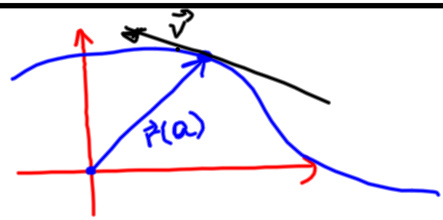
$$\vec{r}(t) = \langle x(t), y(t) \rangle = x(t)\vec{i} + y(t)\vec{j}$$



3.7: Derivatives of the vector functions

DEFINITION 1. *If* $\mathbf{r}(t)$ *is a vector function then the* **tangent vector v** *at* $t = a$ *is found by*

$$\mathbf{v} = \lim_{t \to a} \frac{1}{t-a} [\mathbf{r}(t) - \mathbf{r}(a)] = \vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$$

DEFINITION 2. *The* **derivative of a vector function** $\mathbf{r}(t)$ *at a number* a, *denoted by* $\mathbf{r}'(t)$, *is*

$$\mathbf{r}'(t) = \lim_{t \to a} \frac{1}{t-a} [\mathbf{r}(t) - \mathbf{r}(a)] \,,$$

if this limit exists.

One can show that this definition implies that

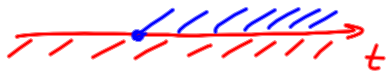
If $(r)(t) = \langle x(t), y(t) \rangle$ *is a vector function, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

if both $x'(t), y'(t)$ *exist.*

EXAMPLE 3. *If* $\mathbf{r}(\mathbf{t}) = \langle t^2, \sqrt{t-5} \rangle$ *find the domain of* $\mathbf{r}(t)$ *and* $\mathbf{r}'(t)$.

$D(t^2) = (-\infty, \infty)$



$D(\sqrt{t-5}) = [5, \infty)$

$D(\vec{r}(t)) = (-\infty, \infty) \cap [5, \infty) = \boxed{[5, \infty)}$

$\vec{r}\,'(t) = \langle (t^2)', (\sqrt{t-5})' \rangle = \langle 2t, \frac{1}{2\sqrt{t-5}} \rangle$

$D(2t) = (-\infty, \infty)$

$D\left(\frac{1}{2\sqrt{t-5}}\right) = (5, \infty)$

$\Rightarrow D(\vec{r}\,'(t)) = \boxed{(5, \infty)}$

EXAMPLE 4. *Given curve* $\mathbf{r}(t) = \langle 2t, 10t - t^2 \rangle$.

(a) *Find a vector tangent to the curve at the point* $(4, 16)$.

Tangent vector $\vec{v}(t) = \vec{r}\,'(t) = \langle 2, 10 - 2t \rangle$

Note that $\vec{r}(2) = \langle 4, 16 \rangle$, so $t = 2$

$$\vec{v}(2) = \langle 2, 10 - 2 \cdot 2 \rangle = \langle 2, 6 \rangle$$

(b) *Find parametric equations of the tangent line to* $\mathbf{r}(t)$ *at* $t = 2$.

$$x = 4 + 2t$$
$$y = 16 + 6t$$

(c) *Find a Cartesian equation of this tangent line.*

Way 1 Use part (b) eliminating parameter t:

$$x = 4 + 2t \Rightarrow 2t = x - 4$$
$$y = 16 + 6t \Rightarrow 6t = y - 16$$
$$\Rightarrow y - 16 = 3(x - 4)$$

Way 2 $y - 16 = m(x - 4)$, where $m = \frac{6}{2} = 3$

DEFINITION 5. *If* $\vec{r}(t) = \langle x(t), y(t) \rangle$ *is a vector function representing the position of a particle at time* t*, then*

- **instantaneous velocity** *at time* t *is* $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$
- **instantaneous speed** *at time* t *is* $|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

EXAMPLE 6. *The vector function* $\mathbf{r}(t) = \left\langle t, \sqrt{t^2+9} \right\rangle$ *represents the position of a particle at time* t*. Find the velocity and speed of the particle at time* $t = 4$*.*

velocity $= \vec{r}\,'(4)$

speed $= |\vec{r}\,'(4)|$

$$\vec{r}\,'(t) = \left\langle 1, \frac{1}{2\sqrt{t^2+9}} \cdot 2t \right\rangle = \left\langle 1, \frac{t}{\sqrt{t^2+9}} \right\rangle$$

$$\vec{r}\,'(4) = \left\langle 1, \frac{4}{\sqrt{4^2+9}} \right\rangle = \left\langle 1, \frac{4}{5} \right\rangle \text{ velocity}$$

$$|\vec{r}\,'(4)| = \left| \left\langle 1, \frac{4}{5} \right\rangle \right| = \sqrt{1^2 + \left(\frac{4}{5}\right)^2} = \sqrt{1 + \frac{16}{25}} = \frac{\sqrt{41}}{5} \text{ speed.}$$