Section 3.5: Chain Rule

Question: How to find the derivatives of the following functions:

$$y = (x^6 + 4x^2 + 12)^{15}; \qquad y = \sec(12x^2) + \tan^3(x) \qquad y = \sqrt[3]{4+x}$$

Review of Composite Functions:

$$[f \circ g](x) = f(g(x))$$

If $f(x) = x^{15}$ and $g(x) = x^6 + 4x^2 + 12$ then

$$[f \circ g](x) = f(g(x)) = f(x^6 + 4x^2 + 12)$$
$$= (x^6 + 4x^2 + 12)^{15}$$

Conversely, if $\underbrace{[f \circ g](x) = \sec(12x^2)}_{f(g(x))}$ then $f(x) = \sec x$ and $g(x) = 12x^2$

The CHAIN RULE: *If the derivatives $g'(x)$ and $f'(x)$ both exist, and $F = f \circ g$ is the composite defined by*

$$F(x) = f(g(x))$$

then

$$F'(x) = f'(g(x))g'(x).$$

$f' \cdot g'$

In Leibniz notation: If the derivatives of $y = f(u)$ and $u = g(x)$ both exist then

$$y = f(g(x))$$

is differentiable function of x and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}. \quad = y' \cdot u'$$

$y = \underbrace{f(\underbrace{u(x)}_{\text{inner function}})}^{\text{outer function}}$ $f' \cdot u' = y' \cdot u'$

$y = f(x)$	$u(x)$	$f(u)$	$\frac{dy}{dx}$
$y = (\underline{x^6 + 4x^2 + 12})^{15}$	$u = x^6 + 4x^2 + 12$ $u' = 6x^5 + 8x$	$y = u^{15}$ $y' = 15u^{14}$	$\frac{dy}{dx} = 15u^{14} \cdot (6x^5 + 8x)$ $= 15(x^6 + 4x^2 + 12)^{14}(6x^5 + 8x)$
$y = \sec(12x^2)$	$u = 12x^2$ $u' = 24x$	$y = \sec u$ $y' = \sec u \tan u$	$\frac{dy}{dx} = (\sec u \tan u)(24x)$ $= 24x \sec(12x^2)(\tan(12x^2))$
$y = \tan^3(x)$ $= (\tan x)^3$	$u = \tan x$ $u' = \sec^2 x$	$y = u^3$ $y' = 3u^2$	$\frac{dy}{dx} = 3u^2 \sec^2 x$ $= 3 \tan^2 x \sec^2 x$
$y = \sqrt[3]{4+x}$	$u = 4 + x$ $u' = 1$	$y = \sqrt[3]{u} = u^{\frac{1}{3}}$ $y' = \frac{1}{3}u^{\frac{1}{3}-1} = \frac{1}{3}u^{-\frac{2}{3}} =$ $= \frac{1}{3\sqrt[3]{u^2}}$	$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{u^2}} \cdot 1$ $= \frac{1}{3\sqrt[3]{(4+x)^2}}$
$y = [g(x)]^n$	$u = g(x)$ $u' = g'(x)$	$y = u^n$ $y' = nu^{n-1}$	$\frac{dy}{dx} = nu^{n-1} g'(x)$ $\boxed{\frac{d}{dx}[g(x)]^n = n(g(x))^{n-1} g'(x)}$ Generalized Power Rule

EXAMPLE 1. *Find the derivative:*

(a) $f(x) = \dfrac{1}{(x^3+5x^2+12)^{2012}} = \underline{(x^3+5x^2+12)}^{-2012}$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

G.P.R.

$f'(x) = -2012\,(x^3+5x^2+12)^{-2012-1} \cdot (x^3+5x^2+12)'$

$= -\dfrac{2012\,(3x^2+10x)}{(x^3+5x^2+12)^{2013}}$

Note : $u = x^3+5x^2+12$
$f(u) = u^{-2012}$

(b) $h(x) = x^8(3\sqrt{x}-11)^8 = \left(x(3\sqrt{x}-11)\right)^8$

$= (3x\sqrt{x}-11x)^8 = (\underline{3x^{\frac{3}{2}}-11x})^8$

By G.P.R.

$h'(x) = 8\left(3x^{\frac{3}{2}}-11x\right)^{8-1} \cdot \left(3x^{\frac{3}{2}}-11x\right)'$

$= 8\left(3x^{\frac{3}{2}}-11x\right)^7\left(3\cdot\frac{3}{2}x^{\frac{3}{2}-1}-11\right)$

$= 8\left(3x\sqrt{x}-11x\right)^7\left(\frac{9}{2}\sqrt{x}-11\right)$

(c) $f(x) = \cos(5x) + \cos^5 x = \overset{\text{outer}}{\cos}(\underbrace{5x}_{\text{inner function}}) + (\underbrace{\cos x}_{\text{inner}})^5$

$= -\sin(5x)\,(5x)' + 5\cos^4 x \cdot (\cos x)'$

$= -5\sin(5x) - 5\cos^4 x\,\sin x$

$\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$

$\boxed{(\sqrt{u(x)})' = \frac{1}{2\sqrt{u}}\cdot u'}$

(d) $f(x) = \sqrt{\underbrace{x^3+\sqrt{x^2+\sqrt{x}}}_{u(x)}}$

$f'(x) = \dfrac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}} \cdot \left(x^3+\sqrt{\underbrace{x^2+\sqrt{x}}_{u(x)}}\right)'$

$= \dfrac{3x^2 + \frac{1}{2\sqrt{x^2+\sqrt{x}}}\cdot(x^2+\sqrt{x})'}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}$

$= \dfrac{3x^2 + \dfrac{2x+\frac{1}{2\sqrt{x}}}{2\sqrt{x^2+\sqrt{x}}}}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}$

EXAMPLE 2. *Find F' and G' if*

$$F(x) = f(\underbrace{\sin x}_{u(x)}), \qquad G(x) = \sin(\underbrace{f(x)}_{u(x)}),$$

where $f(x)$ is a differentiable function.

$$F'(x) = f'(u) \cdot u' = f'(\sin x)(\sin x)' = f'(\sin x)\cos x$$

$$G'(x) = (\sin u)' \cdot f'(x) = \cos(f(x))\, f'(x)$$

EXAMPLE 3. *Let $f(x)$ and $g(x)$ be given differentiable functions satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	3	12
3	1	2	-2	8

Suppose that $h = f \circ g$. Find $h'(1)$.

$h(x) = f(g(x))$. By Chain Rule,

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$h'(1) = f'(\underbrace{g(1)}_{3}) \underbrace{g'(1)}_{12} = f'(3) \cdot 12 = 2 \cdot 12 = \boxed{24}$$