3.8: Higher Derivatives

The derivative of a differentiable function f is also a function and it may have a derivative of its own:

$$(f')' = f'' \qquad \textbf{second derivative}$$

$$f'' = f^{(2)} = \frac{d^2 f}{dx^2} = f''(x)$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f'(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$$

`Alternative Notation:` If $y = f(x)$ then

$$y'' = f''(x) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D^2 f(x) = D\left(Df(x)\right)$$

Similarly, the **third derivative** $f''' = (f'')'$ or

$$y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = D^3 f(x).$$

In general, the n^{th} derivative of $y = f(x)$ is denoted by $f^{(n)}(x)$:

$$y^{(n)} = f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right) = D^n f(x).$$

$$= D\left(D^{n-1}y\right) = D^{n-1}\left(Dy\right) = \frac{d^n y}{dx^n}$$

EXAMPLE 1. *If* $y = \overbrace{x^5 + 3x + 1}^{f(x)}$ *find* $f^{(n)}(x)$ $n = 1, 2, 3, \ldots$

$$f'(x) = 5x^4 + 3$$

$$f''(x) = (5x^4+3)' = 20x^3$$

$$f'''(x) = (20x^3)' = 60x^2$$

$$f^{(4)}(x) = (60x^2)' = 120x$$

$$f^{(5)}(x) = (120x)' = 120$$

$$f^{(6)}(x) = 0$$

$f^{(k)}(x) = 0$ foll all $k \geq 6$ $(6 = \deg f(x) + 1)$

CONCLUSION: If $p(x)$ *is a polynomial of degree* n *then,* $p^{(k)}(x) = 0$ *for* $k \geq n+1$.

EXAMPLE 2. *Find the second derivative of* $f(x) = \tan(x^3)$.

$$f'(x) = \frac{d}{dx}\left(\tan(x^3)\right) = \sec^2(x^3)\ (3x^2) = 3x^2 \sec^2(x^3)$$

$$f''(x) = 3\left(x^2 \sec^2(x^3)\right)'$$

$$= 3\left(2x \sec^2(x^3) + x^2 \cdot 2 \sec(x^3) \sec(x^3) \tan(x^3) \cdot 3x^2\right)$$

$$\sec^2(x^3) = \left(\sec(x^3)\right)^2$$

EXAMPLE 3. *Find* $D^{2016}\sin x$.

$$D\sin x = (\sin x)' = \cos x$$

$$D^2\sin x = D(D\sin x) = D(\cos x) = -\sin x$$

$$D^3\sin x = D(D^2\sin x) = D(-\sin x) = -\cos x$$

$$D^4\sin x = D(D^3\sin x) = D(-\cos x) = \sin x$$

Cycle of length 4

$$D^5\sin x = D(D^4\sin x) = D(\sin x)$$

$$D^8\sin x = D^4(D^4\sin x) = D^4(\sin x) = \sin x$$

$$D^{4k}\sin x = \sin x$$

Since $2016 = 4\cdot 504$ we have $D^{2016}\sin x = \boxed{\sin x}$

Remark

$$D^{4k}\sin x = \sin x$$

$$D^{4k+1}\sin x = D\left(D^{4k}\sin x\right) = D(\sin x) = \cos x$$

$$D^{4k+2}\sin x = D\left(D^{4k+1}\sin x\right) = D(\cos x) = -\sin x$$

$$D^{4k+3}\sin x = D\left(D^{4k+2}\sin x\right) = D(-\sin x) = -\cos x$$

EXAMPLE 4. *If* $f(x) = \dfrac{1}{x}$ *find a general formula for its* n^{th} *derivative.*

$$f^{(0)}(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = -x^{-2} = -1 \cdot x^{-2}$$

$$f''(x) = -(-2)x^{-3} = 2x^{-3}$$

$$f'''(x) = -2 \cdot 3\, x^{-4}$$

$$f^{(4)}(x) = -2 \cdot 3 \cdot (-4)x^{-5} = 2 \cdot 3 \cdot 4\, x^{-5}$$

$$f^{(n)}(x) = (-1)^n \cdot \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots n}_{n!}\, x^{-(n+1)}$$

$$(0! = 1)$$

$$\boxed{f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}}$$

rectilinear motion

Acceleration: If $s(t)$ is the position of an object then the acceleration of the object is the first derivative of the velocity (consequently, the acceleration is the second derivative of the position function.)

$s'(t) = v(t)$

$$a(t) = v'(t) = s''(t).$$

EXAMPLE 5. *If $s(t) = t^3 - \frac{9}{2}t^2 - 30t + 12$ is the position of a moving object at time t (where $s(t)$ is measured in feet and t is measured in seconds) find the acceleration at the times when the velocity is zero.*

First find time t such that $v(t) = 0$.

$$v(t) = s'(t) = \left(t^3 - \frac{9}{2}t^2 - 30t + 12\right)' = 3t^2 - 9t - 30$$

$$= 3(t^2 - 3t - 10) = 3(t-5)(t+2) = 0$$

$\boxed{t=5}$ or $\underbrace{t=-2}_{\text{negative time? No.}}$

It remains to find $a(5)$.

$$a(t) = v'(t) = 3(t^2 - 3t - 10)' = 3(2t - 3)$$

$$a(5) = 3(2 \cdot 5 - 3) = \boxed{21 \text{ ft}/s^2}$$

circular motion

EXAMPLE 6. *Sketch the curve traced by* $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ *and plot the position, tangent and acceleration vectors at* $t = \frac{\pi}{4}$.

$\vec{r}(t) = \langle \cos t, \sin t \rangle$ position

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$ tangent vector

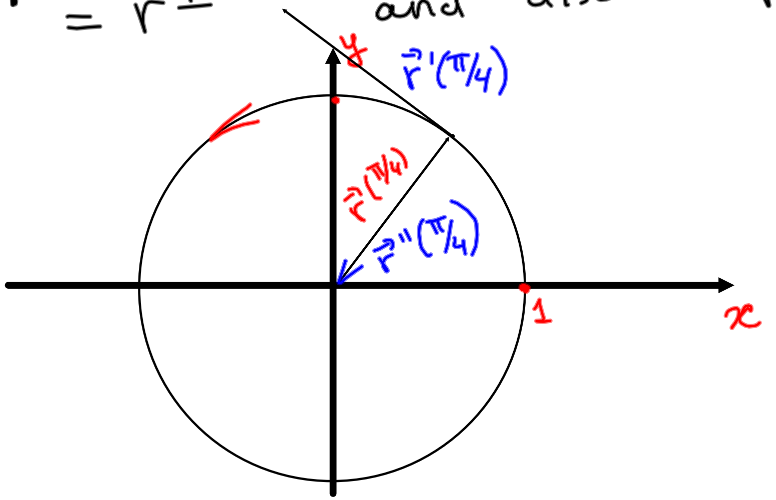
$\vec{r}\,''(t) = \langle -\cos t, -\sin t \rangle$ acceleration

$\vec{r}(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\vec{r}\,'(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\vec{r}\,''(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$

Note $\vec{r}\,' = \vec{r}^{\perp}$ and also $\vec{r}\,'' = -\vec{r}$



$\vec{a}(\frac{\pi}{4}) = -\vec{r}(\frac{\pi}{4})$ points radially inwards (centripetal) and $\vec{a}(\frac{\pi}{4})$ is perpendicular to velocity $\vec{v}(\frac{\pi}{4}) = \vec{r}\,'(\frac{\pi}{4})$.