Math 220 Exam 3 Practice Problems
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The following are some representative problems, from old exams, on the material for Exam 3. They are not meant to include examples of all possible problems that may be on the exam. You will also want to be prepared to work any problems similar to homework problems or those from class.

domain, codomain

Examples
$f(1)=2$
$f(2)=4$
$f(3)=4$

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases}$

(a) Is f one-to-one? Justify your answer.

No.

$f(2) = f(3)$ and $2 \neq 3$

(b) Is f onto? Justify your answer.

No.

For example, $23 \in \mathbb{Z}$ and $23 \notin \operatorname{ran} f$ since $\operatorname{ran} f = \mathbb{Z}_{even}$ (the set of even integers)

Proof that $\operatorname{ran} f = \mathbb{Z}_{even}$: Let $m \in \mathbb{Z}_{even}$. Then $m = f(m-1)$ ~~for an odd integer n~~ since $m-1$ is an odd integer.
So $m \in \operatorname{ran} f$, which shows that $\mathbb{Z}_{even} \subseteq \operatorname{ran} f$.
Let $m \in \operatorname{ran} f$, so $m = 2n$ for an even integer n, or $m = n+1$ for an odd integer n, and in either case, $m \in \mathbb{Z}_{even}$. So $\operatorname{ran} f \subseteq \mathbb{Z}_{even}$.
It follows that $\operatorname{ran} f = \mathbb{Z}_{even}$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (-y, x) \quad \text{and} \quad g(x, y) = (x + 2, y - 1)$$

for all $(x, y) \in \mathbb{R}^2$. Find $f \circ g$ and $g \circ f$. (That is, find formulas for $(f \circ g)(x, y)$ and $(g \circ f)(x, y)$.)

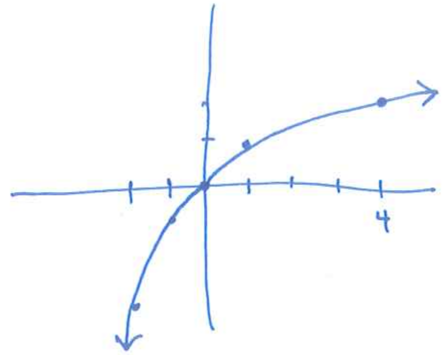
$$(f \circ g)(x,y) = f(g(x,y)) = f(x+2, y-1) = (-(y-1), x+2) = (-y+1, x+2)$$

$$(g \circ f)(x,y) = g(f(x,y)) = g(-y, x) = (-y+2, x-1)$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

Is f invertible? If so, find f^{-1}. If not, explain why not.



yes.

Case $x \geq 0$. $f^{-1}(x) = x^2$

(To find this: Write $y = \sqrt{x}$, switch order for $x = \sqrt{y}$, solve for y
$y = x^2$)

Case $x < 0$ $f^{-1}(x) = -\sqrt{-x}$

(Write $y = -x^2$, switch order for $x = -y^2$, solve for y

$y^2 = -x$

$y = \pm\sqrt{-x}$

take $y = -\sqrt{-x}$ since we want the inverse of f, which takes negative numbers to negative numbers.)

$$f^{-1}(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0 \end{cases}$$

(Prove f^{-1} is indeed the inverse of f:

If $x \geq 0$: $f(f^{-1}(x)) = f(x^2) = \sqrt{x^2} = x$

$f^{-1}(f(x)) = \cdots = x$

If $x < 0$: $f(f^{-1}(x)) = \cdots = x$

$f^{-1}(f(x)) = \cdots = x$)

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$f(n) = \begin{cases} n, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd} \end{cases}$$

$\text{ran} f = \mathbb{Z}_{\text{even}}$

(a) Find $f(\{1, 2, 3, 4\})$.

$= \{2, 6, 4\}$

(b) Find $f^{-1}(\{1, 2, 3, 4\})$. $= \{n \in \mathbb{Z} \mid f(n) \in \{1, 2, 3, 4\}\}$

not function value

$f^{-1}(\{1,2,3,4\}) = \{1, 2, 4\}$

since $f(1) = 2$, $f(2) = 2$, $f(4) = 4$

(c) Is f one-to-one? Justify your answer.

No:

$f(1) = f(2)$ and $1 \neq 2$

(d) Is f onto? Justify your answer.

No, $\text{ran} f = \mathbb{Z}_{\text{even}}$

so e.g. $1 \notin \text{ran} f$

5. Let A and B be sets and $f : A \to B$. Let X and Y be subsets of A.

(a) Prove that if f is *one-to-one*, then $f(X \cap Y) = f(X) \cap f(Y)$.

Prove that $f(X \cap Y) \subseteq f(X) \cap f(Y)$: Let $b \in f(X \cap Y)$, i.e. $b = f(z)$ for some $z \in X \cap Y$. Since $z \in X \cap Y$, we know that $z \in X$ and $z \in Y$.
Since $b = f(z)$ and $z \in X$, it follows that $b \in f(X)$.
Since $b = f(z)$ and $z \in Y$, it follows that $b \in f(Y)$.
Therefore $b \in f(X) \cap f(Y)$, and so $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

Prove that $f(X) \cap f(Y) \subseteq f(X \cap Y)$: Let $b \in f(X) \cap f(Y)$, i.e. $b \in f(X)$ and $b \in f(Y)$.
So $b = f(x)$ for some $x \in X$ and $b = f(y)$ for some $y \in Y$.
It follows that $f(x) = f(y)$. Since f is one-to-one, this implies $x = y$.
So $x \in X \cap Y$. That is, $b = f(x)$ for $x \in X \cap Y$. This means that $b \in f(X \cap Y)$. Therefore $f(X) \cap f(Y) \subseteq f(X \cap Y)$.

We have proven that $f(X \cap Y) = f(X) \cap f(Y)$.

(b) Give an example of sets A, B, X, Y and a function f for which $f(X \cap Y) \neq f(X) \cap f(Y)$. (By (a), f cannot be one-to-one.)

Try
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$

Let
$X = \{2\}$, $Y = \{-2\}$.
So $X \cap Y = \phi$, and $f(X \cap Y) = \phi$. Then
$f(X) = \{4\}$, $f(Y) = \{4\}$, so
$f(X) \cap f(Y) = \{4\}$.
Therefore $f(X \cap Y) \neq f(X) \cap f(Y)$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ x+1, & \text{if } x < 0 \end{cases}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 1 - x$ for all $x \in \mathbb{R}$. Find $f \circ g$.

$$(f \circ g)(x) = f(g(x))$$
$$= f(1-x)$$

Case 1 $\quad 1-x \geq 0$ (i.e. $x \leq 1$)

$$f(1-x) = (1-x)^2$$

Case 2 $\quad 1-x < 0$ (i.e. $x > 1$)

$$f(1-x) = (1-x)+1 = 2-x$$

$$(f \circ g)(x) = \begin{cases} (1-x)^2 & \text{if } x \leq 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

7. Let X, Y, Z be sets. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Prove that $g \circ f : X \to Z$ is invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let $x \in X$. Then

$$
\begin{aligned}
(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) &= (f^{-1} \circ g^{-1})\big((g \circ f)(x)\big) \\
&= f^{-1}\big(g^{-1}(g(f(x)))\big) \\
&= f^{-1}(f(x)) && \text{since } g^{-1} \text{ is the inverse of } g \\
&= x && \text{since } f^{-1} \text{ is the inverse of } f.
\end{aligned}
$$

Let $z \in Z$. Then

$$
\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1})(z) &= g\big(f(f^{-1}(g^{-1}(z)))\big) \\
&= g(g^{-1}(z)) && \text{since } f^{-1} \text{ is the inverse of } f \\
&= z && \text{since } g^{-1} \text{ is the inverse of } g.
\end{aligned}
$$

Therefore $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$, and $g \circ f$ is invertible.