When x is the Exponent

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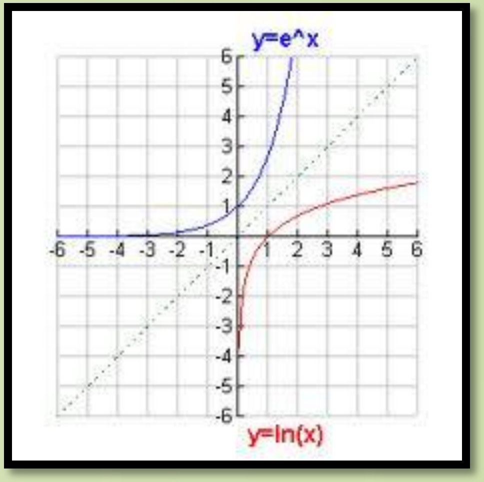
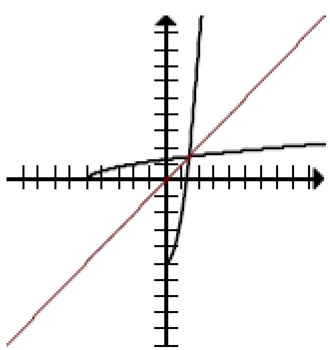
Overview: Inverses

- Indicated by the function $y = f^{-1}(x)$, inversely related to $y = f(x)$.
- Domain of the inverse corresponds to the Range of the original function, and vice versa.
- Can be shown as a reflection across $y = x$



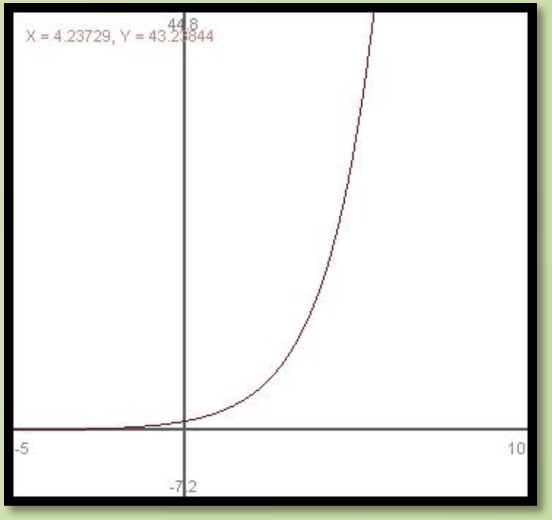
x	f(x)
0	–5
1	–2
2	7
3	22
4	43

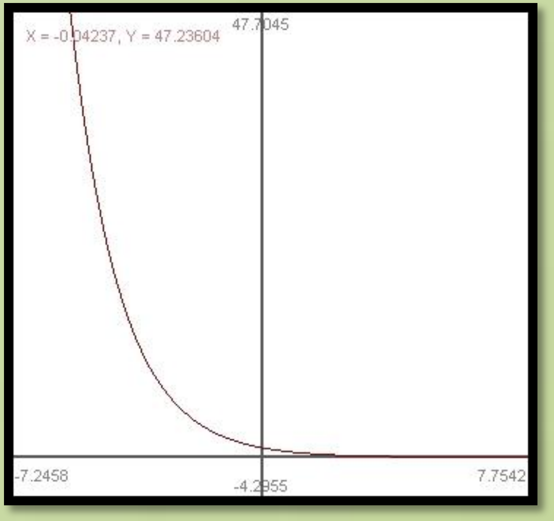
x	f^{-1}
–5	0
–2	1
7	2
22	3
43	4



Overview: Exponentials

- Indicated by the Function $f(x) = a^x$ such that a is a positive number.

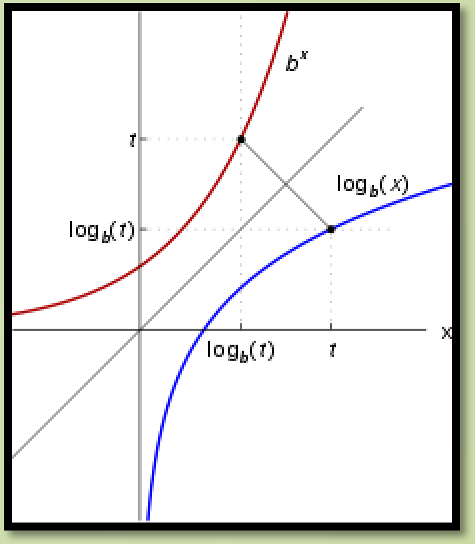


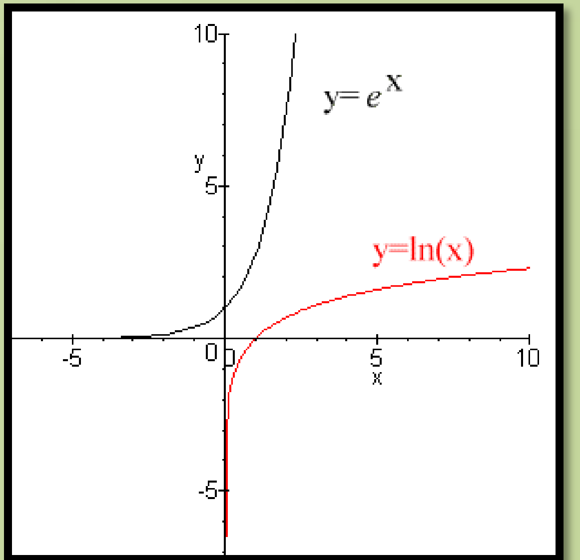


- Used for modeling Exponential Growth/Decay, or Continuous Compound Interest

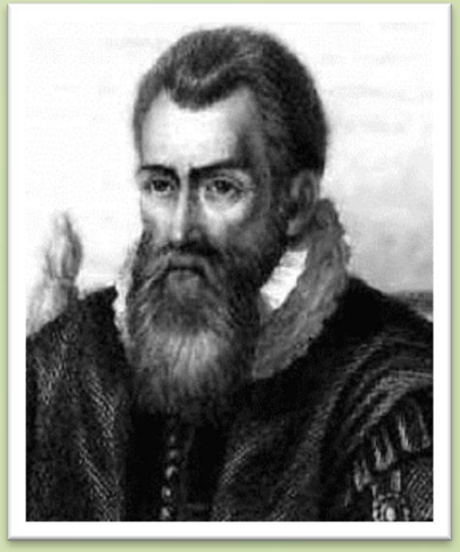
Overview: Logarithms

- Inverse of exponential function;
  if $f(x) = a^x$, $x = log_a f(x)$
- Logarithm with base *e* known as "natural logarithm", represented by *ln*





History of the Log

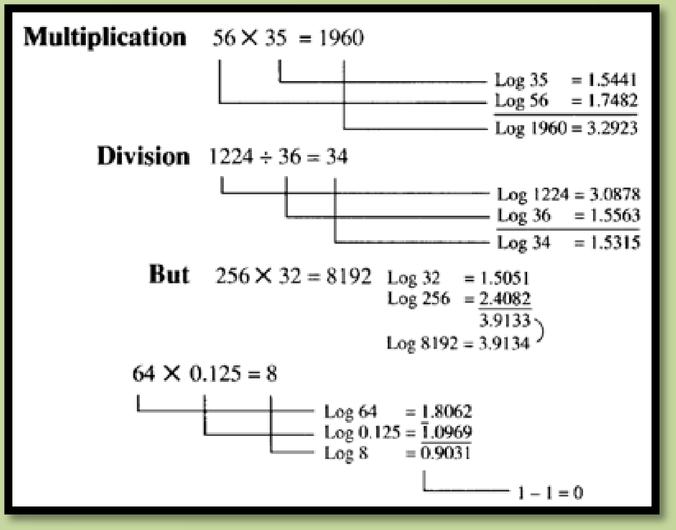


The idea that a mathematical inverse to exponential growth and decay existed had been understood even by the earliest of mathematicians.

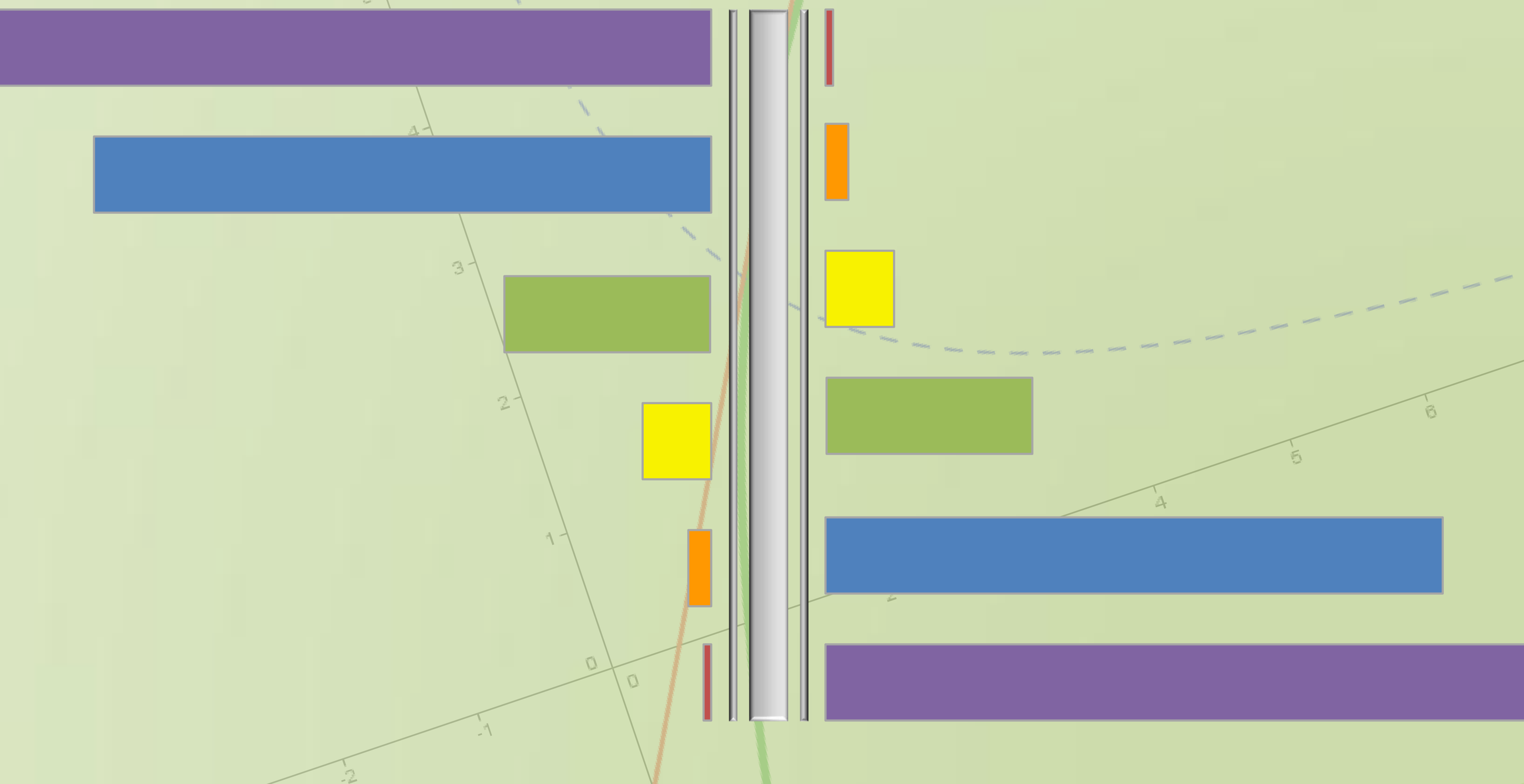
However, it was not until the early 1600s that the principles of logarithms were published by John Napier of Scotland.

The availability of Logarithms opened up new frontiers in mathematics, while simplifying old formulas.

This advancement took place due to the fact that previously difficult multiplications and divisions could be represented by the addition or subtraction of logarithms due to the law of exponents.



Inverses

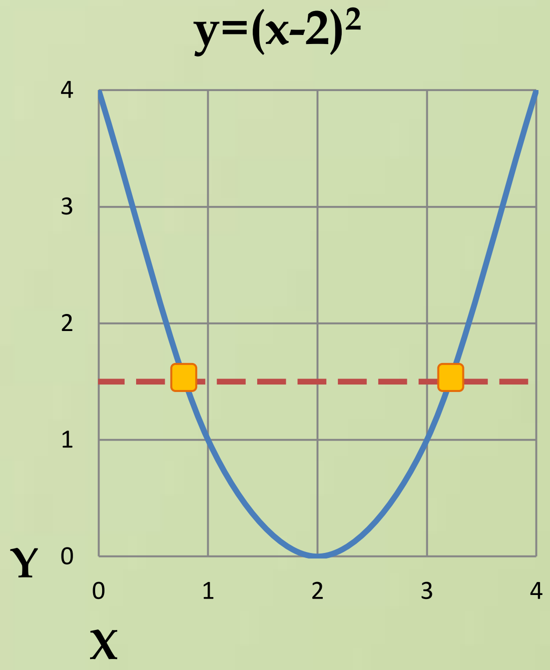


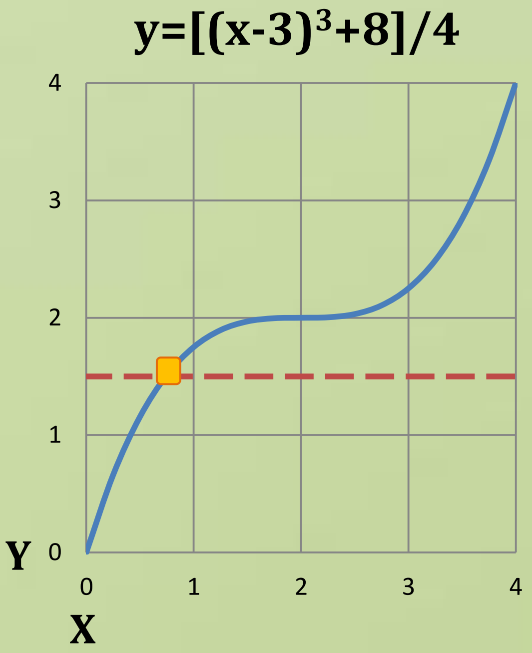
One to One

- **Key term: <u>One-to-one Function-</u>** A function in which there are no repeated x values for any given y value.

- Given a graph, the best way to determine whether or not a function is one-to-one is to perform the “horizontal line test,” which is similar to the “vertical line test” for determining if a graph represents a function. An example of the horizontal line test is shown on the next slide:

Horizontal Line Test





Since the graph intercepts the horizontal line at multiple points, the function is not considered one-to-one.

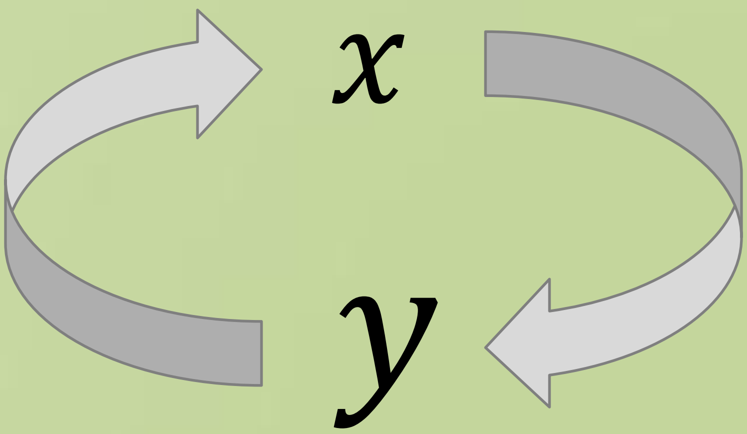
For each y value, there is only one corresponding x value; the function graphed is one-to-one.

The Inverse Itself

If $f(x)$ is a one-to-one function such that $f(x) = y$, then the inverse function is defined to be the function such that $f^{-1}(y) = x$. Therefore, $f^{-1}\big(f(x)\big) = x$.

Conversely, if $f(x)$ is one-to-one such that $f(y) = x$, then $f^{-1}(x) = f(y)$, and thus $f(f^{-1}(x)) = x$.

Commonly, the inverse is found by solving the current equation for x and then interchanging the x's and y's.



Warning!

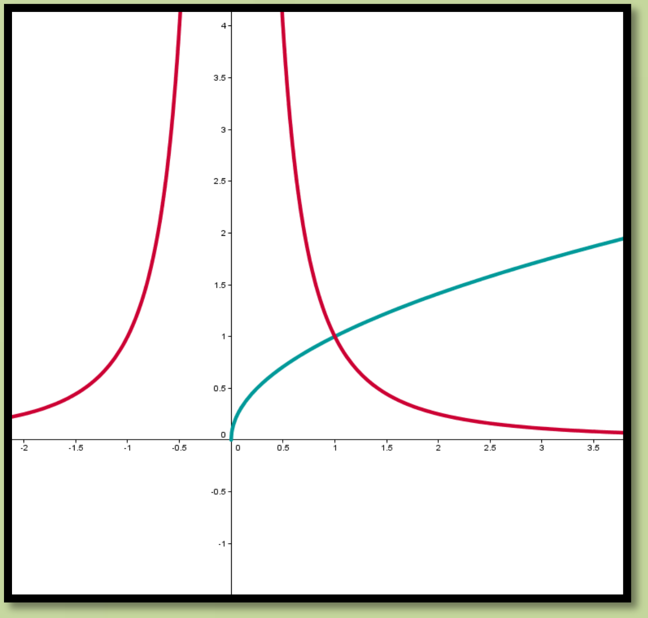
Students commonly confuse inverse notation with exponential notation. Remember:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

For instance, if $f(x) = x^2$, then:

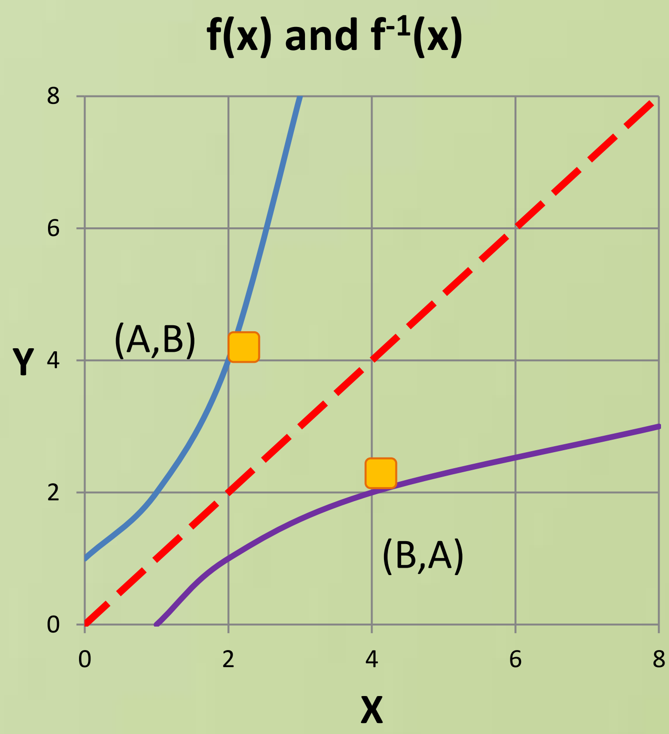
$f^{-1}(x) = \sqrt{x}$ $\qquad \frac{1}{f(x)} = \frac{1}{x^2}$

$$\sqrt{x} \neq \frac{1}{x^2}$$



Graphically

To graph inverse functions, always remember that the graph of f^{-1} is a reflection of the graph of f about the line $y = x$. For instance, if point (A,B) belongs to function f, point (B,A) belongs to function f^{-1}.



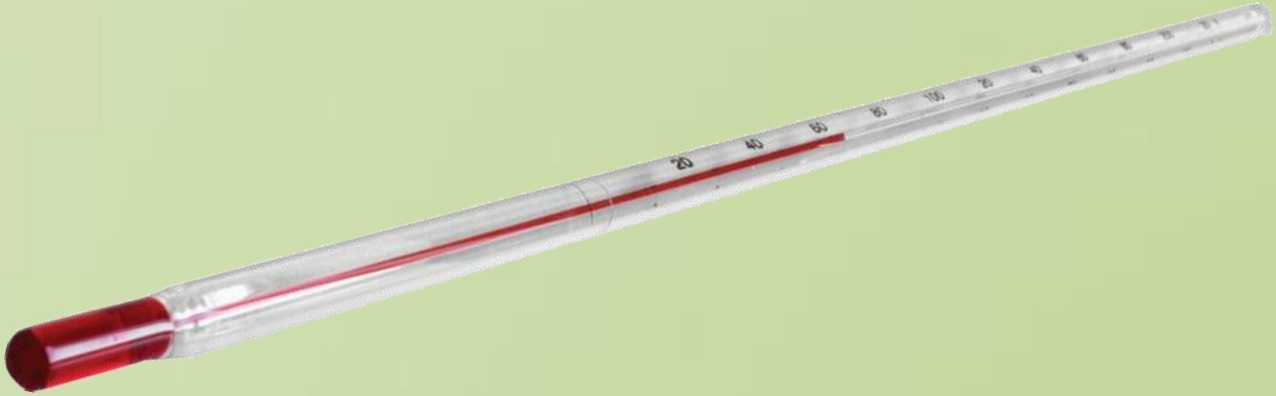
Derivative

To find the derivative of the inverse of $f(x)$, let $g(x) = f(x)$. Then, use the following equality:

$$g'(x) = \frac{1}{f'(g(x))}$$

Problem 1: Temperature

At present, there are two main scales used for measuring temperature: the Fahrenheit scale and the Kelvin scale (which is the same as the Celsius scale, calibrated so that 0 is absolute zero).



Given the temperature in degrees Fahrenheit, the Kelvin equivalent can be expressed with the formula below:

$$K = \frac{5}{9}(F - 32) + 273$$

http://www.lightlabsusa.com/images/P/lab-thermometerjpg.jpeg

Problem 1: Temperature

Find the inverse function of the Fahrenheit to Kelvin conversion formula.

Solution:

Solve for the independent variable, in this case F.

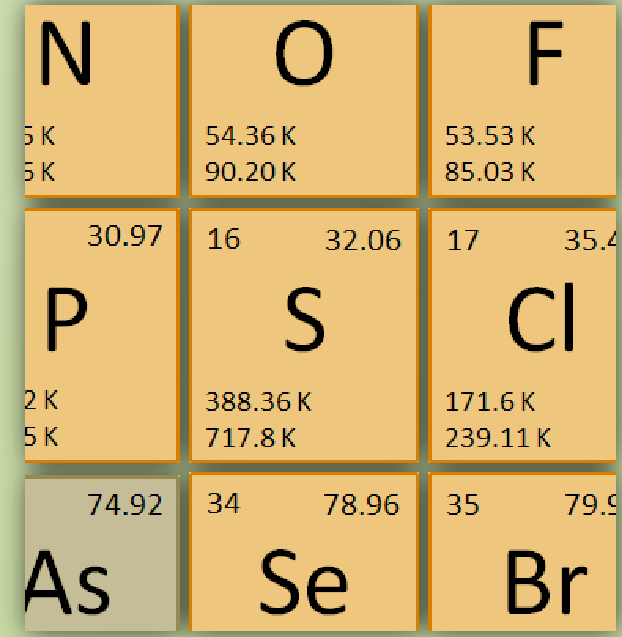
$$K = \frac{5}{9}(F - 32) + 273$$

$$K - 273 = \frac{5}{9}(F - 32)$$

$$\frac{9}{5}(K - 273) + 32 = F = K^{-1}$$

Now the temperature in Fahrenheit is a function of the temperature in Kelvins. Finding the inverse allows us to now determine the temperature in Fahrenheit if given Kelvins instead.

Problem 2: Application



The periodic table to the right displays the melting and boiling points of Sulfur. What are these temperatures in Fahrenheit?

Solution:

Since we are given Kelvins, we need to use the K^{-1} formula.

$$K^{-1} = F = \frac{9}{5}(K - 273) + 32$$

$$F = \frac{9}{5}(388.36 - 273) + 32$$

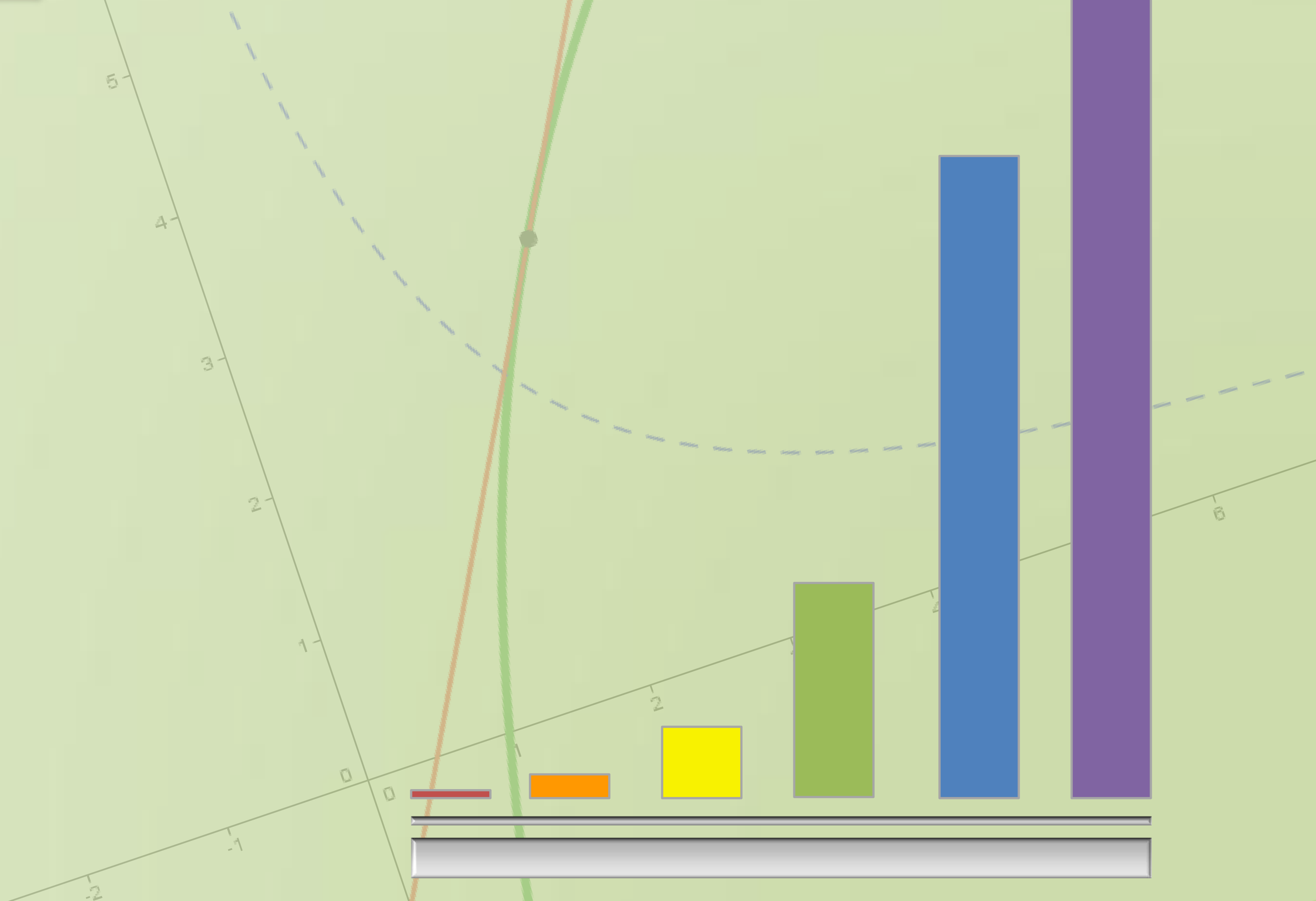
$$F = 239.65$$

$$F = \frac{9}{5}(717.8 - 273) + 32$$

$$F = 832.6$$

The melting point of sulfur is 239.65°F, and the boiling point is 832.6°F.

Exponentials



The Basics

The basic exponential form is $f(x) = a^x$ in which a is a constant.

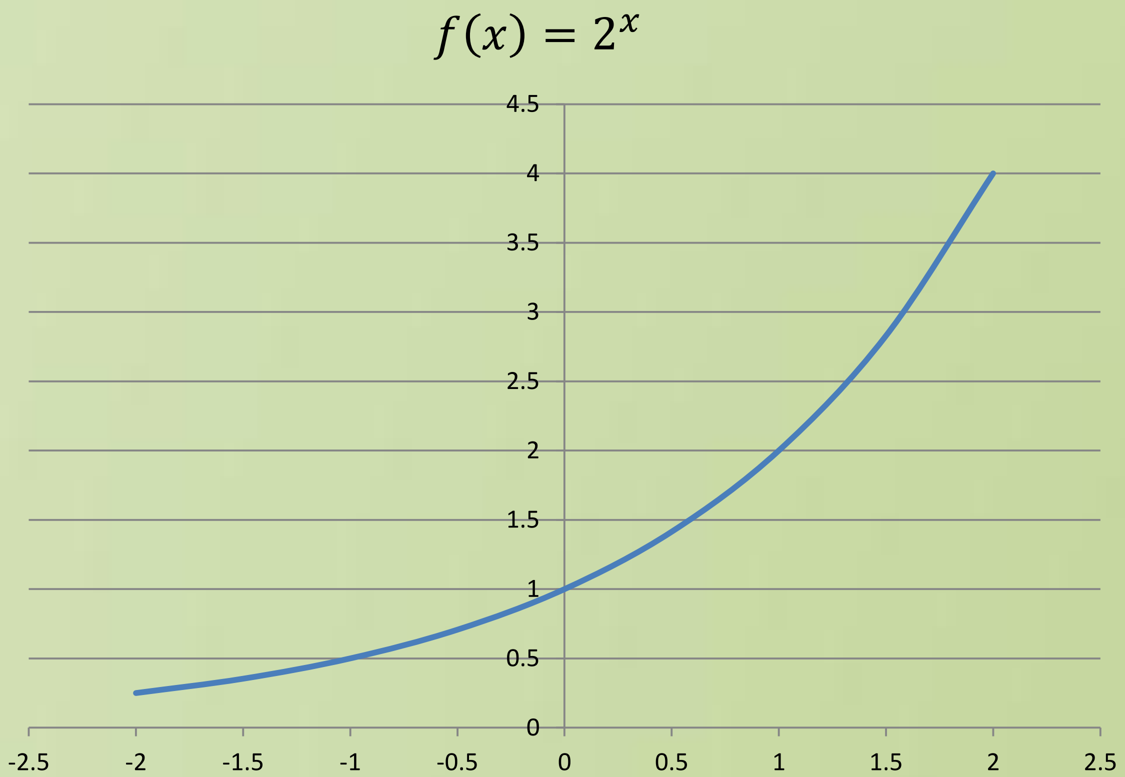
It is useful to note the following ($a > 1$):

- When $x < 0$, $f(x) < 1$
- When $x = 0$, $f(x) = 1$
- When $x > 0$, $f(x) > 1$
- $f(x) > 0$ for all x

http://www.proteanservices.com/wp-content/uploads/2010/08/bacteriaCulture.jpg

Exponential Growth

Occurs when $a > 1$



Domain: $(-\infty, \infty)$

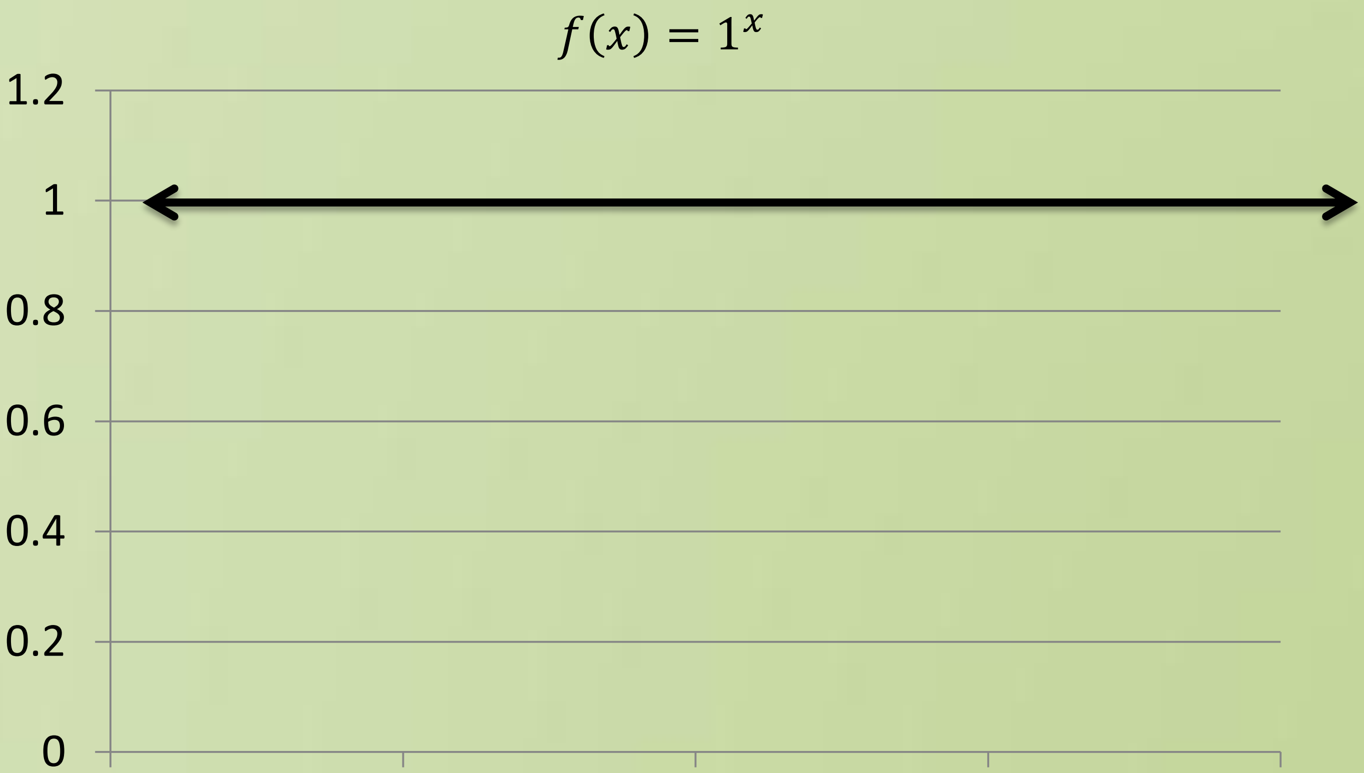
Range: $(0, \infty)$

$\lim_{x \to \infty} a^x = \infty$

$\lim_{x \to -\infty} a^x = 0$

Constant

Occurs when $a = 1$

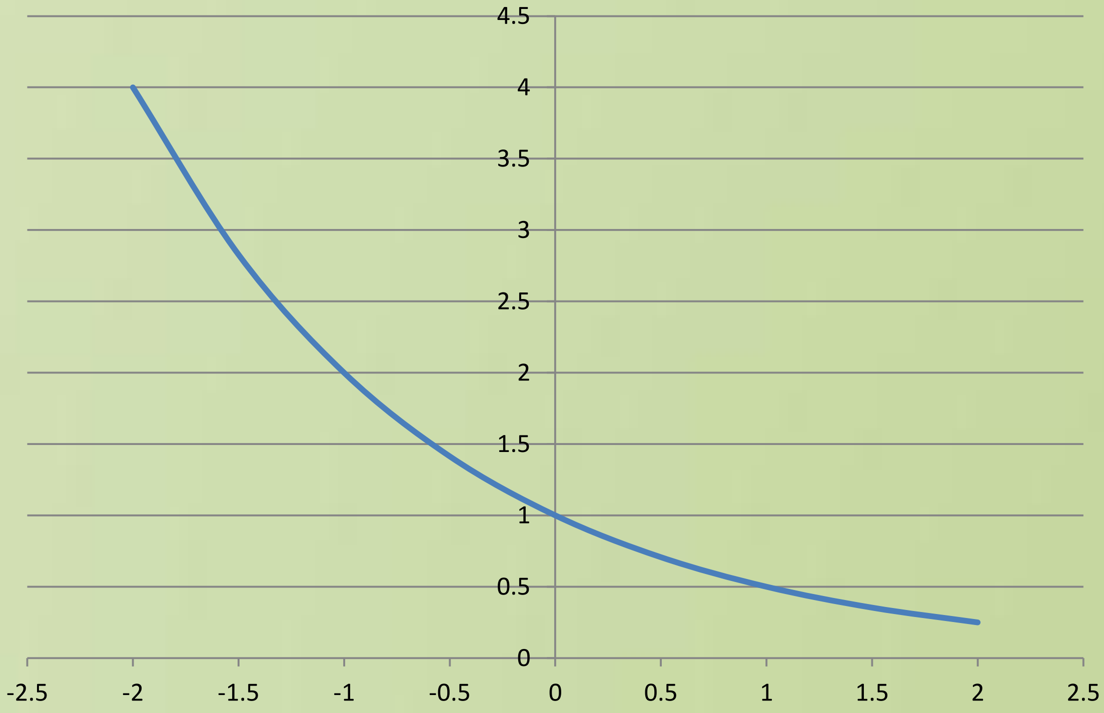


Domain: $(-\infty, \infty)$
Range: $\{1\}$

Exponential Decay

Occurs when $0 < a < 1$

$$f(x) = \left(\frac{1}{2}\right)^x$$



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

$$\lim_{x \to \infty} a^x = 0$$

$$\lim_{x \to -\infty} a^x = \infty$$

e, The Specialest Number

What is e?

- e is an irrational number that is approximately equal to 2.71828182845904523536
- It can be more accurately defined as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

e has been found in the growth patterns of nature and has many applications in mathematics.

Properties of e

Let's apply e to the concept of exponential growth:

$$f(x) = e^x$$

This graph is special because e is the only value for a in which the slope at $x = 0$ is 1.

Proof:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^h - e^0}{h}$$

$$= \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Differentiation

Derivative of $f(x) = e^x$ using the Definition of the Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = \boxed{e^x}$$

Differentiation

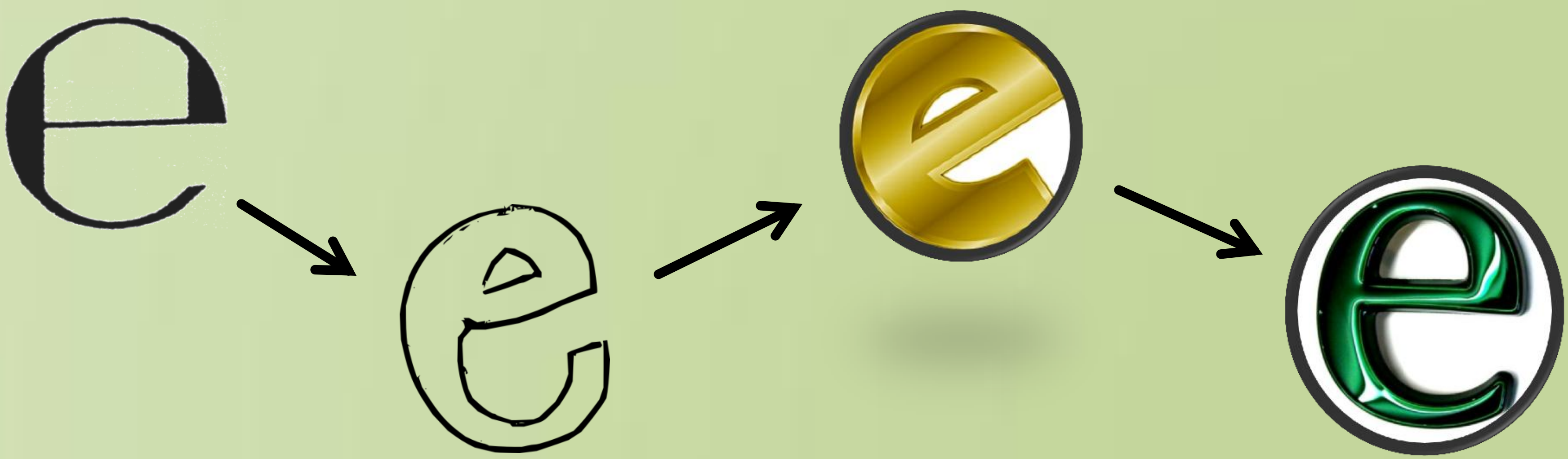
What does this tell us?

That e also has special properties regarding its differentiation.

$$(e^x)' = e^x$$

Or, more formally using the Chain Rule:

$$\frac{d}{dx}e^{u(x)} = e^{u(x)} \cdot u'$$



Compound Interest

When money is invested in the bank, it is compounded using the following formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Here, P is the principal amount, r is the interest rate, and n is the number of times compounded per year. If the money is compounded continuously (n approaches infinity), then the following is used instead:

$$A = Pe^{rt}$$

Continuously Compounded

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$$

$$= \lim_{n\to\infty} \ln\left(P\left(1+\frac{r}{n}\right)^{nt}\right)$$

$$= \lim_{n\to\infty} \ln(P) + nt\ln\left(1+\frac{r}{n}\right)$$

$$= \ln(P) + \lim_{n\to\infty} \frac{\ln\left(1+\frac{r}{n}\right)}{\frac{1}{nt}}$$

(L'Hospital) -> $= \ln(P) + \lim_{n\to\infty} -\frac{\frac{\frac{r}{n^2}}{1+\frac{r}{n}}}{-\frac{1}{n^2 t}}$

$$= \ln(P) + \lim_{n\to\infty} \frac{rt}{1+\frac{r}{n}} = \ln(P) + rt \to e^{\ln(P)+rt} = \boxed{Pe^{rt}}$$

Problem 1: Accumulation

You invest $5000 in a bank which offers a 0.5% fixed interest rate compounded monthly. How much more money will you make if you decide to wait 20 years rather than 10?

Here…

$$P = 5000$$
$$r = 0.005$$
$$n = 12$$
$$t = \{10, 20\}$$

http://i.istockimg.com/file_thumbview_approve/3294158/2/stock-photo-3294158-chart-of-coin-s-stacks.jpg

Problem 1: Accumulation

Plugging in the values and using a calculator to estimate, we get:

$$A = 5000\left(1 + \frac{.005}{12}\right)^{12\cdot\{10,\,20\}}$$

After 10 years, the amount of money in the bank is \$5256. After 20 years, though, the amount is \$5526. In 10 years, you make \$256, but in twice the time, you make more than twice the profit.

Problem 2: Double Time

You research a different bank and find that it offers 0.55% fixed interest also compounded monthly. How long would it take for your $5000 to double at this bank?

Now…

$$P = 5000$$

$$r = 0.0055$$

$$n = 12$$

$$t = ?$$

$$A = 10000$$

Problem 2: Double Time

Plugging in these values we get

$$10000 = 5000\left(1 + \frac{.0055}{12}\right)^{12\cdot t}$$

Now solve for t

$$2 = \left(1 + \frac{.0055}{12}\right)^{12t}$$

$$\ln(2) = 12(t)\ln(1.0004583)$$

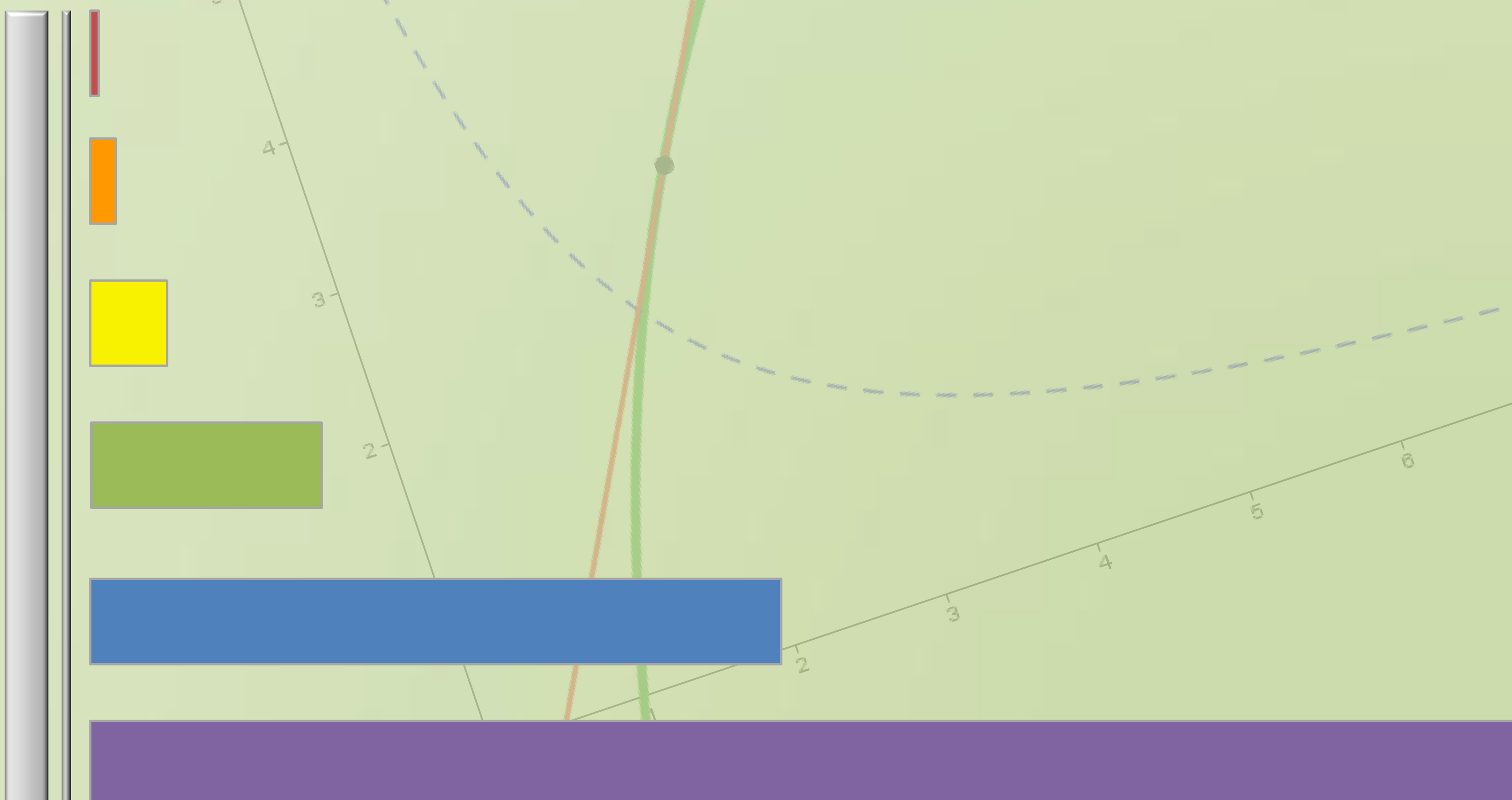
$$t = \frac{\ln(2)}{12\ln(1.0004583)} = 126.06$$

Problem 2: Double Time

Conclusion:

At a fixed interest rate of .55% compounded monthly it would take 126 years and 24 days for your deposit to double. This is true for any starting amount at this interest rate.

Logarithms

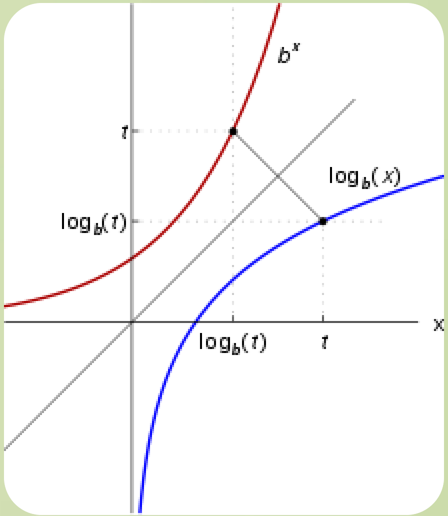


What are they?

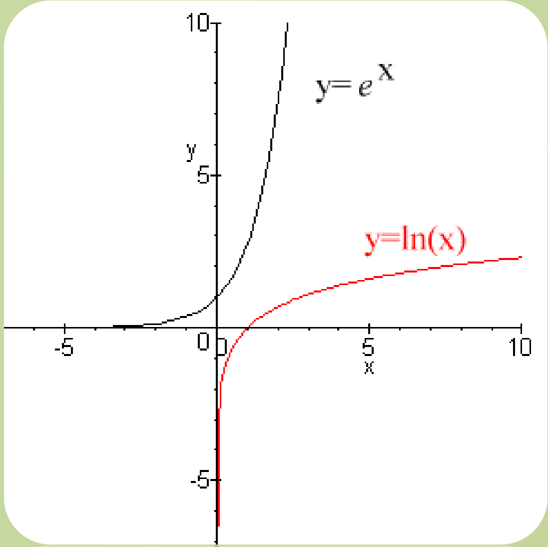
The Logarithm is simply the name of the inverse operation of exponentiation. That is…

$$\log_b b^x = x$$

Whereas the exponential functions asks how many times to multiply a certain base, logarithms ask how many times was a base multiplied to get the result.



The natural log, ln(x), is of the base *e*.



Logarithms can be used to reverse exponentiation or make what are otherwise exponential models seems linear.

Important Rules

Logarithms have very important and useful properties. If it were not for the following relationships, there would have been no need to adopt logarithms in the first place.

$$b^x = y \ \rightarrow \log_b y = x$$

$$\log_b mn = \log_b m + \log_b n$$

$$\log_b \frac{m}{n} = \log_b m - \log_b n$$

$$\log_b a^x = x \log_b a$$

$$\log_b x = \frac{\ln x}{\ln b}$$

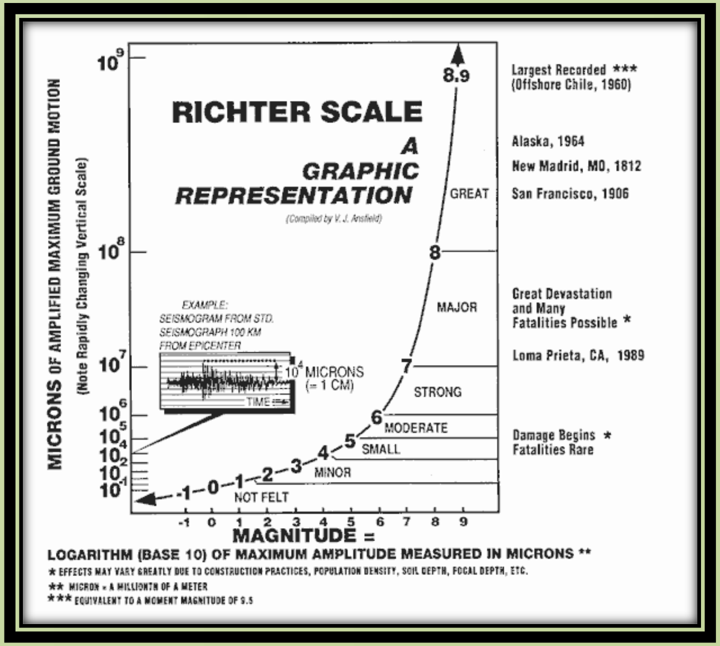
$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

Problem 1: Earthquakes

The relative strength of earthquakes is measured by a logarithmic scale known as the Richter Scale. In basic terms, a unit increment on the scale represents a tenfold multiplication in strength.

http://static.ddmcdn.com/gif/richter-scale-sam.jpg



http://www.sdgs.usd.edu/publications/maps/earthquakes/images/RichterScale.gif

Problem 1: Earthquakes

Los Angeles experiences a foreshock that registers a 5.7 on the Richter scale. It is expected that the main earthquake will be anywhere from 95 to 125 times stronger than the foreshock. Give the predicted range of the magnitude of the earthquake according to the Richter scale.

Start with a simpler problem:

A 2.0 on the Richter scale is 10 times stronger than a 1.0, and a 3.0 is 100 times stronger. Therefore, the strength of an earthquake relative to 1.0 is the following:

$$S = 10^{r-1}$$

So given the relative strength, the Richter value can be obtained by taking the $\log_{10}$ of both sides. Remember, taking the log of an exponential expression undoes the exponential if the bases are the same!

$$r = 1 + \log S$$

6.0, Nevada 2008

7.0, Haiti 2010

Note! For base 10, the subscript is usually left unwritten.

Problem 1: Earthquakes

Los Angeles experiences a foreshock that registers a 5.7 on the Richter scale. It is expected that the main earthquake will be anywhere from 95 to 125 times stronger than the foreshock. Give the predicted range of the magnitude of the earthquake according to the Richter scale.

Since the answer is a range, we have to find the magnitude given the lowest estimate and a magnitude for the highest estimate. Both of these can be found by means of the formula on the previous page…

Except!

Remember that the formula before was relative to 1.0. Since this is relative to 5.7, we must modify the formula accordingly.

$$r = 5.7 + \log S$$

$$r = 5.7 + \log 95 \qquad\qquad r = 5.7 + \log 125$$

Substitute values of S

$$\boxed{r \approx 7.68 \quad \text{to} \quad r \approx 7.80}$$

Simplify, using the calculator to approximate the log

The magnitude of the main shock is predicted to be within the range of 7.68 and 7.80 on the Richter scale.

Problem 2: Sounds

The intensity of sound is measured by a logarithmic unit called the decibel (dB). A sound of 0 dB is chosen to represent the threshold of hearing (smallest audible sound). All other sounds are then measured according to their relative intensity to that smallest sound according to the following equation:

$$dB = 10\log I$$

Therefore, a sound 10 times as intense is 10 dB. A sound 1,000,000 times as intense is 60 dB (the normal volume of a typical conversation).

As a corollary, if there are multiple sound sources, then the intensities of the sounds combine in the following way:

$$dB = 10\log(I_1 + I_2 + \cdots + I_n)$$

Problem 2: Sounds

In a filled cafeteria, every person is conversing with a volume if 62 dB. If the total sound intensity in the room is 86 dB, how many people are in the cafeteria?

Since this is a sound question dealing with decibels and more than one sound source, we will be using the second equation on the last slide.

$$dB = 10\log(I_1 + I_2 + \cdots + I_n)$$

All of the sound sources have the same volume, and therefore the number of people can be represented by one multiplier, n.

$$dB = 10\log(nI)$$

Here is the impasse. We are only given the decibels of the room (dB) and the decibels of each person. We are NOT given the *Intensity* of the sound of each person, so we actually do not have I.

http://watermarked.cutcaster.com/cutcaster-photo-100534570-Everybodys-talking-speech-bubble-communication-network.jpg

http://www.howstuffworks.com/question124.htm

http://www.engineeringtoolbox.com/adding-decibel-d_63.html

Problem 2: Sounds

In a filled cafeteria, every person is conversing with a volume if 62 dB. If the total sound intensity in the room is 86 dB, how many people are in the cafeteria?

$$62 = 10\log I$$

However, since we have the decibel level of each person, we can find I using the first sound formula. But, there's an easier way.

$$86 = 10\log(n) + 10\log(I)$$

Since $\log_b mn = \log_b m + \log_b n$

Using the log rules, the original log expression on the first page can be decomposed into two separate logs: one representing n and the other I.

$$86 = 10\log(n) + 62$$

$$24 = 10\log(n)$$

$$2.4 = \log(n)$$

$$\boxed{n \approx 251 \text{ people}}$$

As indicated by the arrow, we can now substitute the $10\log(I)$ for the decibel level of each sound, 62. From there, solve for n.

There are 251 people in the cafeteria.

http://www.howstuffworks.com/question124.htm

http://www.engineeringtoolbox.com/adding-decibel-d_63.html

Derivatives

The derivative of a logarithmic function with constant base b is the following:

$$\frac{d}{dx}\log_b f(x) = \frac{1}{f(x)\ln b} \cdot \frac{df}{dx}$$

This helps find the rate of change of, for instance, the dissolution of an acid.

The above is found by using the chain rule.

$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

If the quantity being logged were a function of x, then the chain rule says to multiply the derivative with respect to f(x) by the derivative of f(x) itself, hence the above.

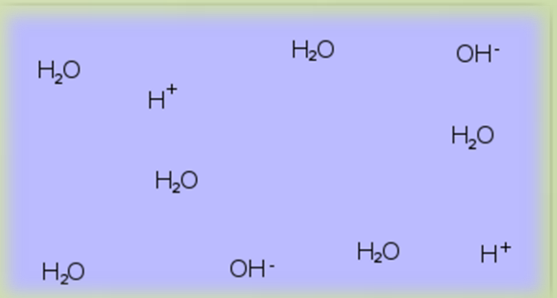
http://scotrstone.com/sitebuilder/wp-content/uploads/2010/04/chain.jpg

Problem 3: Acids and Bases

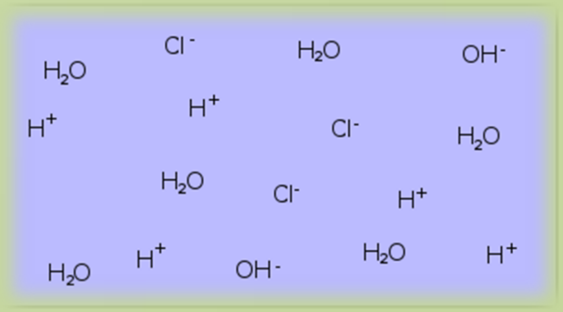
The pH scale is used to measure the acidity or alkalinity of a solution, ranging from 0 to 14. A 0 is a perfect acid and holds the greatest concentration of H^+ ions possible. A 14 is a perfect base and holds the least concentration of such ions. A solution of pH 7 is perfectly neutral, like water.

The relationship between pH and the concentration of H^+ ions is an inverse logarithmic scale. H^+ is in moles per liter:

$$pH = -\log H^+$$



Water



Acid

http://water.me.vccs.edu/courses/env211/changes/acid.gif

http://www.engineeringtoolbox.com/acids-ph-d_401.html

http://hyperphysics.phy-astr.gsu.edu/hbase/chemical/ph.html

Problem 3: Acids and Bases

Vinegar (pH 3) is being poured into a one-liter beaker of water at a rate of 5 mL per second. What rate is the pH of the mixture changing when the solution has a pH of 5?

To solve this, we need to find the pH of the mixture as a function of time. From the equation for pH, we know that it is going to be the negative logarithm of the concentration (C) of the solution.

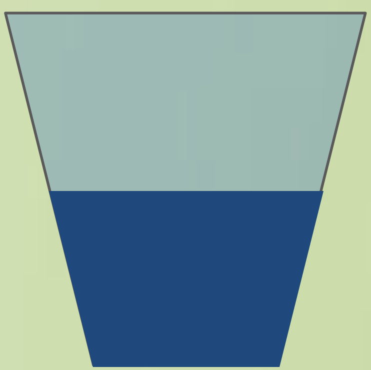
Before:

$$7 = -\log C$$

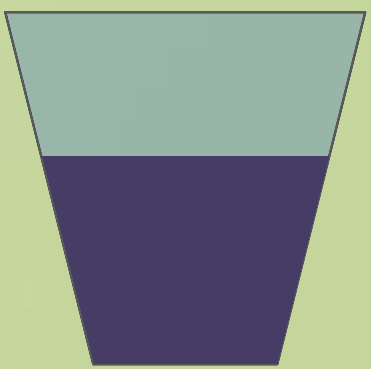
$$C = 10^{-7} \text{ mols/L}$$

After:

$$C = \frac{10^{-7} + .005(10^{-3})t}{1 + .005t}$$



Concentration after adding acid is the original amount of H^+ ions plus the amount of ions added per second all over the total volume of the solution (which is also increasing).



Problem 3: Acids and Bases

Vinegar (pH 3) is being poured into a one-liter beaker of water at a rate of 5 mL per second. What rate is the pH of the mixture changing when the solution has a pH of 5?

Therefore, the pH of the solution as a function of time is:

$$pH(t) = -\log\left(\frac{10^{-7} + .005(10^{-3})t}{1 + .005t}\right)$$

Since we need to know the *rate* of which the pH is changing when the pH is 5, we need to find both the derivative of the pH and the time at which the pH becomes 5.

$$5 = -\log\left(\frac{10^{-7} + .005(10^{-3})t}{1 + .005t}\right)$$

$$10^{-5} = \frac{10^{-7} + .005(10^{-3})t}{1 + .005t}$$

The time is found by solving the equation for t when pH(t) = 5.

$$10^{-5} + .005(10^{-5})t = 10^{-7} + .005(10^{-3})t$$

$$(10^{-5})t - (10^{-3})t = -0.00198$$

After 2 seconds, the pH is 5.

$$t = 2$$

Problem 3: Acids and Bases

Vinegar (pH 3) is being poured into a one-liter beaker of water at a rate of 5 mL per second. What rate is the pH of the mixture changing when the solution has a pH of 5?

Finally, we need to find the derivative of the logarithmic function to determine the rate of change at a particular time.

$$pH(t) = -\log\left(\frac{10^{-7} + .005(10^{-3})t}{1 + .005t}\right)$$

First, decompose the log into two logs using the division log rule.

$$pH(t) = -(\log(10^{-7} + .005(10^{-3})t) - \log(1 + .005t))$$

Begin taking the derivative. Each log is treated separately, and each uses the logarithmic differentiation rule.

$$pH(t) = \log(1 + .005t) - \log(10^{-7} + .005(10^{-3})t)$$

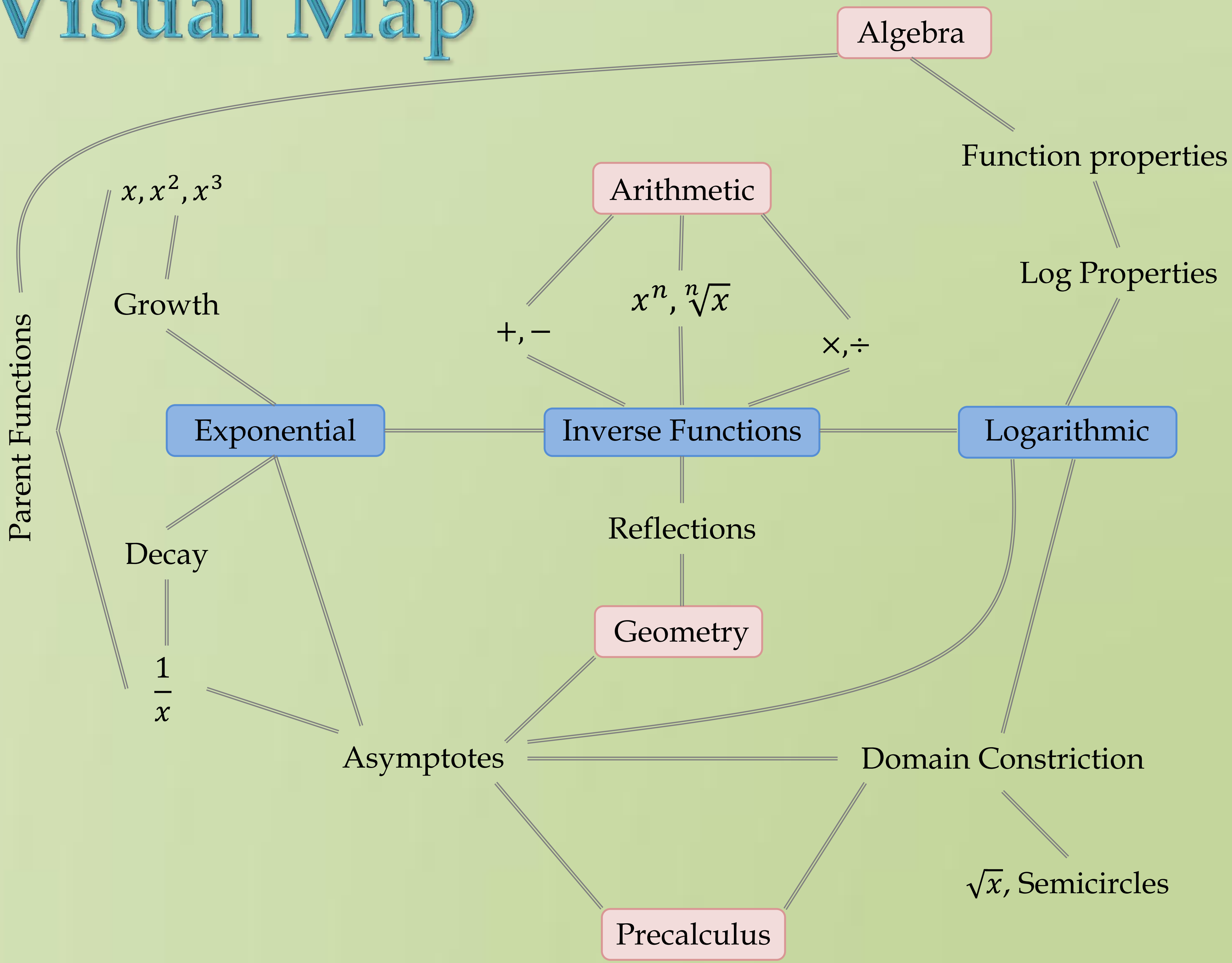
$$pH'(t) = \frac{1}{(1 + .005t) \cdot \ln(10)} \cdot .005 - \frac{1}{(10^{-7} + .005(10^{-3})t) \cdot \ln(10)} \cdot .005(10^{-3})$$

Finally, plug in 2 seconds, the time when the pH is 5:

$$\boxed{pH'(2) \approx -0.213}$$

When the pH of the solution is 5, the pH is decreasing by 0.213 every second.

Visual Map



Algebra
Function properties
Arithmetic
x, x^2, x^3
Log Properties
Growth
$x^n, \sqrt[n]{x}$
$+, -$
$\times, \div$
Parent Functions
Exponential
Inverse Functions
Logarithmic
Reflections
Decay
Geometry
$\frac{1}{x}$
Asymptotes
Domain Constriction
$\sqrt{x}$, Semicircles
Precalculus

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