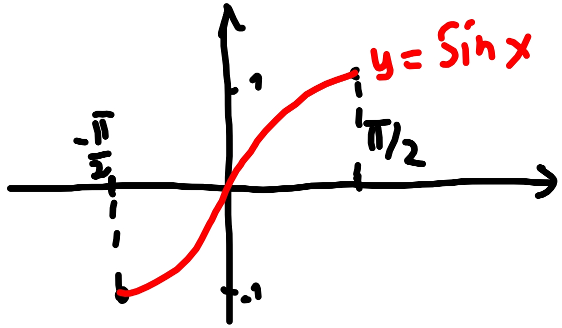
$$\sin^{-1} x \neq \frac{1}{\sin x}$$

4.6: Inverse trigonometric functions

- **INVERSE SINE:** If $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$, then $f(x) = \sin x$ is one-to-one, thus the inverse exists, denoted by $\sin^{-1}(x)$ or $\arcsin x$.

	$y = \sin x$	$y = \arcsin x$
Domain	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
Range	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$



$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

Cancellation equations:

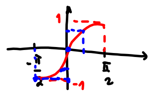
$$\boxed{\arcsin(\sin x) = x \quad \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}}$$

and

$$\boxed{\sin(\arcsin x) = x \quad \text{if} \quad -1 \leq x \leq 1.}$$

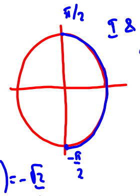
EXAMPLE 1. *Find the exact values of the expression:*

(a) $\sin^{-1} 0 = 0$, because $\sin 0 = 0$



(b) $\arcsin(-1) = -\frac{\pi}{2}$

(c) $\sin^{-1}(0.5) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $\sin\frac{\pi}{6} = \frac{1}{2}$



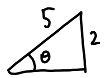
I & IV quadrants

(d) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

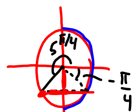
(e) $\sin\left(\arcsin\frac{2}{7}\right) = \frac{2}{7}$ by Cancelation rule because $-1 < \frac{2}{7} < 1$

(f) $\tan\underbrace{\left(\arcsin\frac{2}{5}\right)}_{\theta} = \tan\theta = \boxed{\frac{2}{\sqrt{21}}}$

$\theta = \arcsin\frac{2}{5} \Rightarrow \sin\theta = \frac{2}{5} \Rightarrow \tan\theta = \frac{2}{\sqrt{21}}$



$\sqrt{5^2 - 2^2} = \sqrt{21}$



$\sin\frac{5\pi}{4} = \sin\left(-\frac{\pi}{4}\right)$

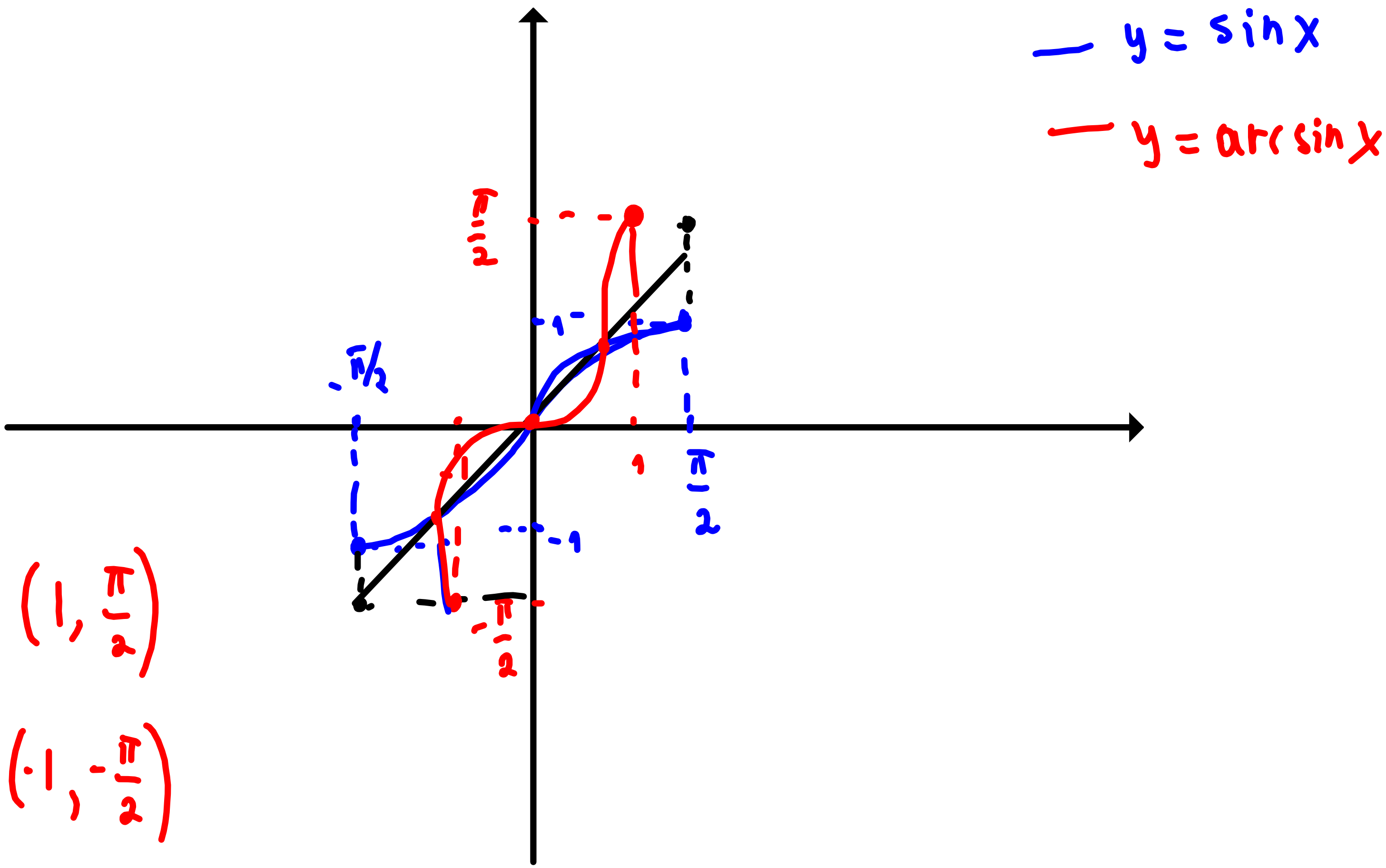
(g) $\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(\sin\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

by Cancelation rule because $-\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$

(h) $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$ by Canc. rule $-\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2}$

(i) $\arcsin\left(\sin\frac{\pi}{150}\right) = \frac{\pi}{150}$

EXAMPLE 2. *Sketch the graph of* $\arcsin(x)$.

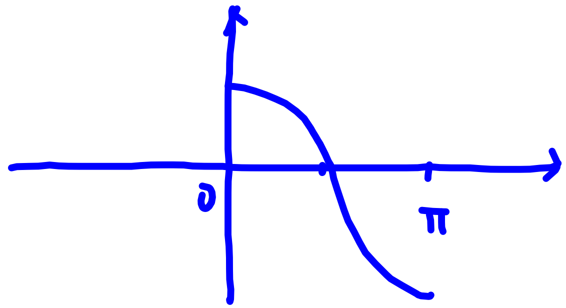


$\left(\frac{\pi}{2}, 1\right) \to \left(1, \frac{\pi}{2}\right)$

$\left(-\frac{\pi}{2}, -1\right) \to \left(-1, -\frac{\pi}{2}\right)$

- **INVERSE COSINE:** If $0 \le x \le \pi$, then $f(x) = \cos x$ is one-to-one, thus the inverse exists, denoted by $\cos^{-1}(x)$ or $\arccos x$.

	$y = \cos x$	$y = \arccos x$
Domain	$[0, \pi]$	$[-1, 1]$
Range	$[-1, 1]$	$[0, \pi]$



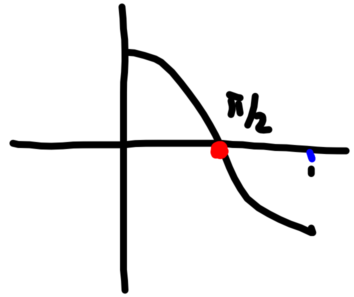
Cancellation equations:

$$\arccos(\cos x) = x \quad \text{if} \quad 0 \le x \le \pi$$

and

$$\cos(\arccos x) = x \quad \text{if} \quad -1 \le x \le 1.$$

EXAMPLE 3. *Find the exact values of the expression:*



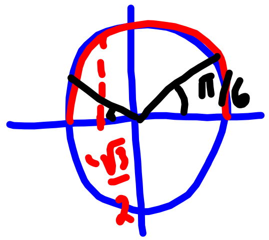
(a) $\arccos 0 = \frac{\pi}{2}$, $\cos\frac{\pi}{2} = 0$

(b) $\cos^{-1} 1 = 0$, $\cos 0 = 1$

(c) $\arccos(-1) = \pi$, $\cos\pi = -1$

(d) $\arccos 0.5 = \arccos\frac{1}{2} = \frac{\pi}{3}$

(e) $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

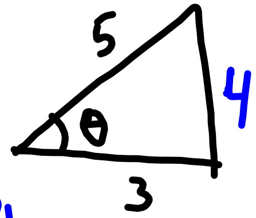


(f) $\sin\left(2\arccos\frac{3}{5}\right) = 2\underbrace{\overbrace{\sin\left(\arccos\frac{3}{5}\right)}^{4/5}}\cdot\underbrace{\cos\left(\arccos\frac{3}{5}\right)}_{\frac{3}{5}} =$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\theta = \arccos\frac{3}{5}$

$\cos\theta = \frac{3}{5}$



$\sin\theta = \frac{4}{5} = \sin\left(\arccos\frac{3}{5}\right)$

$= 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \boxed{\frac{24}{25}}$

(g) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

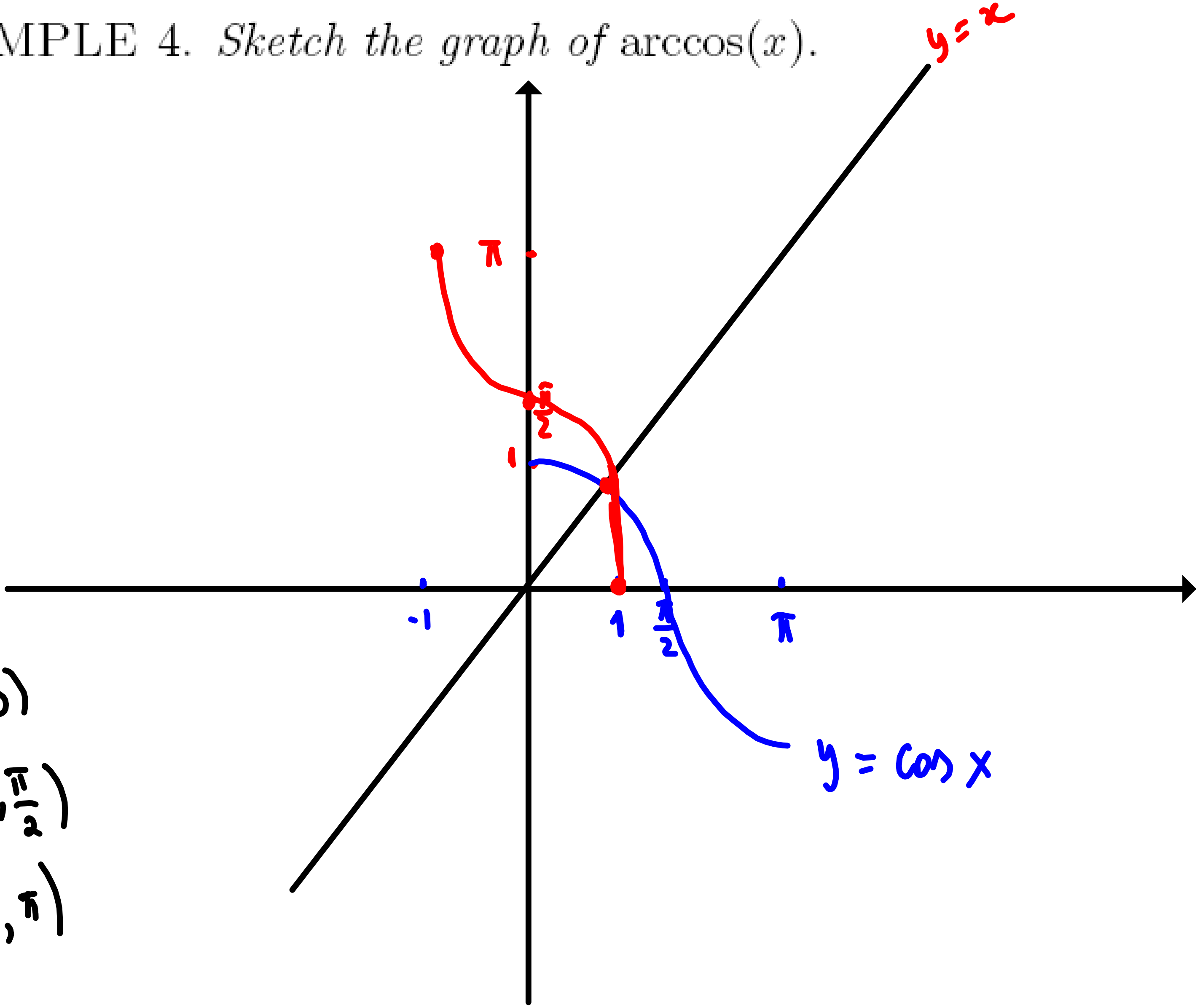
$0 \leq \frac{\pi}{6} \leq \pi$ Cancel. Rule

(h) $\arccos\left(\cos\frac{7\pi}{6}\right) = \arccos\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6}$

(i) $\cos(\arccos 2)$ undefined, 2 is not in the domain of arccos

(j) $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right) = \arccos\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$

EXAMPLE 4. *Sketch the graph of* $\arccos(x)$.



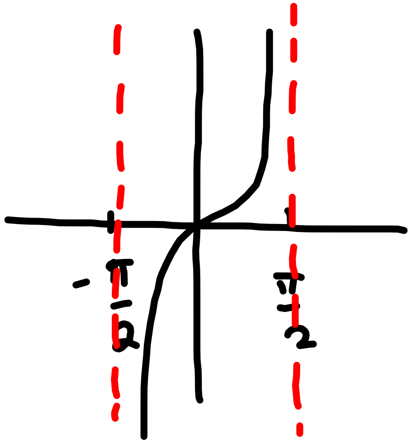
$(0,1) \to (1,0)$

$(\frac{\pi}{2},0) \to (0,\frac{\pi}{2})$

$(\pi,-1) \to (-1,\pi)$

- **INVERSE TANGENT:** If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $f(x) = \tan x$ is one-to-one, thus the inverse exists, denoted by $\tan^{-1}(x)$ or $\arctan x$.

	$y = \tan x$	$y = \arctan x$ $= \tan^{-1} x$
Domain	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
Range	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$



Cancellation equations:

$$\arctan(\tan x) = x \quad \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

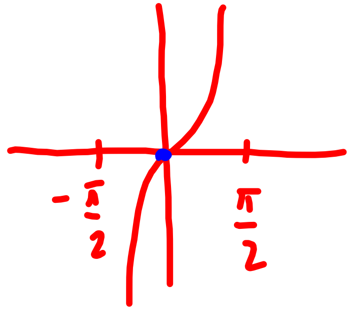
$x = \pm\frac{\pi}{2}$ are vertical asymptotes for $y = \tan x$

and

$$\tan(\arctan x) = x \quad \text{for all } x.$$

EXAMPLE 5. *Find the exact values of the expression:*

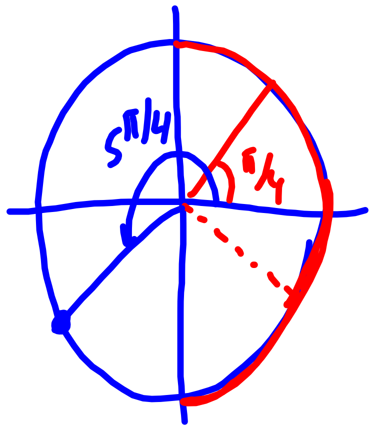
(a) $\arctan 0 = 0$, $\tan 0 = 0$



(b) $\arctan(-1) = -\frac{\pi}{4}$ $\tan\frac{\pi}{4} = 1$

(c) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

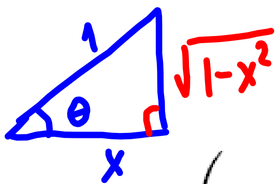
$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$



(d) $\tan(\underbrace{\arccos x}_{\theta}) = \tan\theta$

$\theta = \arccos x$

$\cos\theta = x = \frac{x}{1}$ $\Rightarrow \tan\theta = \boxed{\frac{\sqrt{1-x^2}}{x}}$



(e) $\arctan\left(\tan\frac{5\pi}{4}\right) = \arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ by Cancel. Rule

$\tan\frac{5\pi}{4} = \tan\frac{\pi}{4}$

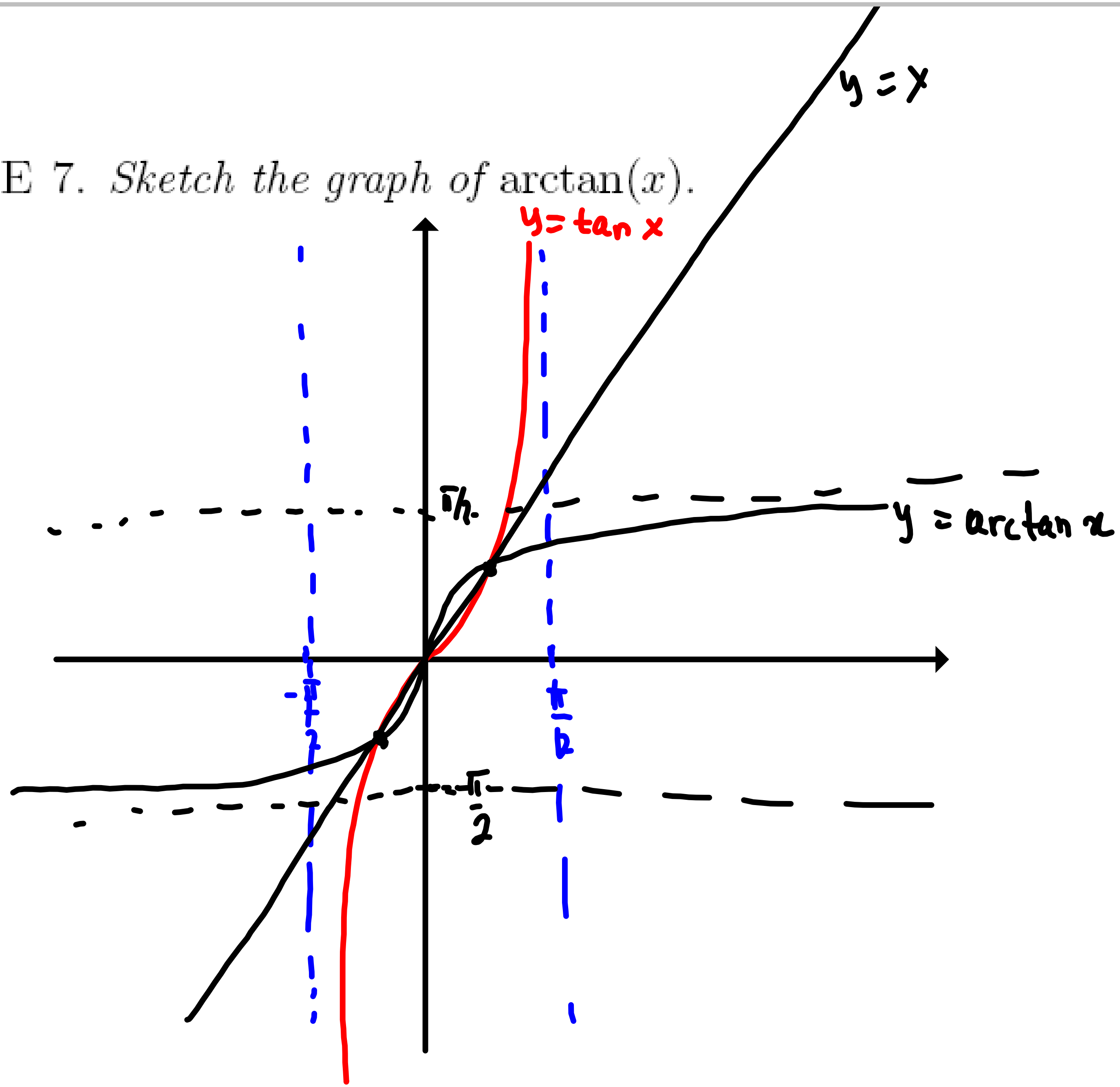
EXAMPLE 6. *Find the following limits:*

(a) $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

(b) $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

(see the graph on the next page)

EXAMPLE 7. *Sketch the graph of* $\arctan(x)$.



Derivatives of Inverse Trigonometric Functions:

EXAMPLE 8. **(a)** *Find the derivative of* $f(x) = \arcsin x$.

$y(x) = \arcsin x \Rightarrow (\sin y(x))' = (x)'$

Implicit differ.

$y' \cos y = 1 \Rightarrow y' = \dfrac{1}{\cos y}$

$\sin y = x$

$\Downarrow$

$\cos y = \sqrt{1-x^2}$

$y' = \dfrac{1}{\sqrt{1-x^2}}$

$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

(b) *Find* $\frac{d}{dx}\left(\frac{1}{\underbrace{\arcsin(3x+1)}_{u(x)}}\right) = \frac{d}{dx}\left(\frac{1}{u(x)}\right) = -\frac{1}{[u(x)]^2} \cdot u'(x) =$

$$= -\frac{1}{(\arcsin(3x+1))^2} \cdot \frac{d}{dx}\left(\arcsin(\underbrace{3x+1}_{g(x)})\right) =$$

$$= \boxed{-\frac{1}{(\arcsin(3x+1))^2} \cdot \underbrace{\frac{1}{\sqrt{1-(3x+1)^2}}}_{\sqrt{1-g^2}} \cdot \underbrace{3}_{g'(x)}}$$

TABLE OF DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad , \quad -1 < x < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad , \quad -1 < x < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Memorize!

EXAMPLE 9. *Find the derivative of* $f(x) = \sin^{-1}(\arctan x)$

$$f'(x) = \frac{d}{dx}\left(\sin^{-1}(\underbrace{\arctan x}_{g(x)})\right) \overset{\text{chain rule}}{=} \frac{1}{\sqrt{1-g^2}}\, g'(x) =$$

$$= \frac{1}{\sqrt{1-g^2}} \cdot (\arctan x)' = \frac{1}{\sqrt{1-(\arctan x)^2}} \cdot \frac{1}{1+x^2}$$

EXAMPLE 10. *Find domain of the following functions:*

(a) $f(x) = \arcsin(\underbrace{4x+2}_{u})$ = arc sin u

$-1 \le u \le 1$

$\underbrace{-1} \le \underbrace{4x+2} \le \underbrace{1}$ (-2)

$-1-2 \le 4x+2-2 \le 1-2$

$-3 \le 4x \le -1$ $\left(\times \frac{1}{4}\right)$

$\boxed{-\frac{3}{4} \le x \le -\frac{1}{4}}$ domain of $f(x)$

(b) $f(x) = \arctan(4x+2)$

$-\infty < 4x+2 < \infty \Rightarrow \boxed{-\infty < x < \infty}$

domain of f