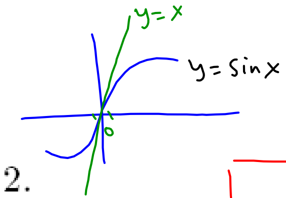
3.4 Derivatives of trigonometric functions.

It is important to remember that everything for six trigonometric functions ($\sin x, \cos x, \tan x, \cot x, \csc x, \sec x$ will be done in radians.

EXAMPLE 1. *Compute:*

(a) $\lim_{x\to 0}\sin x = \sin 0 = 0$ (b) $\lim_{x\to 0}\cos x = \cos 0 = 1$



THEOREM 2.

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1,} \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Proof

$$\lim_{x\to 0}\frac{(\cos x - 1)}{x}\frac{(\cos x + 1)}{(\cos x + 1)} = \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x\to 0} -\frac{\sin^2 x}{x(\cos x + 1)}$$

$$= -\lim_{x\to 0}\overset{\to 1}{\frac{\sin x}{x}}\cdot\overset{\to \frac{0}{2}}{\frac{\sin x}{\cos x + 1}} = -\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{\cos x + 1}\right)$$

$$= -1\cdot\frac{0}{1+1} = 0.$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \Rightarrow \lim_{x\to 0}\frac{x}{\sin x}=1$$

EXAMPLE 3. *Find these limits:*

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x} = \lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot 5 \underset{\substack{u=5x\to 0\\ x\to 0}}{=} 5\underbrace{\lim_{u\to 0}\frac{\sin u}{u}} = 5\cdot 1 = 5$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(9x)}{\sin(7x)} = \lim_{x\to 0}\underbrace{\frac{\sin(9x)}{9x}}_{1}\cdot\underbrace{\frac{7x}{\sin(7x)}}_{1}\cdot\frac{9}{7} = \frac{9}{7}$

(c) $\displaystyle\lim_{x\to 0}\frac{x}{\sin(4x)} = \underbrace{\lim_{x\to 0}\frac{4x}{\sin(4x)}}_{1}\cdot\frac{1}{4} = \frac{1}{4}$

Conclusion: If $a, b \neq 0$ then

$$\lim_{x\to 0}\frac{\sin(ax)}{x} = a, \qquad \lim_{x\to 0}\frac{x}{\sin(ax)} = \frac{1}{a}, \qquad \lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$$

(d) $\displaystyle\lim_{x\to 0}\frac{1}{x^2\cot^2(3x)} = \lim_{x\to 0}\frac{1}{x^2\,\frac{\cos^2(3x)}{\sin^2(3x)}} = \lim_{x\to 0}\frac{\sin^2(3x)}{x^2\cos^2(3x)}$

$$= \lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right)^2\cdot\frac{1}{\cos^2(3x)} = \left(\lim_{x\to 0}\frac{\sin(3x)}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{\cos^2(3x)}$$

$$= 3^2\cdot\frac{1}{1} = 9$$

(e) $\displaystyle\lim_{x\to 0}\frac{(\cos x - 1)\,x}{(\sin x\quad)\,x} = \lim_{x\to 0}\left(\frac{\cos x-1}{x}\cdot\frac{x}{\sin x}\right) = \lim_{x\to 0}\frac{\cos x-1}{x}\cdot\lim_{x\to 0}\frac{x}{\sin x}$

$$= 0\cdot 1 = 0.$$

EXAMPLE 4. *Find the following derivatives:*

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim\limits_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim\limits_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$

$$= \lim_{h\to 0}\left(\frac{\sin x\,(\cos h - 1)}{h} + \frac{\sin h}{h}\cos x\right)$$

$$= \sin x \underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{=0} + \cos x \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{=1} = \cos x$$

$$\boxed{(\sin x)' = \cos x}$$

Remark Similarly one can get $\boxed{(\cos x)' = -\sin x.}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \left(\frac{\sin x}{\cos x}\right)' \underset{=}{\overset{Q.R.}{=}} \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Derivatives of Trig Functions (memorize these!)

$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$	$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$	$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$	$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$

EXAMPLE 5. *Find the derivative of these functions.*

(a) $y' = (\cot x)' + 5(\sec x)' + (\underbrace{x\sqrt{x}}_{x^{3/2}})' = -\csc^2(x) + 5\sec x\tan x + \frac{3}{2}\sqrt{x}$

$$\left(x^{3/2}\right)' = \frac{3}{2}\underbrace{x^{\frac{3}{2}-1}}_{x^{\frac{1}{2}}} = \frac{3}{2}\sqrt{x}$$

$$(x^n)' = n x^{n-1}$$

(b) $f'(x) = \left(\dfrac{\cos x}{1+\sin x}\right)' = \dfrac{(\cos x)'(1+\sin x) - \cos x(1+\sin x)'}{(1+\sin x)^2}$

$$= \frac{-\sin x(1+\sin x) - \cos x\cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(\sin x + 1)}{(1+\sin x)^2}$$

$$= -\frac{1}{1+\sin x}$$

EXAMPLE 6. *Find the equation of the tangent line to the graph of function* $y = x^2 \sin x$ *at* $x = \frac{\pi}{4}$.

$f(x) = x^2 \sin x$

$f\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \sin\frac{\pi}{4} = \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2} = \frac{\pi^2\sqrt{2}}{32}$

$f'(x) = (x^2 \sin x)' \overset{\text{P.R.}}{=} (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x$

$f'\left(\frac{\pi}{4}\right) = 2\frac{\pi}{4} \sin\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 \cos\frac{\pi}{4} = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\pi\sqrt{2}}{4} + \frac{\pi^2\sqrt{2}}{32}$

$y - f\left(\frac{\pi}{4}\right) = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$

$y - \frac{\pi^2\sqrt{2}}{32} = \left(\frac{\pi\sqrt{2}}{4} + \frac{\pi^2\sqrt{2}}{32}\right)\left(x - \frac{\pi}{4}\right)$