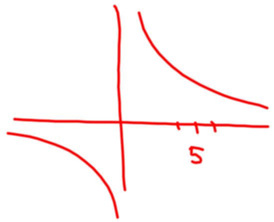
3.6: Implicit differentiation

EXAMPLE 1. *Find y' if the $y = y(x)$ satisfies the equation $xy = 5$ for all values of x in its domain and evaluate $y'(5)$.*

represents curve



Solution 1 (by explicit differentiation):

$$xy = 5 \Rightarrow y = \frac{5}{x} = 5x^{-1}$$

$$y' = 5\cdot(-1)\cdot x^{-1-1} = -\frac{5}{x^2}$$

$$y'(5) = -\frac{5}{5^2} = \boxed{-\frac{1}{5}}$$

Solution 2 (by implicit differentiation):

$$x\,y(x) = 5$$

Using Chain Rule (and Product Rule) differentiate both sides of the equation w.r.t. x:

$$x'\,y(x) + x\,y'(x) = 5'$$

$$y(x) + x\,y'(x) = 0$$

$$x\,y'(x) = -y(x)$$

$$y'(x) = -\frac{y(x)}{x}$$

$x = 5$

$y(x) = \frac{5}{x} \Rightarrow y(5) = 1$

$$y'(5) = -\frac{1}{5}$$

EXAMPLE 2. (a) *If* $x^2 + y^2 = 16$ *find* $\frac{dy}{dx} = y'$ $y = y(x)$

$$\frac{d}{dx}\left[x^2 + [y(x)]^2\right] = \frac{d}{dx}[16]$$

$$\cancel{2}x + \cancel{2}y \cdot y' = 0$$

$$y y' = -x$$

$$\boxed{y' = -\frac{x}{y}}$$

(b) *Find the equation of the tangent line to* $x^2 + y^2 = 16$ *at the point* $(2, 2\sqrt{3})$.

$$y - 2\sqrt{3} = y'(2)(x-2)$$

$$y'(2) = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\boxed{y - 2\sqrt{3} = -\frac{1}{\sqrt{3}}(x-2)}$$

EXAMPLE 3. *Find* $\dfrac{dy}{dx}$ *for the following:*

(a) $4x^3 + 2y^2 = 4xy^5 + y$

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$$\frac{d}{dx}\left[4x^3 + 2[y(x)]^2\right] = \frac{d}{dx}\left[4x[y(x)]^5 + y(x)\right]$$

$$4 \cdot 3x^2 + 2 \cdot 2yy' = 4y^5 + 4x \cdot 5y^4 \cdot y' + y'$$

$$4yy' - 20xy^4y' - y' = 4y^5 - 12x^2$$

$$y'(4y - 20xy^4 - 1) = 4(y^5 - 3x^2)$$

$$\boxed{y' = \frac{4(y^5 - 3x^2)}{4y - 20xy^4 - 1}}$$

(b) $x^3 - \cot(xy^2) = x\cos y$

$$\frac{d}{dx}\left(x^3 - \cot\left(\overbrace{x[y(x)]^2}\right)\right) = \frac{d}{dx}\left[x \cos y(x)\right]$$

$$3x^2 - \left(-\csc^2(xy^2)\right)\cdot\frac{d}{dx}\left(x[y(x)]^2\right) = \cos y + x(-\sin y)y'$$

$$3x^2 + \csc^2(xy^2)\left[y^2 + 2xyy'\right] = \cos y - x\sin y\ y'$$

$$3x^2 + y^2\csc^2(xy^2) + \underline{2xy\csc^2(xy^2)y'} = \cos y - \underline{x\sin y\cdot y'}$$

$$2xy\csc^2(xy^2)y' + x\sin y\ y' = \cos y - 3x^2 - y^2\csc^2(xy^2)$$

$$xy'\left(2y\csc^2(xy^2) + \sin y\right) = \cos y - 3x^2 - y^2\csc^2(xy^2)$$

$$y' = \frac{\cos y - 3x^2 - y^2\csc^2(xy^2)}{x(2y\csc^2(xy^2 + \sin y)}$$

(c) $(x^2+y^2)^5 = x^2y^3$ Find $\frac{dx}{dy}$ $x = x(y)$

$$\frac{d}{dy}\left(\left[x(y)\right]^2 + y^2\right)^5 = \frac{d}{dy}\left[\underbrace{\left[x(y)\right]^2}\, y^3\right]$$

$$5(x^2+y^2)^4\left(2xx' + 2y\right) = \underbrace{2xx'}y^3 + x^2\,3y^2$$

$$\underbrace{10(x^2+y^2)^4 xx'} + 10y(x^2+y^2)^4 = \underbrace{2xx'y^3} + 3x^2y^2$$

$$10(x^2+y^2)^4 xx' - 2xx'y^3 = 3x^2y^2 - 10y(x^2+y^2)^4$$

$$2xx'\left(5(x^2+y^2)^4 - y^3\right) = 3x^2y^2 - 10y(x^2+y^2)^4$$

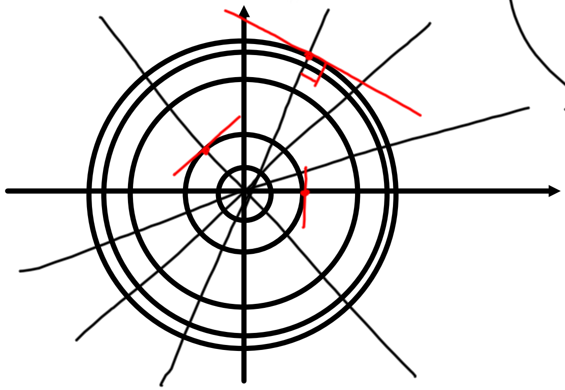
$$\frac{dx}{dy} = x' = \frac{3x^2y^2 - 10y(x^2+y^2)^4}{2x\left(5(x^2+y^2)^4 - y^3\right)}$$

DEFINITION 4. *Two curves are said to be* **orthogonal** *if at the point(s) of their intersection, their tangent lines are orthogonal(perpendicular). In this case we also say that the angle between these curves is* $\frac{\pi}{2}$.

Illustration: Consider two families of curves:

$$\underbrace{x^2 + y^2 = r^2}_{m_2 = y'}, \quad \underbrace{y = kx}_{\text{slope } m_1 = k},$$

where r and k are real parameters.



$2x + 2yy' = 0$

$y' = -\frac{x}{y}$

$m_2 = -\frac{x}{y}$ $\qquad$ $m_1 = k = \frac{y}{x}$

$m_1 \cdot m_2 = \frac{y}{x} \cdot \left(-\frac{x}{y}\right) = -1$

The curves from this families are orthogonal at intersection points.