Section 3.2: Differentiation formulas

The properties and formulas in this section will be given in both "prime" notation and "fraction" notation.

PROPERTIES:

1. *Constant rule*: If f is a constant function, $f(x) = c$, then $f'(x) = 0$, or $\frac{\mathrm{d}c}{\mathrm{d}x} = 0$.

   $(2015)' = 0$

2. *Power rule:* If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$, or $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$.

   $(x^{-2015})' = -2015\,x^{-2015-1} = -2015\,x^{-2016}$

3. *Constant multiple rule:* If c is a constant and $f'(x)$ exists then

$$(cf(x))' = cf'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x}.$$

$$\frac{d}{dx}(2015 \tan x) = 2015 \frac{d}{dx}(\tan x)$$

4. *Sum/Difference rule*: If $f'(x)$ and $g'(x)$ exists then

$$(f(x) \pm g(x))' = f'(x) \pm g'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(f \pm g) = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x}.$$

$$(3\cos x - 7e^x)' = 3(\cos x)' - 7(e^x)'$$

5. *Product rule:* If $f'(x)$ and $g'(x)$ exists then Note $(f(x)g(x))' \neq f'(x)g'(x)$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f f}{\mathrm{d}x}.$$

$(fg)' = f'g + fg'$

$(\cos x \cdot \sin x)' = (\cos x)'\sin x + \cos x\,(\sin x)'$

Note $\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$

6. *Quotient rule:* If $f'(x)$ and $g'(x)$ exists then $(g(x) \neq 0)$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{[g(x)]^2}.$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\left(\frac{x^2}{\cos x}\right)' = \frac{(x^2)'\cos x - x^2(\cos x)'}{\cos^2 x}$$

EXAMPLE 1. *Find the derivatives of the following functions:*

(a) $f'(x) = \left(x^{10} + 3x^5 - 12x + 44 - \pi^5\right)' \underset{=}{③+④}$

$= (x^{10})' + 3(x^5)' - 12(x)' + (44 - \pi^5)' \underset{=}{②+①}$

$= 10x^{10-1} + 3 \cdot 5x^{5-1} - 12 \cdot \cancel{x}^{1-1} + 0$

$= 10x^9 + 15x^4 - 12$

(b) $g(t) = (1 + \sqrt{t})^2 = 1 + 2\sqrt{t} + t = 1 + 2t^{\frac{1}{2}} + t$

$g'(t) = \left(1 + 2t^{\frac{1}{2}} + t\right)' = 1' + 2\left(t^{\frac{1}{2}}\right)' + t' = 0 + 2 \cdot \frac{1}{2} t^{\frac{1}{2}-1} + 1$

$= t^{-\frac{1}{2}} + 1 = \frac{1}{\sqrt{t}} + 1 \quad (t \neq 0)$

(c) $F(s) = \left(\frac{s}{3}\right)^4 - s^{-5} = \frac{s^4}{81} - s^{-5}$

$F'(s) = \frac{1}{81}(s^4)' - (s^{-5})' = \frac{1}{81} \cdot 4 \cdot s^{4-1} - (-5) \cdot s^{-5-1}$

$= \frac{4s^3}{81} + 5s^{-6}$

(d) $y = \dfrac{u^5+1}{u^2\sqrt{u}} = \dfrac{u^5+1}{u^2\, u^{\frac{1}{2}}} = \dfrac{u^5+1}{u^{\frac{5}{2}}} = \dfrac{u^5}{u^{5/2}} + \dfrac{1}{u^{5/2}} = u^{\frac{5}{2}} + u^{-\frac{5}{2}}$

$$y' = \left(u^{\frac{5}{2}} + u^{-\frac{5}{2}}\right)' = \left(u^{\frac{5}{2}}\right)' + \left(u^{-\frac{5}{2}}\right)' = \frac{5}{2}u^{\frac{5}{2}-1} + \left(-\frac{5}{2}\right)u^{-\frac{5}{2}-1}$$

$$= \frac{5}{2}u^{\frac{3}{2}} - \frac{5}{2}u^{-\frac{7}{2}}$$

$(fg)' = f'g + fg'$

(e) $f'(x) = \Big(\overbrace{(x^4 - 3x^2 + 11)}^{f}\overbrace{(3x^3 - 5x^2 + 22)}^{g}\Big)'$ $\underline{\underline{PR}}$

$$= (x^4-3x^2+11)'(3x^3-5x^2+22) + (x^4-3x^2+11)(3x^3-5x^2+22)'$$

$$= (4x^3-\underline{6x})(3x^3-5x^2+22) + (x^4-3x^2+11)(9x^2-\underline{10x})$$

$\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

(f) $\dfrac{d}{dz}\big(g(z)\big) = \left(\dfrac{4-z^2}{4+z^2}\right)' = \dfrac{(4-z^2)'(4+z^2) - (4-z^2)(4+z^2)'}{(4+z^2)^2}$

$$= \frac{-2z(4+z^2) - (4-z^2)\cdot 2z}{(4+z^2)^2}$$

$$= \frac{-8z - \cancel{2z^3} - 8z + \cancel{2z^3}}{(4+z^2)^2} = -\frac{16z}{(4+z^2)^2}$$

EXAMPLE 2. *The functions f and g satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	5	12
3	1	2	-2	8

Find the indicated quantity: (a) $h'(3)$ *if* $h(x) = (3x^2+1)g(x)$

$$h'(x) \overset{\text{P.R.}}{=} (3x^2+1)' g(x) + (3x^2+1) g'(x)$$

$$= 6x\, g(x) + (3x^2+1) g'(x)$$

$$h'(3) = 6 \cdot 3\, g(3) + (3 \cdot 3^2 + 1) g'(3)$$

$$= 18 \cdot (-2) + 28 \cdot 8 = -36 + 224 = 188$$

(b) $H'(1)$ *if* $H(x) = \dfrac{x^2}{f(x)}$

$$H'(x) \overset{\text{Q.R.}}{=} \frac{(x^2)' f(x) - x^2 f'(x)}{(f(x))^2} = \frac{2x f(x) - x^2 f'(x)}{(f(x))^2}$$

$$H'(1) = \frac{2 f(1) - f'(1)}{(f(1))^2} = \frac{2 \cdot (-5) - 8}{(-5)^2} = \frac{-10-8}{25} = -\frac{18}{25}$$

EXAMPLE 3. *Given* $f(x) = x^3 - 5x^2 + 6x - 3$

$f(1)$

(a) *Find the equation of the tangent line to the graph of* $f(x)$ *at the point* $(1, -1)$.

Equation of tangent line to the graph of f(x) at x=a :

$$\boxed{y - f(a) = f'(a)(x-a)}$$

$f'(x) = 3x^2 - 10x + 6$; $f'(1) = 3 - 10 + 6 = -1$

$$y - (-1) = -1(x-1)$$

$$y + 1 = -x + 1 \Rightarrow \boxed{y = -x}$$

(b) *Find the value(s) of* x *where* $f(x)$ *has a tangent line that is parallel to* $y = 6x + 1$.

In other words, find the value(s) of x where

$$\underbrace{f'(x)}_{\text{slope of tangent}} = \underbrace{6}_{\text{slope of the given line.}}$$

$$3x^2 - 10x + 6 = 6$$

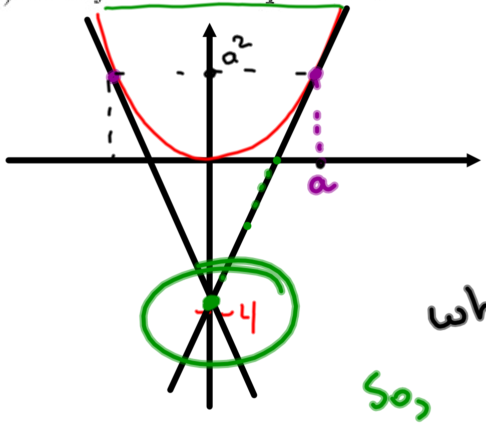
$$3x^2 - 10x = 0 \Rightarrow x(3x - 10) = 0$$

$x = 0$ or $3x = 10$

$x = \frac{10}{3}$

Answer: $x = 0 ; \frac{10}{3}$

EXAMPLE 4. *Show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$ and find their equations.*



Let $x = a$ be a tangent point to the graph of $f(x) = x^2$. $(f'(x) = 2x)$

Then equation of tangent line at $x = a$:

$$y - f(a) = f'(a)(x - a)$$

where $f'(a) = 2a$, $f(a) = a^2$

So,

$$y - a^2 = 2a(x - a)$$

But $(0, -4)$ belongs to this tangent line, i.e.

$$-4 - a^2 = 2a(0 - a)$$

$$-4 - a^2 = -2a^2$$

$$a^2 = 4 \Rightarrow a = \pm 2$$ means that there are two tangent points and then two tangent lines, namely,

If $a = 2$ then $y - 4 = 4(x - 2)$

If $a = -2$ then $y - 4 = -4(x + 2)$

EXAMPLE 5. *Let* $f(x) = \begin{cases} -1-2x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$ $\Rightarrow \boxed{f'(x) = \begin{cases} -2, & x < -1 \\ 2x, & -1 < x < 1 \\ 1, & x > 1 \end{cases}}$

$\begin{cases} ?, & \cancel{x = 1} \\ ?, & \cancel{x = -1} \end{cases}$

(a) *Give a formula for* f'.

$f'_-(-1) = (-1-2x)' = -2$

$f'_+(-1) = (x^2)'\Big|_{x=-1} = 2x\Big|_{x=-1} = -2$

$\Rightarrow f'(-1) = -2$

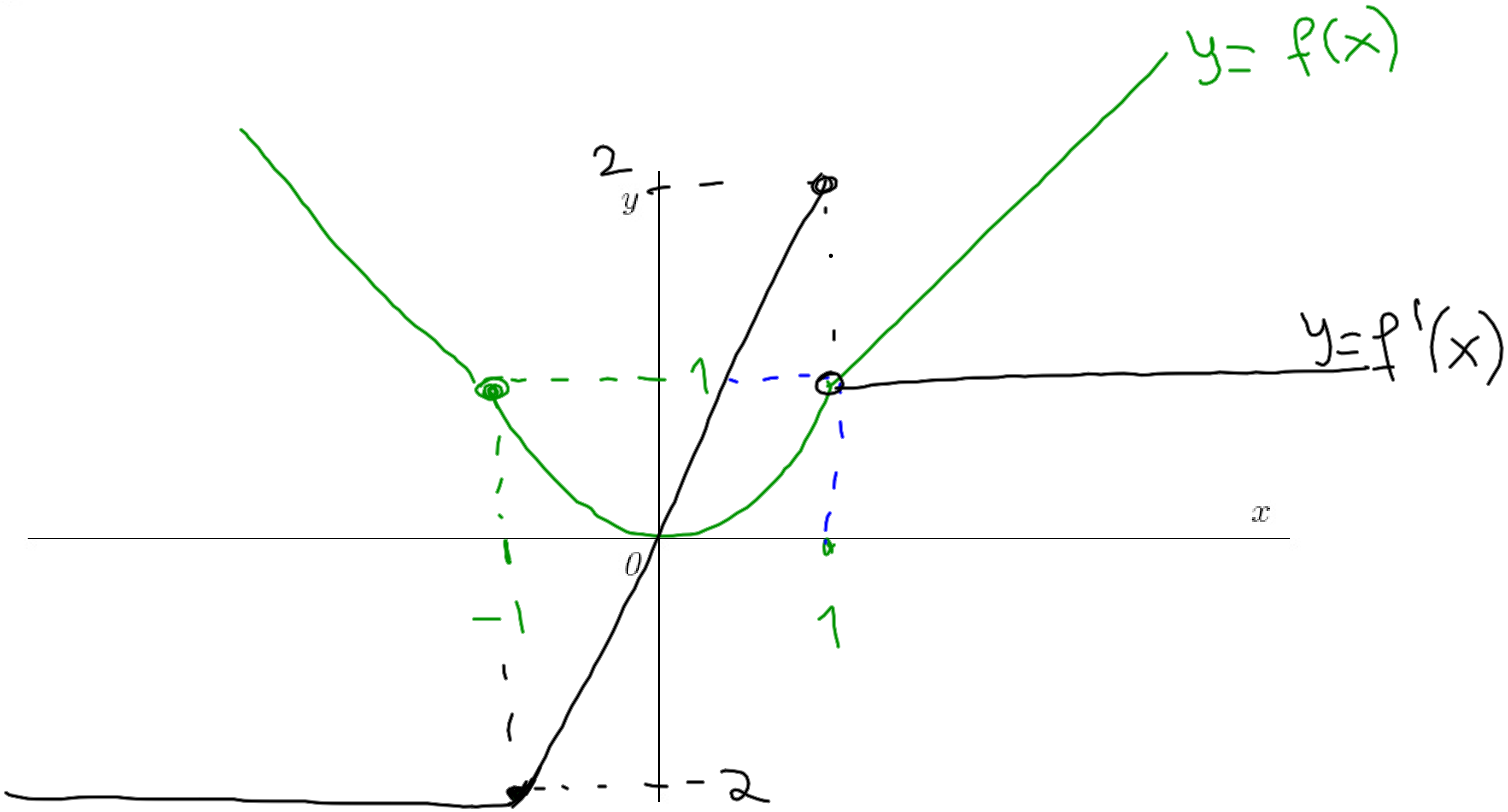
Also $f_-(-1) = -1-2\cdot(-1) = 1$, $f_+(-1) = 1$ (no jump discont.)

$f'_-(1) = (x^2)'\Big|_{x=1} = 2x\Big|_{x=1} = 2$

$f'_+(1) = 1$ So, $f'_-(1) \neq f'_+(1) \Rightarrow f'(1)$ DNE

(b) *For what value(s) of* x *the function is not differentiable?* $x = 1$

(c) *Sketch the graph of f and f' on the same axis.*



EXAMPLE 6. *A ball is thrown into the air. Its position at time t is given by*

$$\vec{s}(t) = \langle 2t, 10t - t^2 \rangle .$$

(a) *Find the velocity of the ball at time $t = 2$.*

$$\vec{v}(t) = \vec{s}\,'(t) = \langle (2t)', (10t - t^2)' \rangle = \langle 2, 10 - 2t \rangle$$

$$\vec{v}(2) = \langle 2, 10 - 2 \cdot 2 \rangle = \boxed{\langle 2, 6 \rangle} = 2\langle 1, 3 \rangle$$

(b) *Find the speed of the ball at time $t = 2$.*

$$\text{speed} = |\vec{v}(2)| = |2\langle 1, 3 \rangle| = 2\sqrt{1^2 + 3^2} = 2\sqrt{10}$$

EXAMPLE 7. *Let*

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 2 \\ ax+b & \text{if} \quad x > 2 \end{cases}$$

Find the values of a and b that makes f differentiable everywhere.

f must be continuous everywhere:

If $x \neq 2$ then continuity of f follows from continuity of $y = x^2$ and $y = ax+b$ (as polynomials).

At $x=2$ we need: $x^2 = ax+b$, or $\boxed{4 = 2a+b}$

To ensure differentiability at $x=2$, we need

$$f'_-(2) = f'_+(2)$$

$$(x^2)'\Big|_{x=2} = (ax+b)'\Big|_{x=2}$$

$$2x\Big|_{x=2} = a \Rightarrow \boxed{a=4}$$

$b = 4-2a = 4-2\cdot 4 = -4$

$\boxed{\text{Answer: } a=4, \ b=-4}$