3.9: Slopes and tangents of parametric curves

Consider a curve C given by the parametric equations

$$x = x(t), \quad y = y(t),$$

or in vector form:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle .$$

If both $x(t)$ and $y(t)$ are differentiable, then

tangent vector $$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

is a vector that is tangent to C. Its slope is:

$$\boxed{\text{slope} = \frac{y'(t)}{x'(t)}}$$

Another way to see this is by using the Chain Rule. We have $y = y(x(t))$ and then $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ which implies $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

EXAMPLE 1. *Find the equation of tangent line to the curve*

$$x(t) = \sin t, \quad y(t) = \tan t$$

at the point corresponding to $t = \frac{\pi}{4}$. is $\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(\sin\frac{\pi}{4}, \tan\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, 1\right)$

slope of the tangent $m = \dfrac{y'(t)}{x'(t)} = \dfrac{\sec^2(t)}{\cos(t)}$

at $t = \frac{\pi}{4}$ $\quad m = \dfrac{\sec^2\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \dfrac{2}{1/\sqrt{2}} = 2\sqrt{2}$

Equation of the tangent line: at $t = t_0$

$$y - y(t_0) = \overset{\text{slope}}{\frac{y'(t_0)}{x'(t_0)}}\left(x - x(t_0)\right)$$

In our case: $\boxed{y - 1 = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right)}$

EXAMPLE 2. *Find the equation of tangent line to the curve*

$$x(t) = t + 1, \quad y(t) = t^2 + 4$$

at the $(2, 5)$. Find t such that $(x(t), y(t)) = (2,5)$, or

$$\left.\begin{array}{l} x(t) = t+1 = 2 \Rightarrow t = 1 \\ y(t) = t^2+4 = 5 \Rightarrow t = \pm 1 \end{array}\right\} \Rightarrow \boxed{t=1}$$

Find slope of tangent:

$$m = \frac{y'(1)}{x'(1)} = \left.\frac{2t}{1}\right|_{t=1} = 2$$

Equation of tangent line: $\boxed{y - 5 = 2(x-2)}$

EXAMPLE 3. *Find the points on the curve*

$$x = t + t^2, \quad y = t^2 - t$$

where the tangent lines are horizontal and there they are vertical.

horizontal means $m = 0$ | vertical means m DNE

$$\boxed{m = \frac{y'(t)}{x'(t)}}$$

$$\boxed{\begin{aligned} x'(t) &= 1 + 2t \\ y'(t) &= 2t - 1 \end{aligned}}$$

$y'(t) = 0$, but $x'(t) \neq 0$

$2t - 1 = 0$ $\qquad$ $1 + 2t \neq 0$

$t = \frac{1}{2}$ $\qquad$ $t \neq -\frac{1}{2}$

Tangent line is horizontal at $t = \frac{1}{2}$

$$\left(x\left(\tfrac{1}{2}\right), y\left(\tfrac{1}{2}\right)\right) = \left(\tfrac{1}{2} + \tfrac{1}{4}, \tfrac{1}{4} - \tfrac{1}{2}\right) = \boxed{\left(\tfrac{3}{4}, -\tfrac{1}{4}\right)}$$

$x'(t) = 0$, but $y'(t) \neq 0$.

$t = -\frac{1}{2}$ $\qquad$ $t \neq \frac{1}{2}$

Tangent line is vertical at $t = -\frac{1}{2}$

$$\left(x\left(-\tfrac{1}{2}\right), y\left(-\tfrac{1}{2}\right)\right) = \left(-\tfrac{1}{2} + \tfrac{1}{4}, \tfrac{1}{4} + \tfrac{1}{2}\right) = \boxed{\left(-\tfrac{1}{4}, \tfrac{3}{4}\right)}$$

EXAMPLE 4. *Show that the curve*

$$x = \cos t, \quad y = \cos t \sin t = \frac{1}{2}\sin 2t$$

has two tangents at $(0,0)$ *and find their equations.*

Tangent at $(0,0)$: $y = mx$, where $m = \dfrac{y'(t_0)}{x'(t_0)}$

$x'(t) = (\cos t)' = -\sin t$

$y'(t) = \left(\frac{1}{2}\sin 2t\right)' = \frac{1}{2}\cdot 2\cos 2t = \cos 2t$

To find t_0 solve: $(x(t_0), y(t_0)) = (0,0)$

$(\cos t_0, \sin t_0 \cos t_0) = (0,0)$

$\cos t_0 = 0$ if $t_0 = \frac{\pi}{2}, \frac{3\pi}{2}$ means we have two tangent with slopes:

$$m_1 = \frac{y'\left(\frac{\pi}{2}\right)}{x'\left(\frac{\pi}{2}\right)} = \frac{\cos \pi}{-\sin\frac{\pi}{2}} = \frac{-1}{-1} = 1$$

and

$$m_2 = \frac{y'\left(\frac{3\pi}{2}\right)}{x'\left(\frac{3\pi}{2}\right)} = \frac{\cos(3\pi)}{-\sin\left(\frac{3\pi}{2}\right)} = \frac{-1}{-(-1)} = -1$$

Tangents at $(0,0)$: $\boxed{y = x \quad \text{and} \quad y = -x}$