MATH 220 Exam 3 Solutions

1. $(f \circ g)(x,y) = f(g(x,y)) = f(x+y) = (x+y, x+y)$

$(g \circ f)(x) = g(x,x) = x+x = 2x$

2. 1-1: Let $x_1, x_2 \in \mathbb{R}$ for which $f(x_1) = f(x_2)$. Then $1 - x_1^3 = 1 - x_2^3$, so $x_1^3 = x_2^3$, which implies $x_1 = x_2$. Therefore f is 1-1.

onto: Let $y \in \mathbb{R}$. Set $x = \sqrt[3]{1-y}$. Then $f(x) = 1 - (\sqrt[3]{1-y})^3 = 1 - (1-y) = y$, so $y \in \operatorname{ran} f$. Therefore $\mathbb{R} \subseteq \operatorname{ran} f$. By definition of f (since $1 - x^3 \in \mathbb{R}$ for all $x \in \mathbb{R}$), $\operatorname{ran} f \subseteq \mathbb{R}$. So $\operatorname{ran} f = \mathbb{R}$, that is, f is onto.

(b) $f^{-1}(x) = \sqrt[3]{1-x}$

3. (a) Assume $g \circ f$ is one-to-one. Let $x_1, x_2 \in X$ for which $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, this implies that $x_1 = x_2$. Therefore f is one-to-one.

(b) $f: [0, \infty) \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x}$, $g(x) = x^2$. Then $(g \circ f)(x) = x$, so $g \circ f$ is one-to-one. However g is not one-to-one since e.g. $g(-2) = g(2)$.

4. $f[A] = \{f(x,y) \in \mathbb{R} \mid (x,y) \in A\} = \{x^2 + y^2 \in \mathbb{R} \mid x \in \mathbb{R}, y = 0\} = \{x^2 \in \mathbb{R} \mid x \in \mathbb{R}\} = [0, \infty)$

$f^{-1}[B] = \{(x,y) \in \mathbb{R} \mid f(x,y) \in B\} = \{(x,y) \in \mathbb{R} \mid f(x,y) \leq 1\} = \{(x,y) \in \mathbb{R} \mid x^2 + y^2 \leq 1\}$

5. Assume that f is 1-1 and $f[A] \subseteq f[B]$. Let $a \in A$. Then $f(a) \in f[A]$. Since $f[A] \subseteq f[B]$, that implies $f(a) \in f[B]$. Therefore there exists an element $b \in B$ such that $f(a) = f(b)$. Since f is 1-1, this implies $a = b$. Therefore $a \in B$, and so $A \subseteq B$.

6. F; Counterexample is any function that is not onto, e.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

T (by definition of $f^{-1}[B]$ and domain of a function)

F; Counterexample is again any function that is not onto, e.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

T (this was a theorem from class)

T (this was a homework problem)