed{y - \frac{3}{2} = -\frac{3}{4}(x+3)}$$

Question: Where does a derivative not exist for a function?



Answer: $f'(a)$, $f'(b)$, $f'(c)$ DNE

DEFINITION 3. *A function $f(x)$ is said to be* **differentiable** *at $x = a$ if $f'(a)$ exists.*

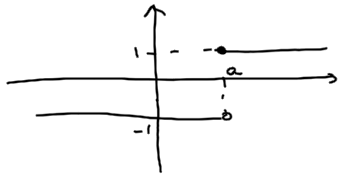
EXAMPLE 4. *Refer to the graph above to determine where $f(x)$ is not differentiable.*

$f(x)$ is not differentiable at $x = a, b, c$

CONCLUSION: A function $f(x)$ is NOT differentiable at $x = a$ if

- $f(x)$ is not continuous at $x = a$;

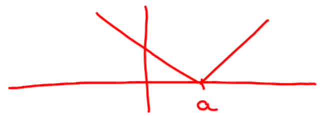
$$f(x) = \begin{cases} 1 & , x \geqslant a \\ -1 & , x < a \end{cases}$$



- $f(x)$ has a sharp turn at $x = a$ (left and right derivatives are not the same);

$$f(x) = |x-a|$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\overbrace{|a+h-a|}^{=h} - \overbrace{|a-a|}^{=0}}{h} = \lim_{h\to 0} \frac{|h|}{h} \text{ DNE}$$



- $f(x)$ has a vertical tangent at $x = a$.

$$f(x) = \sqrt[3]{x-a}$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{\cancel{a}+h-\cancel{a}} - \overbrace{\sqrt[3]{a-a}}^{=0}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} \text{ DNE}$$

$$\tan\alpha = \infty \Rightarrow \alpha = \frac{\pi}{2}$$

THEOREM 5. *If f is differentiable at a then f is continuous at a.*

$\boxed{\text{diff.}} \Rightarrow \boxed{\text{cont.}}$

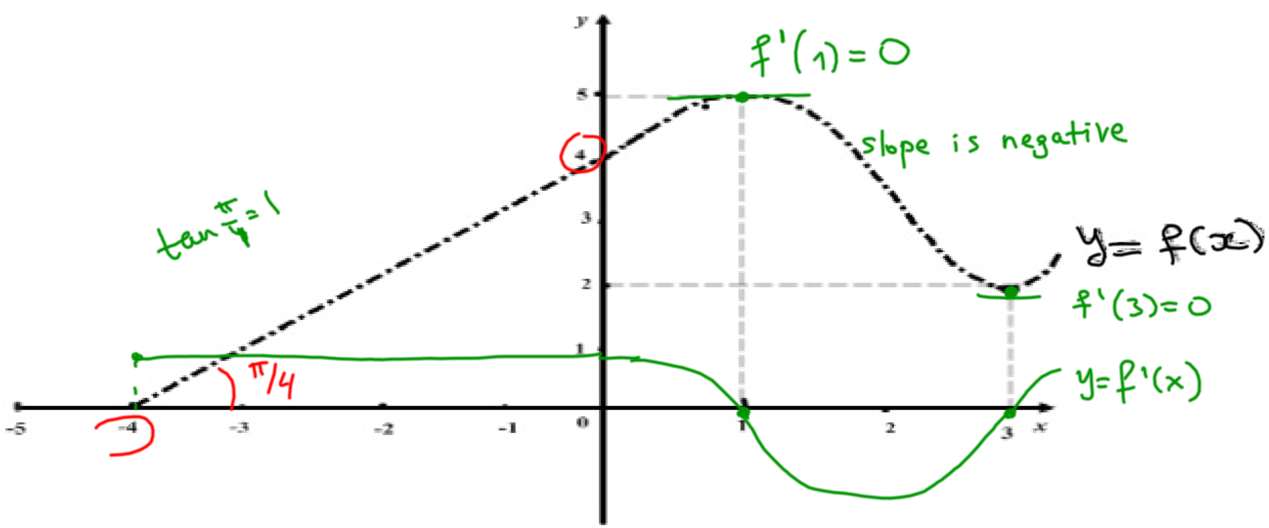
$\Leftarrow$???

The derivative as a function: If we replace a by x in Definition 1 we get:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

A new function $g(x) = f'(x)$ is called the **derivative of** f.

EXAMPLE 6. *Use the graph of $f(x)$ below to sketch the graph of the derivative $f'(x)$.*



EXAMPLE 7. *Use the definition of the derivative to find $f'(x)$ for $f(x) = \sqrt{1+3x}$.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{1+3(x+h)} - \sqrt{1+3x}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{1+3x+3h} - \sqrt{1+3x}\right)}{h} \frac{(\sqrt{\ } + \sqrt{\ })}{(\sqrt{\ } + \sqrt{\ })}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{1+3x+3h}\right)^2 - \overbrace{\left(\sqrt{1+3x}\right)^2}^{-(1+3x)}}{h\left(\sqrt{1+3x+3h} + \sqrt{1+3x}\right)} = \lim_{h\to 0} \frac{\cancel{1} + \cancel{3x} + 3\cancel{h} - \cancel{1} - \cancel{3x}}{\cancel{h}\left(\sqrt{1+3x+3h} + \sqrt{1+3x}\right)}$$

$$= \frac{3}{\sqrt{1+3x+3\cdot 0} + \sqrt{1+3x}} = \frac{3}{2\sqrt{1+3x}}$$

$$\left(\sqrt{1+3x}\right)' = \frac{3}{2\sqrt{1+3x}} \quad \left(x \neq -\frac{1}{3}\right)$$

Note that x should be in the domain of f.

$$f'\left(-\tfrac{1}{3}\right) = \lim_{h\to 0} \frac{f\left(-\frac{1}{3}+h\right) - f\left(-\frac{1}{3}\right)}{h} = \lim_{h\to 0} \frac{\sqrt{\overbrace{1+3\left(-\frac{1}{3}+h\right)}^{1-1+3h=3h}} - \sqrt{\overbrace{1+3\cdot\left(-\frac{1}{3}\right)}^{=0}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3h}}{h} = \lim_{h\to 0} \frac{\sqrt{3}}{\sqrt{h}} \quad \text{DNE}$$

EXAMPLE 8. *Each limit below represents the derivative of function $f(x)$ at $x = a$. State f and a in each case.*

(a) $\lim_{h\to 0} \frac{(3+h)^4 - 81}{h} = f'(3)$, where

$$\boxed{\begin{array}{l} f(x) = x^4 \\ a = 3 \end{array}}$$

$f(3) = 3^4 = 81$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

(b) $\lim_{x\to \frac{3\pi}{2}} \frac{\sin x + 1}{x - \frac{3\pi}{2}} = \lim_{x\to\frac{3\pi}{2}} \frac{\sin x - (-1)}{x - \frac{3\pi}{2}} = f'\left(\frac{3\pi}{2}\right)$,

where

$f(x) = \sin x$

$a = \frac{3\pi}{2}$

$f\left(\frac{3\pi}{2}\right) = -1$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{x\to a} \frac{f(a) - f(x)}{a - x}$$