Rates Of Change

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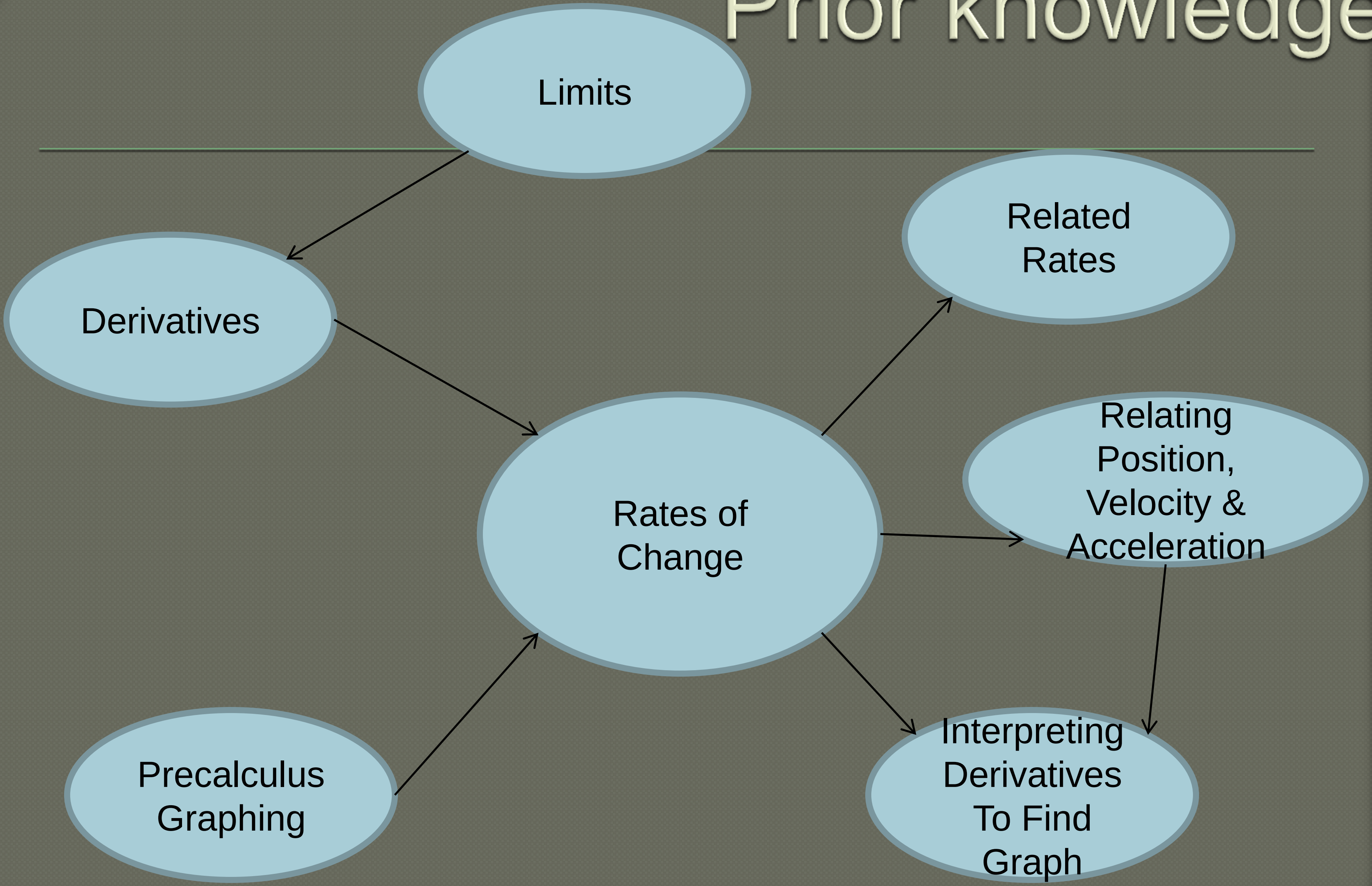
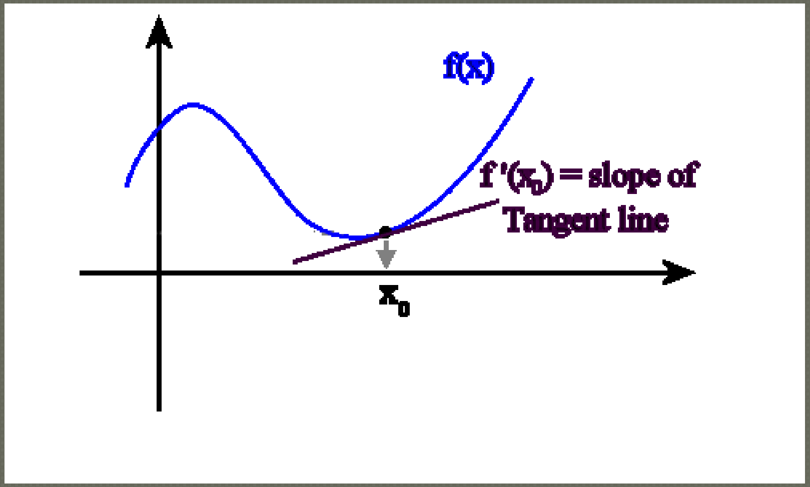
Prior knowledge



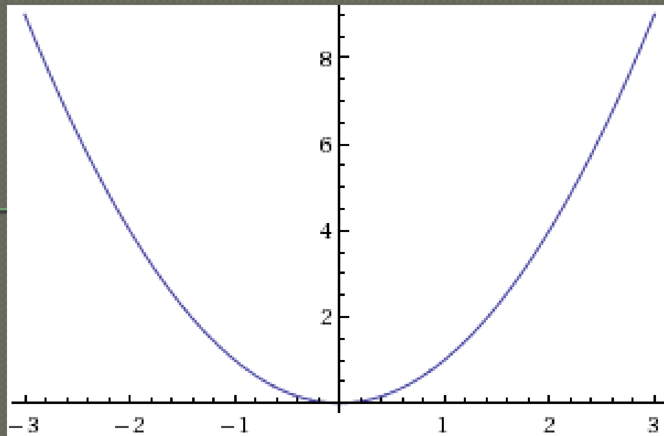
Background info

What is a derivative?
A derivative is the measurement of the instantaneous slope of a curve, also known as a rate of change. A derivative is represented by the slope of the tangent line to a curve at a given point.

How to find a derivative?
You can either use the formal definition of a derivative (using limits), or you can use differentiation rules to find a curves derivative.



Derivative

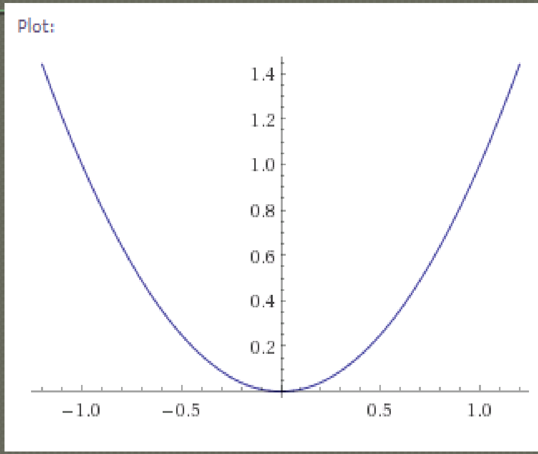


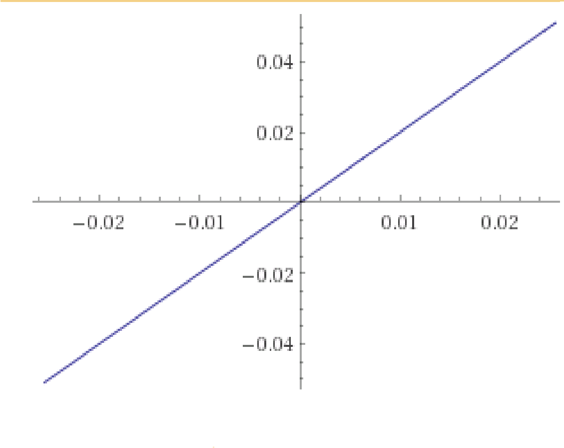
y=x^2

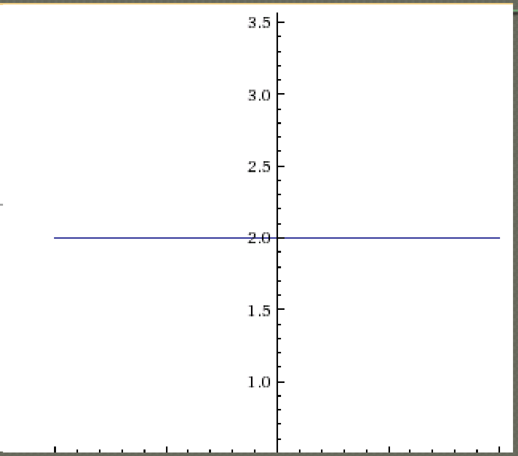
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x
\end{aligned}$$

Relationship of Functions







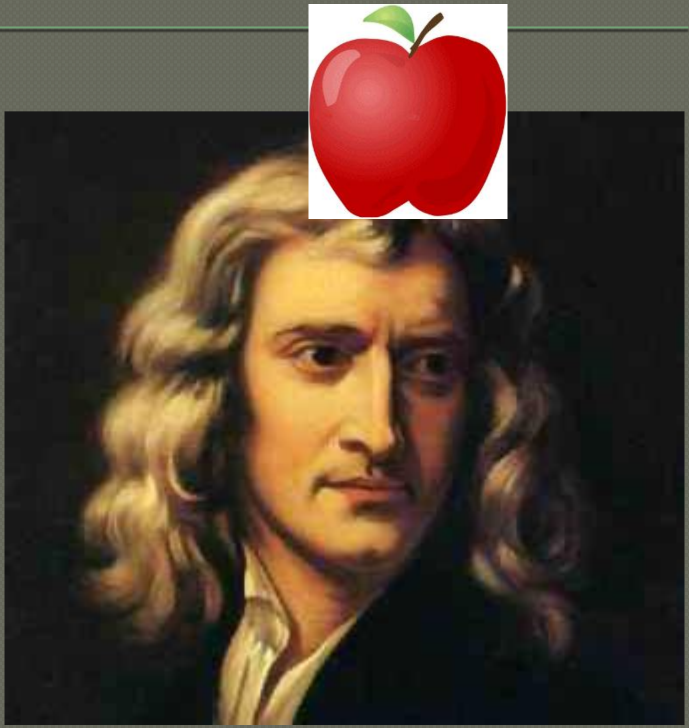
Position:
$s(t)=t^2$

Velocity:
$v(t)=s'(t)$
$v(t)=2t$

Acceleration
$a(t)=v'(t)=s''(t)$
$a(t)=2$

Isaac Newton

- 17^{th} century Englishmen, who was knighted for his revolutionary work and discoveries in the fields of science and math. He is co-credited with the discovery of derivatives and infinitesimal calculus. It is said that Newton discovered derivatives in trying to find the instantaneous slope of curves.



Gottfried Leibniz

- 17th century German mathematician, inventor, and philosopher. Leibniz is well renowned for his meditation on mathematics, as well as being the creator of one of the first computing machines of his time. In 1675, Leibniz stumbled upon integrals while trying to calculate the area under a curve. Subsequently, he also discovered derivatives as a result, and is the founder of the product rule. Also well known for introducing theory of motion utilizing potential and kinetic energy.

Isaac Newton vs Gottfried Leibniz

ISAAC NEWTON

- Published his theories on calculus in 1694.

GOTTFRIED LEIBNIZ

- Published his theories on calculus in 1684.

To this day, the founding of calculus is disputed between Leibniz and Newton. The late years of their lives were spent in controversy over who published the work first.

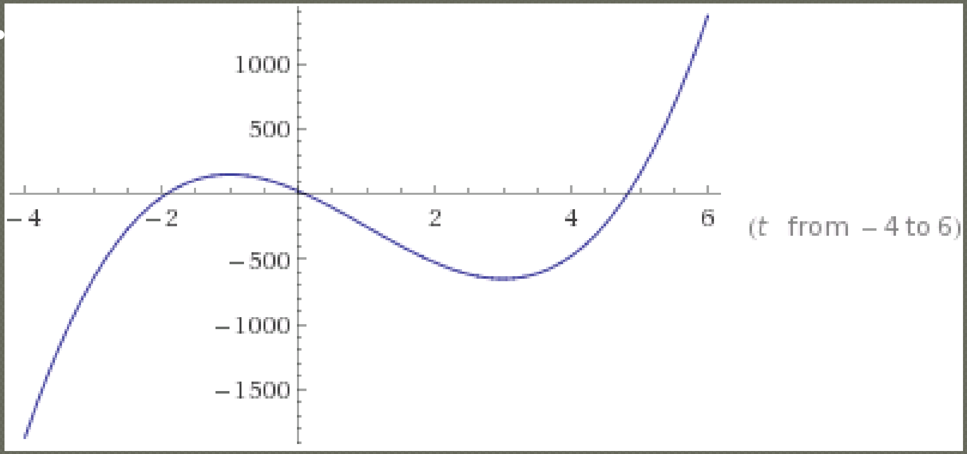
NEWTON NOTATION LEIBNIZ NOTATION

Table of the Foreign, and the corresponding English *Notation.*

F.	dx	d^2x	d^3x	d^4x	dx^2	dx^3
E.	$\dot{x}$	$\ddot{x}$	$\dddot{x}$	$\dot{x}^{\text{iv}}$	$\dot{x}^2$	$\dot{x}^3$
F.	$\frac{du}{dx}$	$\frac{d^2u}{dx^2}$	$\frac{d^3u}{dx^3}$	$\frac{d^2u}{dx.dy}$	$\frac{d^3u}{dx^2.dy}$	$d^3\left(\frac{dy}{dz}\right)$
E.	$\frac{\dot{u}}{\dot{x}}$	$\frac{\ddot{u}}{\dot{x}^2}$	$\frac{\ddot{u}}{\dot{x}^3}$	$\frac{\ddot{u}}{\dot{x}\dot{y}}$	$\frac{\dddot{u}}{\dot{x}^2\dot{y}}$	$\left(\frac{\dot{y}}{\dot{z}}\right)^{\therefore}$

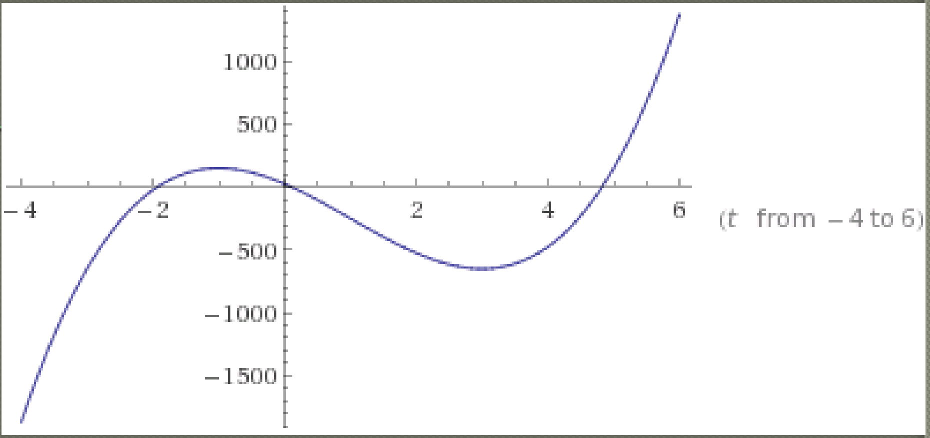
Jet Problem

- A jet is flying along the path described by the function in feet $s(t)=25t^3-75t^2-225t+25$, for $0 \le t \le 6$ seconds.



$s(t)=25t^3-75t^2-225t+25$

- 1. What is the average velocity of the jet in feet per second from t=0 seconds to t=6 seconds?



$s(t)=25t^3-75t^2-225t+25$

SOLUTION:

$s(t)=25t^3-75t^2-225t+25$

$$\text{Average Velocity} = \frac{s(\text{final})-s(\text{initial})}{\Delta t}$$

$s(0)=25(0)^3-75(0)^2-225(0)+25$

$s(0)=25$ feet

$s(6)=25(6)^3-25(6)^2-225(6)+25$

$s(6)=1375$ feet

$$\text{Average Velocity} = \frac{s(6)-s(0)}{6}$$

$$\text{Average Velocity} = \frac{1375-25}{6}$$

Avg Vel = 225 feet

*Note-a common error amongst students is to take the change in velocity over the change in time (giving average acceleration), rather than take the change in distance over the change in time (giving average velocity).

2. When does the velocity of the jet equal zero?

SOLUTION:

$s(t)=25t^3-75t^2-225t+25$

$s'(t)=v(t)$

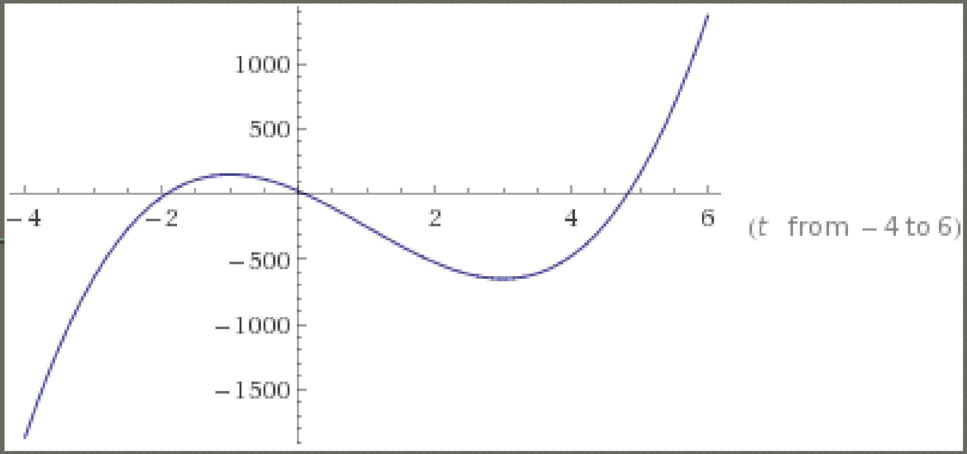
$v(t)=75t^2-150t-225$

$v(t)=75(t^2-2t-3)$

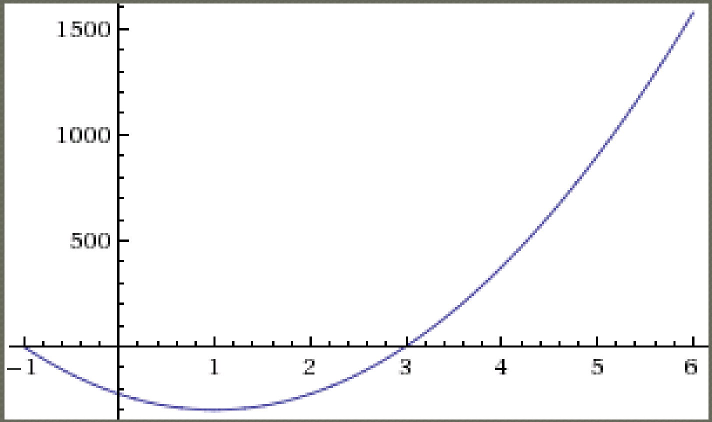
$v(t)=75(t-3)(t+1)$

$v(t)=0$ when $t=3$ seconds

*Note-While our velocity equation provides two solutions of $t=-1,3$ the only solution that is real is $t=3$ because time cannot be negative.



$s(t)=25t^3-75t^2-225t+25$

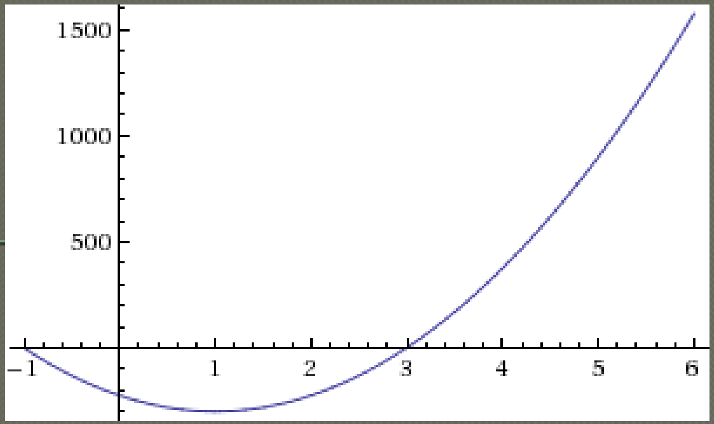


$v(t)=75t^2-150t-225$

3. When is the jet moving left? When is the jet moving right?

SOLUTION:
Jet moves left when $v(t)<0$,right when $v(t)>0$, changing direction when $v(t)=0$
$v(t)=75(t^2-2t-3)$
$v(t) < 0$ from $t=0$ to $t=3$ seconds
$v(t) > 0$ from $t=3$ to $t=6$ seconds

Therefore, the jet is moving left from (0,3), and moving right from (3,6). At t=3, the jet is changing direction because $v(3)=0$.



$v(t)=75t^2-150t-225$

*Note-A common mistake is using the position function rather than the velocity function to determine the motion of the object.

- 4. What is the average acceleration of the jet from t=0 to t=6?

SOLUTION:

$v(t)=75t^2-150t-225$

Avg. Accel. = $\dfrac{\underline{v(final)-v(initial)}}{\Delta t}$

$v(0)=75(0)^2-150(0)-225$

$v(0)=-225$

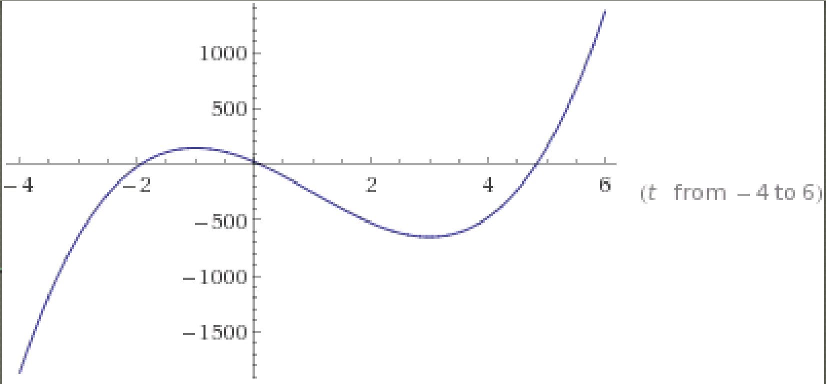
$v(6)=75(6)^2-150(6)-225$

$v(6)=1575$

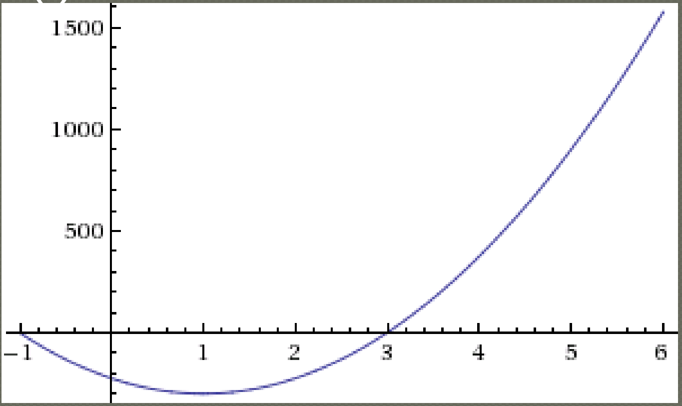
Avg. Accel. = $\dfrac{\underline{v(6)-v(0)}}{6}$

Avg. Accel = $\dfrac{\underline{1575-(-225)}}{6}$

Avg. Accel = 300 $\dfrac{\underline{feet}}{sec^2}$



$s(t)=25t^3-75t^2-225t+25$



$v(t)=75t^2-150t-225$

- 5. When is the jets acceleration zero? Negative? Positive?

SOLUTION:

$v(t)=75t^2-150t-225$

$v'(t)=a(t)$

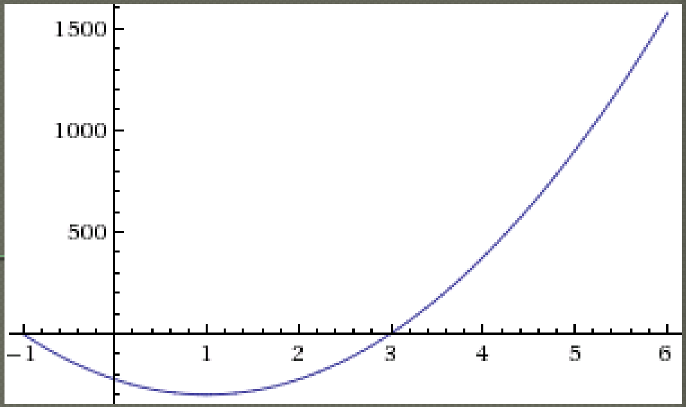
$a(t)=150t-150$

I. $a(t)=0=150(t-1)$

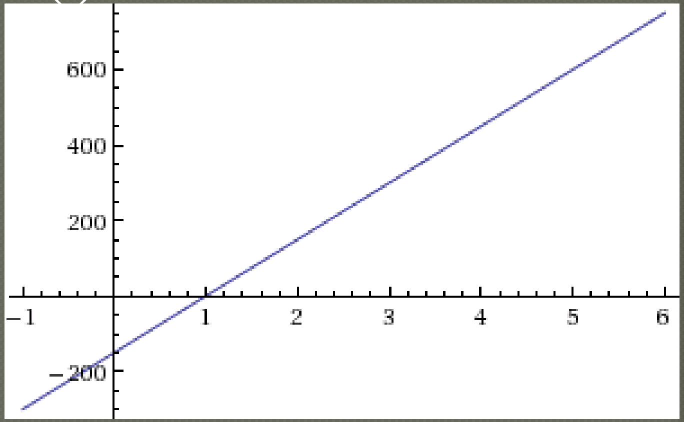
   $a(t)=0$ when $t=1$

II. $a(t) < 0$ from $t=0$ to $t=1$

III. $a(t) > 0$ from $t=1$ to $t=6$



$v(t)=75t^2-150t-225$



$a(t)=150t-150$

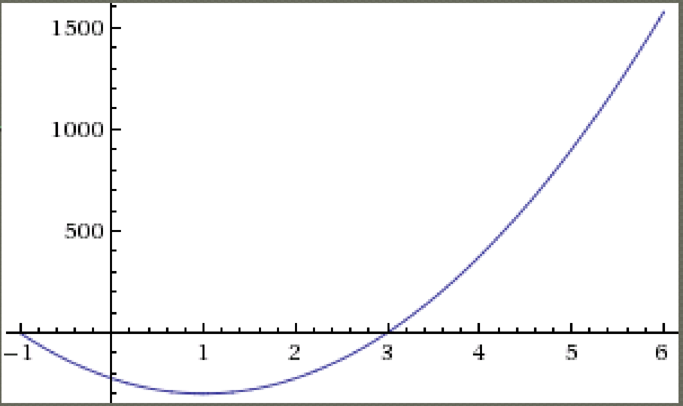
6. When is the jet speeding up? When is it slowing down?
(*hint-use questions 3 and 5)

SOLUTION:
The jet is speeding up when the sign of the velocity and the acceleration is the same. The jet is slowing down when the sign of the velocity and the acceleration are opposite.

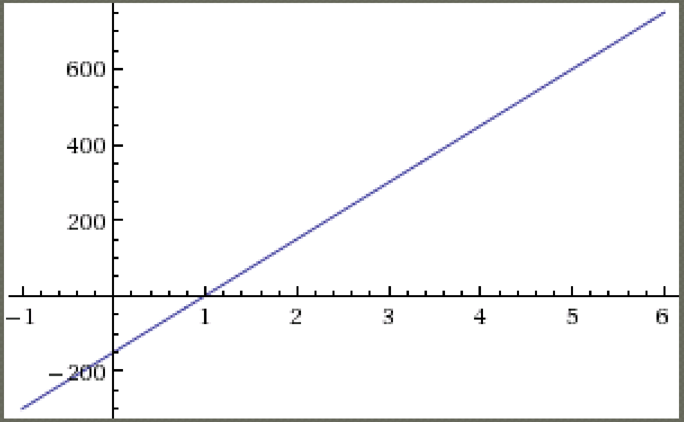
<u>Positive:</u>
$v(t) > 0$ from (3,6)
$a(t) > 0$ from (1,6)

<u>Negative:</u>
$v(t) < 0$ from (0,3)
$a(t) < 0$ from (0,1)

Speeding up from (0,1) and (3,6) because v(t) and a(t) have same sign.
Slowing down from (1,3) because v(t) and a(t) have opposite signs.



$v(t)=75t^2-150t-225$



$a(t)=150t-150$

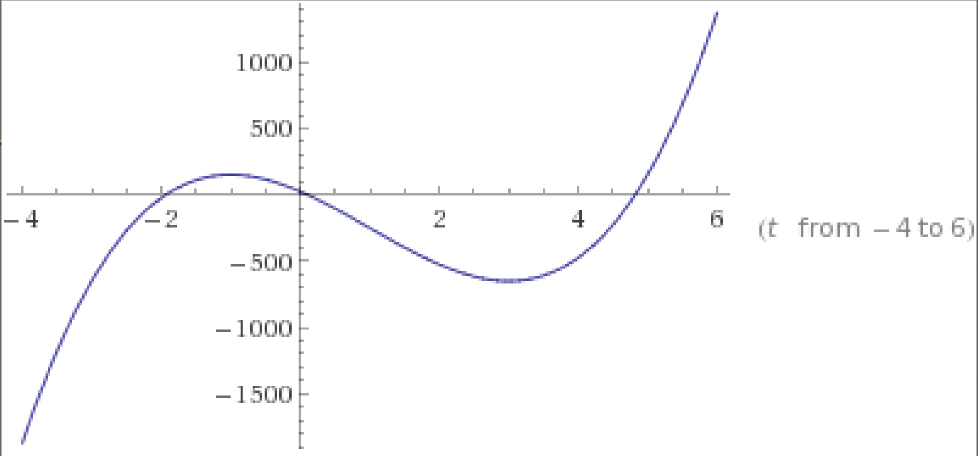
7. What is the displacement of the jet at t=6?

SOLUTION:
Displacement = s(final)-s(initial)

Disp. = s(6)-s(0)

$s(0)=25(0)^3-75(0)^2-225(0)+25$
$s(0)=25$
$s(6)=25(6)^3-75(6)^2-225(6)+25$
$s(6)=1375$

Disp. = 1375-25 = 1350 feet



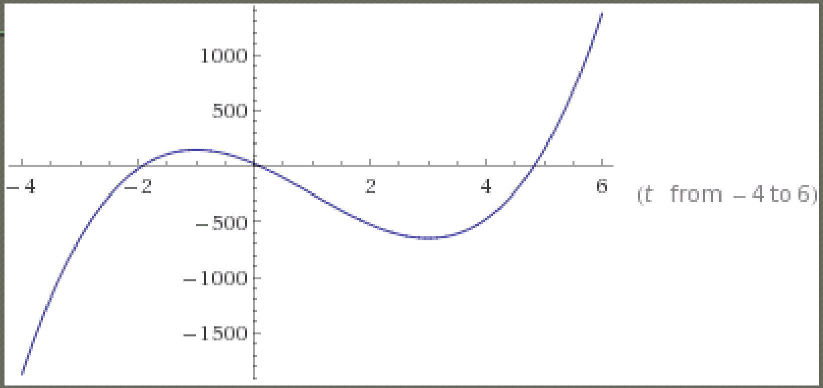
$s(t)=25t^3-75t^2-225t+25$

*Note-the displacement of the jet is the distance from the original starting point. It is not to be confused with the total distance the jet traveled in its flight.

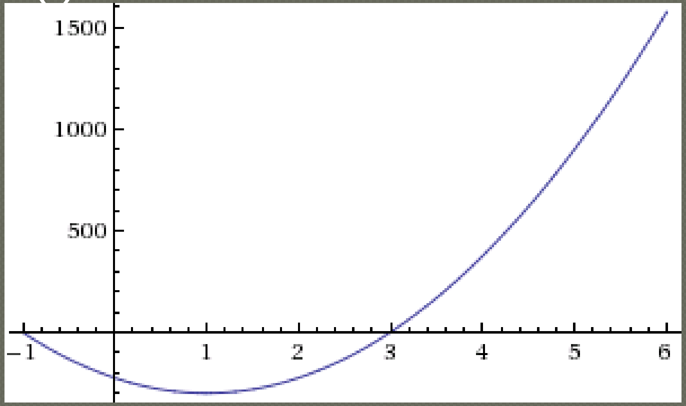
8. What is the total distance the jet traveled from t=0 to t=6? (*hint-use question 3)

SOLUTION:

In order to solve this, we must calculate the distances between the turns in the position graph (where the velocity graph is negative, and where it is positive) and use absolute value to add up total distance.



$s(t)=25t^3-75t^2-225t+25$



$v(t)=75t^2-150t-225$

v(t) < 0 from (0,3)
v(t) > 0 from (3,6)
Therefore, there is a turn between this set of points.

Total Dist. = |s(3)-s(0)|+|s(6)-s(3)|
$s(0)=25(0)^3-75(0)^2-225(0)+25=25$
$s(3)=25(3)^3-75(3)^2-225(3)+25=-650$
$s(6)=25(6)^3-75(6)^2-225(6)+25=1375$

Total Dist. = |-650-25|+|1375+650|
Total Dist. = |675|+|2025|
Total Dist. = 2700 feet

*Note-It is crucial to identify the distance between the turns of the graph. That way we get the total distance traveled, rather than displacement

- Ryan begins his own business making paracord bracelets. Ryan has a starting fixed cost of $100. It costs him $1.50 per bracelet in materials. As Ryan makes each bracelet it costs him $.02 per bracelet squared (with time, Ryan becomes more inefficient and has higher wear on his tools). Ryan plans to charge $5 for each bracelet he makes. What production level will maximize Ryan's profits? What is the maximum profit?

SOLUTION:

$C(x) = 100+1.50x+.02x^2$

$R(x) = 5x$

$P(x) = R(x) – C(x)$

$P(x) = 5x – (100+1.50x+.02x^2)$

$P(x) = -.02x^2 +3.5x-100$

-Max Profit, at Maximum

Take Derivative of Profit, which is "Marginal Profit"

$P'(x) = -.04x+3.5$

$P'(x) = 0$

$.04x = 3.5$

$x = 88$ bracelets

*Note-x=88 is unique critical number

Positive

P′(x) |--------------|---------------|

87.5 Negative

$P(88) = -.02(88)^2+5(88)-100$

$P(88) = \$185.12$

Curiosity Mars Rocket

- The NASA Curiosity rocket is launched from earth on a mission to collect data samples on Mars. It has a mass of 3893 kg, and reaches a speed of 9656 m/s once out of the Earth's atmosphere. What is the rate of change of the gravitational force between the Earth and the Curiosity rocket when the rocket is 100 million miles away (1.61E11 m)?

$$F_g = \frac{GMm}{r^2}$$

Gravity Constant G=6.67E-11 m^3kg^-1s^-2
Mass of Earth M=5.97E24 kg
mass of rocket m=3893 kg
We take the derivative of Force with respect to time using chain rule

$$\frac{dF}{dt} = \frac{-2GMm}{r^3} * \frac{dr}{dt}$$

$$\frac{dF}{dt} = \frac{-2(6.67E-11)(5.97E24)(3893)}{(1.61E11)^3} *(9656)$$

$$\frac{dF}{dt} = -7.18E-12 \text{ N/s}$$

Resources

- http://www.wolframalpha.com/
- http://faculty.wlc.edu/buelow/calc/nt4-7.html
- http://www.nd.edu/~sstarche/Math10550/Lectures/12.Rates%20of%20Change.pdf
- http://www.universetoday.com/89346/assembling-curiosity%E2%80%99s-rocket-to-mars/
- Stewart Calculus Early Vectors Text Book