An Introduction to Hochschild Cohomology

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Abstract. This is an advanced graduate level textbook, designed both as an introduction for students to the subject of Hochschild cohomology, and as a resource for mathematicians who use Hochschild cohomology in their work. The text begins with definitions, properties, and many examples. The structure of Hochschild cohomology as a Gerstenhaber algebra is explored in detail. Many other topics of current interest are presented, including smoothness and duality, algebraic deformation theory, infinity structures, support varieties, and connections to Hopf algebra cohomology. Also included is an appendix containing some needed homological background.

Contents

Introduction

Homological techniques first arose in topology, in work of Poincaré at the end of the 19th century. They appeared in algebra several decades later in the 1940's, when Hochschild [**Hoc45**] introduced (co)homology of algebras and Eilenberg and Mac Lane [**EL47**] introduced (co)homology of groups. Since that time, both Hochschild cohomology and group cohomology, as they came to be called, have become indispensable in algebra, algebraic topology, representation theory, and other fields. They remain active areas of research, with frequent discoveries of new applications. There are excellent books on group cohomology such as [**AM94, Ben91a, Ben91b, Bro82, CTVE03, Eve91**]. These are good references for those working in the field and are also important resources for those learning group cohomology in order to begin using it in their research. There are fewer such resources for Hochschild (co)homology, notwithstanding some informative chapters in the books [**Lod98, Wei94**]. This book aims to begin filling the gap.

Hochschild cohomology records meaningful information about rings and algebras. It is used to understand their structure and deformations, and to identify essential information about their representations. In this book, we take a concrete approach, giving many early examples that reappear later in various applications. We begin in Chapter 1 with Hochshild's own definitions from [**Hoc45**], only slightly rephrased in modern terminology and notation, and then expand to include arbitrary resolutions under suitable conditions. We present important contributions of Gerstenhaber [**Ger63**] beginning in the 1960's that lead us now to think of Hochschild cohomology as a Gerstenhaber algebra; that is, it has both an associative product and a nonassociative Lie bracket. Many properties of Hochschild cohomology that are used today can be seen in these classical definitions of Hochschild and

Gerstenhaber. In Chapter 2 we present several different types of examples: Koszul algebras, algebras given by quivers and relations, and algebras built from others such as skew group algebras and (twisted) tensor products.

Current applications and developments in Hochschild cohomology include the following, guiding our choice of topics to explore in detail in the rest of book.

In noncommutative geometry, notions of smoothness and some other classical geometric notions may be viewed as essentially homological properties of commutative function algebras, allowing interpretations of them in noncommutative settings via Hochschild cohomology. We present these and related ideas in Chapter 3, including noncommutative differential forms, Van den Bergh duality, Calabi-Yau algebras, Connes differential, and Batalin-Vilkovisky structure.

Understanding how some algebras may be viewed as deformations of others calls on Hochschild cohomology, as explained in Chapter 4. It is in algebraic deformation theory that the Lie structure on Hochschild cohomology arises naturally, and we spend some additional time studying this important but elusive structure in detail in Chapter 5. Further probing the associative and Lie algebra structures on Hochschild cohomology and related complexes uncovers infinity algebras. There, the traditional binary operations are layered with newer n-ary operations which in turn have important implications for the traditional algebra structure. We give a brief introduction to infinity structures and their applications to Hochschild cohomology in Chapter 6.

In representation theory, support varieties may sometimes be defined in terms of Hochschild cohomology; these are geometric spaces assigned to modules that encode representation-theoretic information. Support varieties for finite dimensional algebras are introduced and explored in Chapter 7. This theory began as an application of finite group cohomology, and there are strong connections between Hochschild cohomology and group cohomology that we analyze in the more general setting of Hopf algebras in Chapter 8. Hopf algebras are those whose categories of modules are tensor categories, and include many examples of interest such as group algebras, universal enveloping algebras of Lie algebras, and quantum groups. Relationships between Hochschild cohomology and Hopf algebra cohomology lead to better understanding of both and of all their applications.

We include an appendix with needed background material from homological algebra. The appendix is largely self-contained, however, proofs are omitted, and instead the reader is referred to standard homological algebra textbooks such as [**HS71, Wei94**] for proofs and more details.

This book is not intended to be a comprehensive treatment of the whole subject of Hochschild cohomology, as this subject has expanded well beyond the reach of a single book. Necessarily some important topics are left out. For example, we do not treat Tate-Hochschild cohomology, relative Hochschild cohomology, higher Hochschild cohomology, connections to cyclic (co)homology and K-theory, Hochschild cohomology of abelian categories, singular Hochschild cohomology, topological Hochschild cohomology, nor operads. Also, here we will almost exclusively work with algebras over a field, in order to take advantage of a great array of good properties and current applications that we wish to cover, although one can work over a ground ring that is not a field. Hochschild homology is an important subject in its own right, particularly for commutative algebras where it also has a ring structure, and we spend only a little time on it in this book. More can be found in standard references such as [**Lod98, Wei94**].

This book is written for graduate students and working mathematicians to learn about Hochschild cohomology, and for those who want a reference for many facts that are currently only found in research papers. The main prerequisite for students is a graduate course in algebra. It would also be helpful to have taken further introductory courses in homological algebra or algebraic topology and in representation theory, or else have done some reading in these subjects. However, all of the required homological algebra background is summarized in the appendix, with references, and a motivated reader might rely solely on this as homological algebra background. Beyond the first two chapters, the remaining chapters are largely independent of each other, and so there are many choices one can make to give a one semester graduate course based on this book. A one semester course could start with a treatment of Chapter 1 and selected sections from Chapter 2, possibly including material from the appendix depending on the background of the students. Then one could choose to focus on a subset of the remaining chapters: A course with a focus on noncommutative geometry inspired by important results in commutative geometry could continue with Chapter 3; a course with a focus on algebraic deformation theory and related structures could instead continue with Chapter 4 and the related Chapters 5 and 6, as time allowed; a course with a focus on Hopf algebras, group algebras, and support varieties could instead continue with Chapters 7 and 8. A full year course might include the whole book and time for a complete introduction to or review of homological algebra based on the appendix.

This book came into being as an aftereffect of some lecture series that I gave and through interactions with many people. I first thank Universidad de Buenos Aires, and especially Andrea Solotar and her students, postdocs, and colleagues, for hosting me for several weeks in 2010. During that time I gave a short course on Hopf algebra cohomology that led to an early version of Chapter 8 on which they gave me valuable feedback. I thank the Morningside Center in Beijing and the organizers and students of a workshop there in 2011 for the opportunity to give lectures on support varieties that expanded into the current Chapter 7. I thank the Mathematisches Forschungsinstitut Oberwolfach for its hospitality during several workshops where the idea for this book began in discussions with Karin Erdmann and Henning Krause.

Most of this book was written during the academic year 2016–17 that I spent at the University of Toronto visiting Ragnar-Olaf Buchweitz. I am very grateful to the University of Toronto for hosting me and to Buchweitz and his team during my time in Toronto for the many helpful conversations and stimulating seminar talks. This interaction significantly influenced some of my choices of topics for the book. I particularly thank Buchweitz and his students Benjamin Briggs and Vincent Gélinas for many pointers on smooth algebras and infinity algebras that helped me prepare Chapters 3 and 6. I also had fruitful discussions with Cris Negron and Yury Volkov in relation to the material in Chapter 5. Special thanks go to my long time collaborator Anne Shepler for many conversations over the years that led to my current point of view on Chapter 4.

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Chapter 1

Historical Definitions and Basic Properties

We begin with Hochschild's historical definition of cohomology for algebras [**Hoc45**], now called Hochschild cohomology. We present some additional structure found by Gerstenhaber [**Ger63**] under which we now say that Hochschild cohomology is a Gerstenhaber algebra. These early developments were essentially based on one choice of chain complex that we now call the bar complex. Later work invokes many other complexes, depending on the setting, for example, when working over a field. We also include in this chapter some examples and discussion of the structure of Hochschild cohomology that takes advantage of other such chain complexes.

1.1. Definitions of Hochschild homology and cohomology

For now, we let k be a commutative ring (with 1) and A a k-algebra. Denote the multiplicative identity of A also by 1, identified with the multiplicative identity of k via the unit map $k \hookrightarrow A$ given by $c \mapsto c \cdot 1$ for all $c \in k$. Denote by A^{op} the *opposite algebra* of A; this is A as a module over k, with multiplication $a \cdot_{\mathrm{op}} b = ba$ for all $a, b \in A$. Tensor products will be taken over k, unless otherwise indicated, that is, $\otimes = \otimes_k$. Let $A^e = A \otimes A^{\mathrm{op}}$, with the tensor product multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$. (For the notation, we are really taking b_1, b_2 to be elements of A^{op}, but since the underlying vector spaces are the same, we write $b_1, b_2 \in A$ where convenient.) We call A^e the *enveloping algebra* of A.

By an A-*bimodule*, we mean a module M over k that is both a left and a right A-module for which $(a_1m)a_2 = a_1(ma_2)$ for all $a_1, a_2 \in A$ and $m \in M$, and the left and right actions of k induced by the unit map $k \hookrightarrow A$ agree. Thus an A-bimodule M is equivalent to a left A^e-module, where we define

$$(a \otimes b) \cdot m = amb$$

for all $a, b \in A$ and $m \in M$. It is also equivalent to a right A^e-module where the action is defined by $m \cdot (a \otimes b) = bma$ for all $a, b \in A$ and $m \in M$. We will use both structures in the sequel, but for simplicity, when we refer to a module we generally mean a left module unless otherwise specified.

Note that the algebra A is itself a (left) A^e-module (equivalently, an A-bimodule) under left and right multiplication: $(a \otimes b) \cdot c = acb$ for all $a, b, c \in A$. More generally, let $A^{\otimes n} = A \otimes \cdots \otimes A$ (that is, n factors of A). This tensor power of A is an A^e-module (equivalently, an A-bimodule) by letting

$$(a \otimes b) \cdot (c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

Consider the following sequence of A-bimodules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\pi} A \to 0, \tag{1.1.1}$$

where π is multiplication, $d_1(a \otimes b \otimes c) = ab \otimes c - a \otimes bc$ for all $a, b, c \in A$, and in general

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{1.1.2}$$

for all $a_0, \ldots, a_{n+1} \in A$. One may check directly that (1.1.1) is a complex, that is, that $d_{n-1}d_n = 0$ for all n. Moreover, it is exact, as a consequence of existence of a contracting homotopy:

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \tag{1.1.3}$$

for all n and all $a_0, \ldots, a_{n+1} \in A$. We write $B_n(A) = A^{\otimes(n+2)}$ for $n \geq 0$ and often consider the truncated complex associated to (1.1.1):

$$B(A): \qquad \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \to 0. \tag{1.1.4}$$

This is the *bar complex* of the A^e-module A. As a complex, its homology is concentrated in degree 0, where it is simply A, as a consequence of exactness of (1.1.1). Sometimes we use the isomorphism of left A^e-modules

$$A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n}, \tag{1.1.5}$$

given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes (a_1 \otimes \cdots \otimes a_n)$ for all $a_0, \ldots, a_{n+1} \in A$. The action of A^e on $A^e \otimes A^{\otimes n}$ is simply multiplication on the leftmost factor A^e.

Remark 1.1.6. The term bar complex arose historically as some authors abbreviate the tensor products $a_1 \otimes \cdots \otimes a_n$ as $a_1|\cdots|a_n$, particularly when denoting a set of free generators of a free module, as begun by Eilenberg and Mac Lane.

Let M be an A-bimodule. Consider the complex

$$M \otimes_{A^e} B(A) \tag{1.1.7}$$

of k-modules with differentials $1_M \otimes d_n$, where $B(A)$, d_n are defined in (1.1.4), (1.1.2), and 1_M is the identity map on M. We use the k-module isomorphism $M \otimes_{A^e} B_n(A) \xrightarrow{\sim} M \otimes A^{\otimes n}$ given by $m \otimes_{A^e} a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n$ for all $m \in M$ and $a_0, \ldots, a_{n+1} \in A$ (the inverse isomorphism is given by $m \otimes a_1 \otimes \cdots \otimes a_n \mapsto m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$). Combined with the isomorphism (1.1.5), we find that the differential on $M \otimes A^{\otimes n}$ corresponding to $1_M \otimes d_n$ on $M \otimes_{A^e} A^{\otimes(n+2)}$ is

$$\begin{aligned} &\delta_n(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\ &\quad = ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\qquad + (-1)^n a_n m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Definition 1.1.8. The *Hochschild homology* $\mathrm{HH}_*(A, M)$ of A with coefficients in an A-bimodule M is the homology of the complex (1.1.7), equivalently

$$\mathrm{HH}_n(A, M) = \mathrm{H}_n(M \otimes A^{\otimes \bullet}),$$

that is, $\mathrm{HH}_n(A, M) = \mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n+1})$ for all $n \geq 0$, taking $\delta_0 \equiv 0$, and differentials δ_n as given above for $n > 0$. Let $\mathrm{HH}_*(A, M) = \oplus_{n \geq 0} \mathrm{HH}_n(A, M)$.

We have chosen the notation HH to denote Hochschild homology (and cohomology below) in order to distinguish it from other versions of (co)homology that we will use later. In other texts, it is often denoted with a single letter H.

Next consider the complex

$$\mathrm{Hom}_{A^e}(B(A), M) \tag{1.1.9}$$

of k-modules with differentials d_n^*, where $d_n^*(f) = f d_n$ for all functions f in $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, M)$. We use the k-module isomorphism

$$\mathrm{Hom}_{A^e}(B_n(A), M) \xrightarrow{\sim} \mathrm{Hom}_k(A^{\otimes n}, M) \tag{1.1.10}$$

given by $g \mapsto (a_1 \otimes \cdots \otimes a_n \mapsto g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1))$ for all $g \in \mathrm{Hom}_{A^e}(B_n(A), M)$ and $a_1, \ldots, a_n \in A$. (The inverse isomorphism is given by $f \mapsto (a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_0 f(a_1 \otimes \cdots \otimes a_n)a_{n+1})$.) We thus write (abusing notation by identifying functions that correspond under the isomorphism (1.1.10):

$$\begin{aligned} d_n^*(f)(a_1 \otimes \cdots \otimes a_n) &= a_1 f(a_2 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}) a_n \end{aligned}$$

for all $f \in \operatorname{Hom}_k(A^{\otimes n}, M)$ and $a_1, \ldots, a_n \in A$.

Definition 1.1.11. The *Hochschild cohomology* $\mathrm{HH}^*(A, M)$ of A with coefficients in an A-bimodule M is the cohomology of the complex (1.1.9), equivalently

$$\mathrm{HH}^n(A, M) = \mathrm{H}^n(\operatorname{Hom}_k(A^{\otimes \bullet}, M)),$$

that is, $\mathrm{HH}^n(A, M) = \operatorname{Ker}(d_{n+1}^*)/\operatorname{Im}(d_n^*)$ for all $n \geq 0$, where d_0^* is taken to be the zero map, and differentials d_n^* are as given above for $n > 0$. Elements in $\operatorname{Ker}(d_{n+1}^*)$ are *Hochschild n-cocycles* and those in $\operatorname{Im}(d_n^*)$ are *Hochschild n-coboundaries*. Let $\mathrm{HH}^*(A, M) = \oplus_{n \geq 0} \mathrm{HH}^n(A, M)$.

Consider $M = A$ to be an A-bimodule under left and right multiplication. The resulting Hochschild homology and cohomology of A with coefficients in A is sometimes abbreviated

$$\mathrm{HH}_*(A) = \mathrm{HH}_*(A, A) \quad \text{and} \quad \mathrm{HH}^*(A) = \mathrm{HH}^*(A, A).$$

A disadvantage of this notation in the case of cohomology is that it appears to indicate a functor, however HH^* is not a functor: Fixing either of the two arguments, HH^* is a contravariant functor in the first and a covariant functor in the second. For simplicity, we adopt this abbreviated notation in spite of this disadvantage.

Remarks 1.1.12. (i) The definitions show that Hochschild homology and cohomology may be realized as relative Tor and relative Ext. See [**Wei94**, Chapter 9] for details.

(ii) In the case that k is a field (the case of focus in this book), $B(A)$ is a free left A^e-module resolution of A, called the *bar resolution*. Thus

$$\mathrm{HH}_n(A, M) \cong \operatorname{Tor}_n^{A^e}(M, A) \quad \text{and} \quad \mathrm{HH}^n(A, M) \cong \operatorname{Ext}_{A^e}^n(A, M). \tag{1.1.13}$$

More generally, if A is flat over k, the first isomorphism holds, and if A is projective over k, the second holds. We will use these equivalent definitions of Hochschild homology and cohomology. An advantage is that we may thus choose any flat (respectively, projective) resolution of A as an A^e-module to define Hochschild homology (respectively, cohomology). Depending on the algebra A, there may be more convenient resolutions than the bar resolution, which is quite large and not conducive to explicit computation. The bar resolution may also obscure important information that stands out in other resolutions tailored more closely to specific algebras. However, the bar resolution is very useful theoretically, as we will also see. Also useful is a variant of the bar resolution for algebras A free over k, called the *reduced bar resolution*: $\overline{B}(A)$ with $\overline{B}_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A$ where $\overline{A} = A/k$ is a free k-module quotient. One can check that the differentials from the bar resolution $B(A)$ factor through $\overline{B}(A)$ to give differentials under which this is also a free resolution of A as an A^e-module.

Hochschild cohomology $\mathrm{HH}^*(A)$ contains substantial information about the algebra A, some of which we will see here, and some in later chapters. It is invariant under some standard equivalences on rings: In case k is a field, invariance under Morita equivalence is automatic since this is an equivalence of module categories and Hochschild cohomology is given by Ext in this case. See, e.g., [**Ben91b**, Theorem 2.11.1] for details. For tilting and derived category equivalence, see, e.g., [**Hap89**, Theorem 4.2] and [**Ric91**].

We give some examples that take advantage of resolutions smaller than the bar resolution.

Example 1.1.14. Let k be a field and $A = k[x]$. Consider the following sequence

$$0 \longrightarrow k[x] \otimes k[x] \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} k[x] \otimes k[x] \xrightarrow{\pi} k[x] \longrightarrow 0, \tag{1.1.15}$$

where π is multiplication and $(x \otimes 1 - 1 \otimes x)\cdot$ denotes multiplication by the element $x \otimes 1 - 1 \otimes x$. As $k[x]$ is commutative, this sequence is a complex. It is in fact exact, as can be shown directly via a calculation. Alternatively, it can be seen by exhibiting a contracting homotopy: Let $s_{-1}(x^i) = x^i \otimes 1$ and

$$s_0(x^i \otimes x^j) = -\sum_{l=1}^{j} x^{i+j+l} \otimes x^{l-1}$$

for all i, j. A calculation shows that s is a contracting homotopy for the above sequence. Note that for each i, the map s_i is left (but not right) $k[x]$-linear. As the terms in nonnegative degrees are visibly free as A^e-modules, the sequence (1.1.15) is a free resolution of the A^e-module A. Apply the functor $\mathrm{Hom}_{k[x]^e}(-, k[x])$ to the truncation of sequence (1.1.15) given by deleting the term $k[x]$. Identify $\mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x]) \cong \mathrm{Hom}_k(k, k[x])$ with $k[x]$ under the isomorphism in which a function f is sent to $f(1)$. The resulting complex, with arrows reversed, becomes

$$0 \longleftarrow k[x] \longleftarrow k[x] \longleftarrow 0.$$

There is only one map to compute, namely composition with $(x\otimes 1-1\otimes x)\cdot$. Let $a\in k[x]$, identified with the function f_a in $\mathrm{Hom}_{k[x]^e}(k[x]\otimes k[x],k[x])$ taking $1\otimes 1$ to a. Composing with the differential, since f_a is a $k[x]^e$-module homomorphism,

$$f_a((x\otimes 1-1\otimes x)\cdot(1\otimes 1)) = xf_a(1\otimes 1) - f_a(1\otimes 1)x = xa - ax = 0,$$

as $k[x]$ is commutative. Therefore all maps in the above complex are 0, and the homology of the complex in each degree is just the term in the complex. We thus find that $\mathrm{HH}^0(k[x])\cong k[x]$, in accordance with its isomorphism with the center of this commutative algebra, that $\mathrm{HH}^1(k[x])\cong k[x]$, and $\mathrm{HH}^n(k[x])=0$ for $n\geq 2$. A similar argument yields Hochschild homology $\mathrm{HH}_n(k[x])$ by first applying $k[x]\otimes_{k[x]^e} -$ to the truncation of the sequence (1.1.15) and identifying $k[x]\otimes_{k[x]^e}(k[x]\otimes k[x])$ with $k[x]$.

Example 1.1.16. Let k be a field, $n\geq 2$, and $A=k[x]/(x^n)$, called a *truncated polynomial ring*. Consider the following sequence:

$$\cdots \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{\pi} A \longrightarrow 0, \tag{1.1.17}$$

where $u=x\otimes 1-1\otimes x$, $v=x^{n-1}\otimes 1+x^{n-2}\otimes x+\cdots+1\otimes x^{n-1}$, and π is multiplication. This sequence is exact, as can be shown directly. Alternatively, the following is a contracting homotopy: For each i, define a left A-linear map s_i by $s_{-1}(1)=1\otimes 1$ and for all $m\geq 0$,

$$s_{2m}(1\otimes x^j) = -\sum_{l=1}^{j} x^{j-l}\otimes x^{l-1} \quad \text{and} \quad s_{2m+1}(1\otimes x^j)=\delta_{j,n-1}\otimes 1$$

for all j, where $\delta_{j,n-1}$ is the Kronecker delta (that is, $\delta_{j,n-1}=1$ if $j=n-1$ and $\delta_{j,n-1}=0$ otherwise). Since the terms in nonnegative degrees are visibly free as A^e-modules, the sequence (1.1.17) is a free resolution of the A^e-module A.

Apply $\mathrm{Hom}_{A^e}(-,A)$ to (1.1.17) and identify $\mathrm{Hom}_{A^e}(A\otimes A,A)$ with $\mathrm{Hom}_k(k,A)\cong A$. The resulting sequence may be viewed as:

$$\cdots \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \longleftarrow 0$$

If n is divisible by the characteristic of k, then $\mathrm{HH}^n(A)\cong A$ for all n. If n is not divisible by the characteristic of k, then $\mathrm{HH}^0(A)\cong A$, $\mathrm{HH}^{2m+1}(A)\cong (x)$ (the ideal generated by x) for all $m\geq 0$, and $\mathrm{HH}^{2m}(A)\cong A/(x^{n-1})\cong (x)$ for all $m\geq 1$.

We may also use the bar complex to define Harrison cohomology, a variant of Hochschild cohomology for commutative algebras. Let S_n denote the symmetric group on n symbols.

Definition 1.1.18. For nonnegative integers p and q, a (p,q)-*shuffle* is an element σ of the symmetric group S_{p+q} for which $\sigma(i) < \sigma(j)$ whenever $1 \le i < j \le p$ or $p+1 \le i < j \le p+q$. Let $S_{p,q}$ denote the subset of S_{p+q} consisting of all (p,q)-shuffles.

Let $f \in \operatorname{Hom}_k(A^{\otimes n}, A)$. We call f a *Harrison cochain* if

$$\sum_{\sigma \in S_{p,q}} (\operatorname{sgn}\sigma) f(a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}) = 0$$

for each pair p, q for which $p+q = n$ and all $a_1, \ldots, a_n \in A$. It can be shown that the Harrison cochains form a subcomplex of the complex of Hochschild cochains. The cohomology of this subcomplex is the *Harrison cohomology* of A.

1.2. Interpretation in low degrees

The historical Definitions 1.1.8 and 1.1.11 of Hochschild homology and cohomology lead directly to the following observations for small values of n.

Degree 0**.** By definition, $\mathrm{HH}^0(A, M) = \operatorname{Ker}(d_1^*)$. We will determine conditions on a function $f \in \operatorname{Hom}_{A^e}(A \otimes A, M)$ equivalent to being in $\operatorname{Ker}(d_1^*)$. First assume that $d_1^*(f) = 0$, that is, for all $a \in A$,

$$\begin{aligned} 0 \;=\; d_1^*(f)(1 \otimes a \otimes 1) &= f(d_1(1 \otimes a \otimes 1)) \\ &= f(a \otimes 1 - 1 \otimes a) \;=\; af(1 \otimes 1) - f(1 \otimes 1)a. \end{aligned}$$

Then $f(1 \otimes 1)$ is an element m of M for which $am = ma$ for all $a \in A$, and f is determined by this element m: $f(b \otimes c) = bf(1 \otimes 1)c = bmc$ for all $b, c \in A$. Conversely, any such element of M defines a function in $\operatorname{Ker}(d_1^*)$, that is, given $m \in M$ for which $am = ma$ for all $a \in A$, let $f_m \in \operatorname{Hom}_{A^e}(A \otimes A, M)$ be the function given by $f_m(b \otimes c) = bmc$ for all $b, c \in A$. Then $d_1^*(f_m) = 0$. So as a vector space,

$$\mathrm{HH}^0(A, M) \cong \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

In the special case $M = A$, we thus see that $\mathrm{HH}^0(A, A) \cong Z(A)$, the center of the algebra A.

Similarly, one finds that Hochschild homology in degree 0 is

$$\mathrm{HH}_0(A, M) \cong M / \operatorname{Span}_k\{am - ma \mid a \in A,\ m \in M\}.$$

Degree 1**.** By definition, $\mathrm{HH}^1(A, M) = \operatorname{Ker}(d_2^*)/\operatorname{Im}(d_1^*)$. Let $f \in \operatorname{Ker}(d_2^*)$, that is $f \in \operatorname{Hom}_{A^e}(A^{\otimes 3}, M)$ and fd_2 is the zero map on $A^{\otimes 4}$. Equivalently, for all $a, b \in A$,

$$\begin{aligned} 0 &= d_2^*(f)(1 \otimes a \otimes b \otimes 1) \\ &= f(d_2(1 \otimes a \otimes b \otimes 1)) \\ &= f(a \otimes b \otimes 1 - 1 \otimes ab \otimes 1 + 1 \otimes a \otimes b) \\ &= af(1 \otimes b \otimes 1) - f(1 \otimes ab \otimes 1) + f(1 \otimes a \otimes 1)b. \end{aligned}$$

By abuse of notation, we identify f with a function in $\mathrm{Hom}_k(A, M)$ under the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \cong \mathrm{Hom}_k(A, M)$ of (1.1.10), and the above equation becomes $0 = af(b) - f(ab) + f(a)b$, or

$$f(ab) = af(b) + f(a)b$$

for all $a, b \in A$. This is precisely the definition of a *k-derivation* from A to M. The space of all k-derivations from A to M is denoted

$$\mathrm{Der}(A, M).$$

Suppose in addition that $f \in \mathrm{Im}(d_1^*)$, that is $f = d_1^*(g)$ for some g in $\mathrm{Hom}_{A^e}(A^{\otimes 2}, M)$; the function g is defined by its value on $1 \otimes 1$, say m. Then

$$\begin{aligned} d_1^*(g)(1 \otimes a \otimes 1) &= g(d_1(1 \otimes a \otimes 1)) \\ &= g(a \otimes 1 - 1 \otimes a) \\ &= ag(1 \otimes 1) - g(1 \otimes 1)a \quad = \quad am - ma. \end{aligned}$$

That is, $d_1^*(g)$ is the *inner k-derivation* from A to M defined by the element m. Conversely, any inner k-derivation will be an element of $\mathrm{Im}(d_1^*)$. The space of all inner k-derivations from A to M is denoted

$$\mathrm{InnDer}(A, M).$$

We have shown that

$$\mathrm{HH}^1(A, M) \cong \mathrm{Der}(A, M)/\,\mathrm{InnDer}(A, M).$$

In particular, if $M = A$, then $\mathrm{HH}^1(A)$ is isomorphic to the space of derivations of A modulo its inner derivations. If A is commutative, the zero function is the only inner derivation, and in this case, $\mathrm{HH}^1(A)$ is simply the space of derivations of A.

Similarly, one finds that $\mathrm{HH}_1(A, M)$ is the kernel of the map $I \otimes_{A^e} M \to IM$ where I is the kernel of multiplication $\pi : A \otimes A \to A$. For details, see [**Wei94**, Section 9.2], where in case A is commutative, a connection to Kähler differentials is also given. In Section 3.2, we will identify I with the space $\Omega^1_{nc}A$ of noncommutative Kähler differentials defined there.

Degree 2. By definition, $\mathrm{HH}^2(A, M)$ is the quotient $\mathrm{Ker}(d_3^*)/\mathrm{Im}(d_2^*)$. Let $f \in \mathrm{Hom}_{A^e}(A^{\otimes 4}, M)$. Then f is in $\mathrm{Ker}(d_3^*)$ if and only if for all $a, b, c \in A$,

$$\begin{aligned}
0 &= d_3^*(f)(1\otimes a\otimes b\otimes c\otimes 1)\\
&= f(d_3(1\otimes a\otimes b\otimes c\otimes 1))\\
&= f(a\otimes b\otimes c\otimes 1 - 1\otimes ab\otimes c\otimes 1 + 1\otimes a\otimes bc\otimes 1 - 1\otimes a\otimes b\otimes c)\\
&= af(1\otimes b\otimes c\otimes 1) - f(1\otimes ab\otimes c\otimes 1) + f(1\otimes a\otimes bc\otimes 1)\\
&\quad - f(1\otimes a\otimes b\otimes 1)c.
\end{aligned}$$

Identifying f with a function in $\mathrm{Hom}_k(A^{\otimes 2}, M)$ under the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \cong \mathrm{Hom}_k(A^{\otimes 2}, M)$ of (1.1.10), we find that $f \in \mathrm{Ker}(d_3^*)$ if and only if

$$af(b\otimes c) + f(a\otimes bc) = f(ab\otimes c) + f(a\otimes b)c \tag{1.2.1}$$

for all $a, b, c \in A$. A calculation shows that the image of d_2^* may be identified with the space of all functions f in $\mathrm{Hom}_k(A\otimes A, M)$ given by

$$f(a\otimes b) = ag(b) - g(ab) + g(a)b \tag{1.2.2}$$

for some $g \in \mathrm{Hom}_k(A, M)$.

We will see in Section 3.2 that Hochschild 2-cocycles, that is functions satisfying (1.2.1), give the structure of an algebra to the A-bimodule $A\oplus M$, called a square-zero extension. These extensions arise in notions of smoothness of algebras. In the case that $M = A$, equation (1.2.1) gives rise to infinitesimal deformations of A. We will discuss this connection to algebraic deformation theory in Chapter 4, and we will see there that obstructions to lifting a Hochschild 2-cocycle to a formal deformation lie in $\mathrm{HH}^3(A)$. We will also see that functions satisfying (1.2.2) give rise to deformations isomorphic to the original algebra. In this way, each deformation, up to isomorphism, will have associated to it an element of $\mathrm{HH}^2(A)$.

Action of the center of A on $\mathrm{HH}^n(A)$. There is an action of the center $Z(A)$ of A on $\mathrm{Hom}_{A^e}(U, V)$ for any two A^e-modules, given by $(a\cdot f)(u) = af(u)$ for all $a \in Z(A)$, $u \in U$, and $f \in \mathrm{Hom}_A(U, V)$. Taking $U = A^{\otimes(n+2)}$ and $V = A$, this action commutes with the differentials on the bar complex, inducing an action on $\mathrm{HH}^n(A)$ under which $\mathrm{HH}^n(A)$ becomes a $Z(A)$-module. Identifying $Z(A)$ with $\mathrm{HH}^0(A)$ as described above, this is an action of $\mathrm{HH}^0(A)$ on $\mathrm{HH}^n(A)$. In the next section, this action will be extended to a graded product on $\mathrm{HH}^*(A)$. This action has some useful consequences. For example, if $1 = e_1 + \cdots + e_i$ is an expansion of the multiplicative identity of A as the sum of a set of orthogonal central idempotents $e_1, \ldots, e_i$, then

$$\mathrm{HH}^*(A) \cong \oplus_{j=1}^{i} \mathrm{HH}^*(A)e_j.$$

The ideal Ae_j of A is itself an algebra with multiplicative identity e_j, and we may identify $\mathrm{HH}^*(A)e_j$ with $\mathrm{HH}^*(Ae_j)$ to obtain $\mathrm{HH}^*(A) \cong \oplus_{j=1}^{i} \mathrm{HH}^*(Ae_j)$.

1.3. Cup product

Hochschild cohomology $\mathrm{HH}^*(A)$ is a graded vector space by its definition. (That is, it is graded by $\mathbb{N}$, which we understand to include 0 as an element.) It has an associative product making it into a graded commutative algebra, as we see next. We define this product at the chain level for functions on the bar complex (1.1.4) first. Then we give several equivalent definitions. In fact the cup product is the unique associative product on $\mathrm{HH}^*(A)$ satisfying some basic conditions; see Sanada [**San93**]. The many equivalent definitions of the associative product on $\mathrm{HH}^*(A)$, particularly when k is a field, make it very versatile.

From now on, we use the isomorphism (1.1.10) to identify the spaces $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, M)$ and $\mathrm{Hom}_k(A^{\otimes n}, M)$. Set

$$C^*(A, M) = \oplus_{n\geq 0} \mathrm{Hom}_k(A^{\otimes n}, M),$$

the space of *Hochschild cochains on A with coefficients in M*.

We start by taking $M = A$:

Definition 1.3.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The *cup product* $f \smile g$ is the element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ defined by

$$(1.3.2)\quad (f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m)g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

for all $a_1, \ldots, a_{m+n} \in A$. If $m = 0$, we interpret this formula to be

$$(f \smile g)(a_1 \otimes \cdots \otimes a_n) = f(1)g(a_1 \otimes \cdots \otimes a_m),$$

and similarly if $n = 0$.

By its definition, the cup product is associative. A calculation shows that

$$(1.3.3)\qquad d^*_{m+n+1}(f \smile g) = (d^*_{m+1}(f)) \smile g + (-1)^m f \smile (d^*_{n+1}(g)).$$

As a consequence, the space $C^*(A, A)$ of Hochschild cochains is a *differential graded algebra*, that is, a graded algebra with a graded derivation of degree 1 and square 0. Another consequence of equation (1.3.3) is that this cup product $\smile$ induces a well-defined graded associative product on Hochschild cohomology, which we denote by the same notation:

$$\smile\ :\ \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \to \mathrm{HH}^{m+n}(A).$$

Remark 1.3.4. More generally, if B is an A-bimodule that is also an algebra for which $a(bb') = (ab)b'$, $(bb')a = b(b'a)$, and $(ba)b' = b(ab')$ for all $a \in A$ and $b, b' \in B$, a calculation shows that the formula (1.3.2) induces a product on

$\mathrm{HH}^*(A, B)$. This condition is satisfied, for example, in case A is a subalgebra of B and the A-bimodule structure of B is left and right multiplication.

We have seen that in degree 0, Hochschild cohomology $\mathrm{HH}^0(A)$ is isomorphic to $Z(A)$, the center of the algebra A. As a consequence of the definition of cup product, the cup product of two elements in degree 0 is precisely the product of the corresponding elements in $Z(A)$, and the cup product of an element in arbitrary degree n with a degree 0 element corresponds to multiplying the values of a corresponding function by the corresponding element in $Z(A)$. This is the $\mathrm{HH}^0(A)$-module structure on Hochschild cohomology $\mathrm{HH}^*(A)$.

Example 1.3.5. We return to Example 1.1.14, letting $A = k[x]$, and describe the cup product. There we found that $\mathrm{HH}^0(k[x]) \cong k[x]$, $\mathrm{HH}^1(k[x]) \cong k[x]$, and $\mathrm{HH}^n(k[x]) = 0$ for $n > 1$. In degree 0, the cup product is simply multiplication on $k[x]$. Likewise, the product of an element in degree 0 with an element in degree 1 corresponds to multiplication on $k[x]$. Since $\mathrm{HH}^2(k[x]) = 0$, the product of two elements in degree 1 is 0. Thus $\mathrm{HH}^*(k[x]) \cong k[x, y]/(y^2)$ where $|x| = 0$ and $|y| = 1$. (The notation $|x|, |y|$ refers to their homological degrees.)

Since one often works with a resolution other than the bar resolution, other definitions of the cup product are useful for more complicated examples. We give several such equivalent definitions next. We assume here that k is a field and use the definition (1.1.13) of Hochschild cohomology in terms of Ext.

We will use the following construction of chain maps from cocycles for the first equivalent definition of cup product, and elsewhere.

Chain maps from cocycles. Let $P_{\bullet}$ be any projective resolution of A as an A^e-module. Let $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be a cocycle. Extend g to a chain map $g_{\bullet} : P_{\bullet} \to P_{\bullet}$ as follows. Let $K_n = \mathrm{Ker}(d_{n-1}) = \mathrm{Im}(d_n)$ for all n, and note that $K_n \cong P_n/\mathrm{Im}(d_{n+1})$. Since g is a cocycle, that is, $d^*_{n+1}(g) = 0$, it factors through K_n. Denote by $\overline{g}$ the map from K_n to A such that $\overline{g} d_n = g$, that is the following diagram commutes:

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & K_n \\ & \searrow^{g} & \downarrow{\scriptstyle \overline{g}} \\ & & A \end{array}$$

We may consider the sequence

$$\cdots \to P_{n+2} \to P_{n+1} \to P_n \to K_n \to 0$$

to be a projective resolution of K_n as an A^e-module. Now the Comparison Theorem (Theorem A.1.6) guarantees existence of maps $g_i : P_{n+i} \to P_i$ $(i \geq 0)$ that commute with the differentials, that is the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{n+2} & \xrightarrow{d_{n+2}} & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & K_n & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & \searrow^{g} & \downarrow{\scriptstyle \overline{g}} & & \\
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0
\end{array}$$

We give some details of the maps g_i, as we will use them often: Take g_i to be the zero map if $i < 0$. Since ε maps P_0 surjectively onto A, and P_n is projective, there is a map $g_0 : P_n \to P_0$ such that $\varepsilon g_0 = g$. We claim that the image of $g_0 d_{n+1}$ is contained in K_1. To see this, note that $K_1 = \operatorname{Ker} \varepsilon$ and $0 = g d_{n+1} = \varepsilon g_0 d_{n+1}$, so the image of $g_0 d_{n+1}$ is contained in $\operatorname{Ker} \varepsilon = K_1$. Now, since $g_0 d_{n+1}$ maps to K_1, P_1 surjects onto K_1 via d_1, and P_{n+1} is projective, there is a map $g_1 : P_{n+1} \to P_1$ such that $d_1 g_1 = g_0 d_{n+1}$. Once again, we see that the image of $g_1 d_{n+2}$ is contained in K_2, and this ensures existence of $g_2 : P_{n+2} \to P_2$, and so on. The chain map $g_{\bullet}$ is unique up to chain homotopy, as stated in the Comparison Theorem (Theorem A.1.6).

We are now ready for the first equivalent definition of cup product, sometimes called the Yoneda product.

Yoneda product. We will define a product at the chain level on any resolution as a composition of chain maps. Then we will show that for the bar resolution, this definition is equivalent to the cup product as defined in (1.3.2).

Let $P_{\bullet}$ be any projective resolution of A as an A^e-module. Let $f \in \operatorname{Hom}_{A^e}(P_m, A)$ and $g \in \operatorname{Hom}_{A^e}(P_n, A)$ be cocycles. Extend g to a chain map $g_{\bullet} : P_{\bullet} \to P_{\bullet}$ as described above. We define the map $f \smile g \in \operatorname{Hom}_{A^e}(P_{m+n}, A)$ to be the composition $f g_m$:

$$f \smile g = f g_m. \tag{1.3.6}$$

Since $g_{\bullet}$ is unique up to chain homotopy, $f \smile g$ is a cocycle, and if either f or g is a coboundary then so is $f \smile g$. Thus this cup product at the chain level induces a well-defined product on cohomology. Moreover, it does not depend on choice of $g_{\bullet}$, since $g_{\bullet}$ is unique up to chain homotopy.

Equivalently, we may identify both cocycles f, g with the chain maps $f_{\bullet}, g_{\bullet}$ they induce as described above, in which case $f \smile g$ may be identified with a composition of chain maps.

Taking P to be the bar resolution $B(A)$ of (1.1.4), we will show that the product defined above is the same as (1.3.2). Then by the Comparison Theorem (Theorem A.1.6), comparing any other projective resolution to

$B(A)$, the definition (1.3.6) will then indeed give an equivalent definition of cup product. Let us apply the definition (1.3.6) when $P_n = A^{\otimes(n+2)}$ for all n. Given $g : A^{\otimes(n+2)} \to A$ a cocycle, we must choose $g_0 : A^{\otimes(n+2)} \to A \otimes A$ for which $\pi g_0 = g$, that is,

$$\pi g_0(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$$

for all $a_1, \ldots, a_n \in A$. We may choose

$$g_0(1 \otimes a_1 \otimes \cdots a_n \otimes 1) = 1 \otimes g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1),$$

and extend to an A^e-module map by defining

$$g_0(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)a_{n+1},$$

for all $a_0, \ldots, a_{n+1} \in A$. Then we may choose $g_1 : A^{\otimes(n+3)} \to A^{\otimes 3}$ to be

$$g_1(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) = 1 \otimes a_1 \otimes g(1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1),$$

extend to an A^e-module map, and check that $g_0 d_{n+1} = d_1 g_1$ using the definition of g_0 and the assumption that g is a cocycle. In general we may set

$$g_i(1 \otimes a_1 \otimes \cdots \otimes a_{n+i} \otimes 1) = 1 \otimes a_1 \otimes \cdots \otimes a_i \otimes g(1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+i} \otimes 1).$$

We see that if $f \in \mathrm{Hom}_{A^e}(A^{\otimes(m+2)}, A)$, then

$$\begin{aligned}
&(f \smile g)(1 \otimes a_1 \otimes \cdots \otimes a_{m+n} \otimes 1)\\
&= f g_m(1 \otimes a_1 \otimes \cdots \otimes a_{m+n} \otimes 1)\\
&= f(1 \otimes a_1 \otimes \cdots \otimes a_m \otimes g(1 \otimes a_{m+1} \otimes \cdots \otimes a_{m+n} \otimes 1))\\
&= f(1 \otimes a_1 \otimes \cdots \otimes a_m \otimes 1) g(1 \otimes a_{m+1} \otimes \cdots \otimes a_{m+n} \otimes 1).
\end{aligned}$$

This is equivalent to (1.3.2) under the identification of $\mathrm{Hom}_{A^e}(A^{\otimes(m+n+2)}, A)$ and $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ under isomorphism (1.1.10). There is another choice of lifting $g_{\bullet}$ of g to which a comparison leads to a proof that the product on Hochschild cohomology is graded commutative; see, e.g., Solberg [**Sol06**, Theorem 2.1]. For another proof of graded commutativity, see Lemma 1.4.3 below.

Example 1.3.7. We return to Example 1.1.16 in which $A = k[x]/(x^n)$, and compute cup products.

Let $p = \mathrm{char}(k)$. First we assume that p does not divide n. In degree 1, let $g \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function given by $g(1 \otimes 1) = x$. By the description of the cohomology in Example 1.1.16 and of the action of $\mathrm{HH}^0(A)$ described at the end of Section 1.2, g represents an element in $\mathrm{HH}^1(A)$ and generates $\mathrm{HH}^1(A)$ as an $\mathrm{HH}^0(A)$-module. In degree 2, let $f \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function given by $f(1 \otimes 1) = 1$. Then f represents an element in $\mathrm{HH}^2(A)$ and generates $\mathrm{HH}^2(A)$ as an $\mathrm{HH}^0(A)$-module. We will find cup products of these functions as compositions of chain maps, and show that other cup products are determined by these.

In relation to the resolution (1.1.17), let $K_1 = A^e/\operatorname{Im}(v\cdot)$. Since g is a cocycle, it factors through K_1 as in the following diagram. The map from A^e to K_1 in the diagram can be taken to be the quotient map, which can be identified with the map $u\cdot$ to $\operatorname{Im}(u\cdot) \cong K_1$ as a submodule of A^e in degree 0.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A^e & \xrightarrow{u\cdot} & A^e & \xrightarrow{v\cdot} & A^e & \longrightarrow & K_1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & \searrow^{g} & \downarrow{\scriptstyle \overline{g}} & & \\
\cdots & \longrightarrow & A^e & \xrightarrow{v\cdot} & A^e & \xrightarrow{u\cdot} & A^e & \xrightarrow{\pi} & A & \longrightarrow & 0
\end{array}$$

Set $g_{2m}(1 \otimes 1) = x \otimes 1$ and $g_{2m+1}(1 \otimes 1) = 1 \otimes x^{n-1}$ for all $m \geq 0$. A calculation then shows that the above diagram commutes.

We use these maps g_n to find some cup products. First note $gg_1 = 0$, and so $g \smile g = 0$. (This can also be concluded from graded commutativity in case the characteristic p of k is not 2.) We find that

$$(f \smile g)(1 \otimes 1) = fg_2(1 \otimes 1) = f(x \otimes 1) = x,$$

and thus $f \smile g$ represents a generator for $\operatorname{HH}^3(A)$ as an $\operatorname{HH}^0(A)$-module.

Now let $K_2 = A^e/\operatorname{Im}(u\cdot)$. Since f is a cocycle, it factors through K_2 as shown in the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A^e & \xrightarrow{v\cdot} & A^e & \xrightarrow{u\cdot} & A^e & \longrightarrow & K_2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} & & \\
\cdots & \longrightarrow & A^e & \xrightarrow{v\cdot} & A^e & \xrightarrow{u\cdot} & A^e & \xrightarrow{\pi} & A & \longrightarrow & 0
\end{array}$$

Set $f_j(1 \otimes 1) = 1 \otimes 1$ for all $j \geq 0$. A calculation then shows that the above diagram commutes. We find that $(f \smile f)(1 \otimes 1) = ff_2(1 \otimes 1) = 1$, and similarly for all powers i of f, by induction:

$$f^i(1 \otimes 1) = f^{i-1}f_2(1 \otimes 1) = 1,$$

where $f^i = f \smile \cdots \smile f$ (i factors of f). So in each even degree, $\operatorname{HH}^{2i}(A)$ is generated by f^i as an $\operatorname{HH}^0(A)$-module. Similarly, $f^i \smile g = f^i g_2$ generates $\operatorname{HH}^{2i+1}(A)$ as an $\operatorname{HH}^0(A)$-module. Taking y and z to be elements of cohomology represented by g and f, respectively, we see that

$$\operatorname{HH}^*(A) \cong k[x, y, z]/(x^{n-1}y,\ y^2)$$

where $|x| = 0$, $|y| = 1$, and $|z| = 2$, if the characteristic p of k does not divide n.

Next assume that the characteristic p of k divides n. In degree 1, let $g \in \operatorname{Hom}_{A^e}(A^e, A)$ be the function defined by $g(1 \otimes 1) = 1$. In degree 2, let $f \in \mathrm{Hom}_{A^e}(A^e, A)$ be the function defined by $f(1 \otimes 1) = 1$. Let $K_1 = A^e/\mathrm{Im}(v\cdot)$. Set $g_{2m}(1 \otimes 1) = 1 \otimes 1$ and

$$g_{2m+1}(1 \otimes 1) = x^{n-2} \otimes 1 + 2x^{n-3} \otimes x + 3x^{n-4} \otimes x^2 + \cdots + (n-1) \otimes x^{n-2}$$

for all $m \geq 0$. (Note that if $n = 2$, this formula yields $g_{2m+1}(1 \otimes 1) = 1 \otimes 1$.) A calculation shows that, since $n \equiv 0$ in k, the corresponding diagram commutes, that is, $g_\bullet$ is a chain map. Thus we may compute:

$$(1.3.8) \qquad \begin{aligned} (g \smile g)(1 \otimes 1) &= gg_1(1 \otimes 1) = \frac{n(n-1)}{2} x^{n-2}, \\ (f \smile g)(1 \otimes 1) &= fg_2(1 \otimes 1) = 1 \otimes 1. \end{aligned}$$

Further calculations show as before that f and g generate $\mathrm{HH}^*(A)$ as an algebra over its degree 0 component $\mathrm{HH}^0(A) \cong A$. If p is odd, the coefficient $\frac{n(n-1)}{2}$ is 0 in k. As well, if $p = 2$ and $n = 2s$ for some even integer s, the coefficient $\frac{n(n-1)}{2}$ is 0 in k. Thus we find that in odd characteristic p dividing n, or in case $p = 2$ and $n = 2s$ for some even integer s, there is an isomorphism of algebras,

$$\mathrm{HH}^*(A) \cong k[x, y, z]/(y^2),$$

where $|x| = 0$, $|y| = 1$, $|z| = 2$. If $p = 2$ and $n = 2s$ for some odd integer s, the coefficient $\frac{n(n-1)}{2}$ in (1.3.8) is 1. If $n = 2$, this implies that $g \smile g$ generates $\mathrm{HH}^2(A)$ as an $\mathrm{HH}^1(A)$-module, and similarly g generates $\mathrm{HH}^*(A)$ as an algebra over $A \cong \mathrm{HH}^0(A)$. If $n \neq 2$, it does not, and equation (1.3.8) yields a relation among the generators. Thus we find that if $p = 2$ and $n = 2$,

$$\mathrm{HH}^*(A) \cong k[x, y],$$

where $|x| = 0$, $|y| = 1$. If $p = 2$ and $n = 2s$ for some odd integer $s > 1$, then

$$\mathrm{HH}^*(A) \cong k[x, y, z]/(x^2 y^2)$$

where $|x| = 0$, $|y| = 1$, $|z| = 2$.

Tensor product of complexes. Another definition of product on $\mathrm{HH}^*(A)$, that also turns out to be equivalent to the cup product, arises from a tensor product of complexes and a diagonal map, as follows.

Let $P_\bullet$ be any A^e-projective resolution of A. We claim that the total complex of $P_\bullet \otimes_A P_\bullet$ is also an A^e-projective resolution of A: First we will show that for each m, n, the A^e-module $P_m \otimes_A P_n$ is projective. To see this, note that since P_m, P_n are projective, each is a direct summand of a direct sum of copies of A^e. Thus it suffices to show that $A^e \otimes_A A^e$ is projective. But $A^e \otimes_A A^e \cong A^e \otimes_k A$ as an A^e-module, since A^e acts only on the outermost two factors of $A^e \otimes_A A^e$. Since A is free as a k-module, we see that $A^e \otimes_k A$ is free as an A^e-module. Next, to see that $P_\bullet \otimes_A P_\bullet$

has cohomology A concentrated in degree 0, we apply the Künneth Theorem (Theorem A.4.1): The module $A \otimes A^{\mathrm{op}}$, under right multiplication by elements of A in the right factor, is a free right A-module since A is a free k-module. It follows that $P_\bullet$ is, by restriction, a projective resolution of the free right A-module A. So the boundaries are also all projective A-modules, that is the hypotheses of the Künneth Theorem (Theorem A.4.1) hold. The Tor terms in the Künneth sequence (that is, the sequence in the Künneth Theorem statement) vanish: The only term in which both arguments are nonzero is $\mathrm{Tor}_1^A(\mathrm{H}_0(P), \mathrm{H}_0(P)) = \mathrm{Tor}_1^A(A, A) = 0$ (since A is free as an A-module). This implies that $P_\bullet \otimes_A P_\bullet$ is indeed a resolution of $A \otimes_A A \cong A$ by A^e-projective modules.

By the Comparison Theorem (Theorem A.1.6) there is a chain map $\Delta : P_\bullet \to P_\bullet \otimes_A P_\bullet$ lifting the identity map from A to A (where we have identified $A \otimes_A A \cong A$). Such a map Δ is unique up to chain homotopy. Sometimes Δ is called a *diagonal map*.

The cup product on Hochschild cohomology may be defined via a diagonal map $\Delta : P_\bullet \to P_\bullet \otimes_A P_\bullet$ in the following way. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$, $g \in \mathrm{Hom}_{A^e}(P_n, A)$ represent elements of $\mathrm{HH}^m(A, A)$, $\mathrm{HH}^n(A, A)$. We may define

$$f \smile g = \pi(f \otimes g)\Delta, \tag{1.3.9}$$

where π is multiplication on A. For any two resolutions, by the Comparison Theorem (Theorem A.1.6), there are chain maps between them, and one can check it follows that this definition of product does not depend on choices of $P_\bullet$ and Δ. If $P_\bullet$ is the bar resolution, one choice of chain map Δ induces precisely the chain level cup product (1.3.2), namely:

$$\Delta(1 \otimes a_1 \otimes \cdots \otimes a_m \otimes 1) = \sum_{i=0}^{m} (1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_m \otimes 1)$$

for all $a_1, \ldots, a_m \in A$. See also [**San93**]. If $P_\bullet$ is not the bar resolution, one choice of chain map Δ is given by first mapping $P_\bullet$ to $B(A)$, applying this diagonal map on $B(A)$, then mapping $B(A) \otimes_A B(A)$ to $P_\bullet \otimes_A P_\bullet$. Thus on cohomology, our product in (1.3.9) is equivalent to the cup product (1.3.2), justifying our use of the same notation for this product.

Yoneda composition and tensor product of extensions. Two more definitions of the product on Hochschild cohomology $\mathrm{HH}^*(A)$ are given on generalized extensions. One of these is the *Yoneda composition* (or *Yoneda splice*). This definition uses the description of $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A)$ as equivalence classes of n-extensions of A by A as A^e-modules. (See Section A.2 for a description of n-extensions and connections with Ext.)

Let f and g correspond to the m- and n-extensions

$$f: \qquad 0 \longrightarrow A \xrightarrow{\alpha_m} M_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} M_0 \xrightarrow{\alpha_0} A \longrightarrow 0,$$

$$g: \qquad 0 \longrightarrow A \xrightarrow{\beta_n} N_{n-1} \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_2} N_1 \xrightarrow{\beta_2} N_0 \xrightarrow{\beta_0} A \longrightarrow 0,$$

respectively. Then $\beta_n\alpha_0$ is a map from M_0 to N_{n-1}, and so there is a sequence

$$f \smile g:$$

$$0 \longrightarrow A \xrightarrow{\alpha_m} M_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \xrightarrow{\alpha_1} M_0 \xrightarrow{\beta_n\alpha_0} N_{n-1} \xrightarrow{\beta_{n-1}} \cdots$$

$$\xrightarrow{\beta_1} N_0 \xrightarrow{\beta_0} A \longrightarrow 0.$$

A calculation shows that it is exact at M_0 and at N_{n-1}, and so it is an exact sequence, in other words, an $(m+n)$-extension of A. Then $f \smile g$ corresponds to this $(m+n)$-extension. This follows from the definitions and the correspondence between n-extensions and Ext, and is shown in detail for example in [**Aho08**, Theorem 4.3].

We may alternatively take the tensor product of an m-extension with an n-extension to obtain an $(m+n)$-extension as the total complex. This may be seen to be equivalent to Yoneda composition, and thus to cup product, by mapping to two edges. See [**Sch98**].

In the sequel, we will frequently exploit the fact that all of these versions of associative product on $\mathrm{HH}^*(A)$ agree, and we will use the version that is most convenient in each setting.

Properties of cup product. The cup product is associative as a direct consequence of formula (1.3.2). Associativity can also be deduced readily from each of the other equivalent definitions of cup product above.

We next turn to our claim that the cup product is graded commutative. This may be shown in more than one way. It may be proven by induction, as in [**San93**]. Another proof uses two of the equivalent definitions of the product, namely the composition of chain maps and the tensor product of resolutions, together with the Eckmann-Hilton argument relying on the latter being an algebra homomorphism over the former (see [**SA04**]). Yet another proof uses tensor products of generalized extensions and an argument similar to the proof of Theorem 1.6.3 below (see [**SS04**, Theorem 1.1]). We will instead give a proof in the next section using the more concrete historical approach of Gerstenhaber [**Ger63**]. This proof is connected to the first natural appearance of the graded Lie bracket on Hochschild cohomology, defined next.

1.4. Gerstenhaber bracket

In addition to an associative product, Hochschild cohomology $\mathrm{HH}^*(A)$ has another binary operation that is a graded derivation with respect to the cup product. We define this operation at the chain level on the bar complex (1.1.4) here, allowing k to be an arbitrary commutative ring. In Chapter 5, under the assumption that k is a field, we will consider equivalent definitions by way of other projective resolutions and exact sequences.

Definition 1.4.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The *Gerstenhaber bracket* $[f, g]$ is defined as the element of $\mathrm{Hom}_k(A^{\otimes(m+n-1)}, A)$ given by

$$[f,g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

where the *circle product* $f \circ g$ generalizes composition of functions and is defined by

$$\begin{aligned}&(f\circ g)(a_1\otimes\cdots\otimes a_{m+n-1})\\ &= \sum_{i=1}^{m}(-1)^j f(a_1\otimes\cdots\otimes a_{i-1}\otimes g(a_i\otimes\cdots\otimes a_{i+n-1})\otimes a_{i+n}\otimes\cdots\otimes a_{m+n-1}),\end{aligned}$$

in which $j = (n-1)(i-1)$, and similarly $g \circ f$. If $m = 0$, then $f \circ g = 0$ (as indicated by the empty sum), while if $n = 0$, then the formula should be interpreted by taking the value $g(1)$ in place of $g(a_i \otimes \cdots \otimes a_{i+n-1})$.

The following lemmas may be proven by direct computation on the bar complex as in [**Ger63**].

Lemma 1.4.2. *Let* $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$*,* $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$*, and* $h \in \mathrm{Hom}_k(A^{\otimes p}, A)$*. Then*

(i) $[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$,

(ii) $(-1)^{(m-1)(p-1)}[f,[g,h]] + (-1)^{(n-1)(m-1)}[g,[h,f]]$
$+(-1)^{(p-1)(n-1)}[h,[f,g]] = 0$,

(iii) $d^*([f,g]) = (-1)^{n-1}[d^*f, g] + [f, d^*g]$.

Property (i) is graded anti-commutativity of the bracket, where we shift the homological degrees of f, g by -1. Property (ii) is the graded Jacobi identity. These first two properties make $C^*(A,A) = \oplus_{n\geq 0}\mathrm{Hom}_k(A^{\otimes n}, A)$ into a graded Lie algebra. Property (iii) further makes it a *differential graded Lie algebra*, that is a graded Lie algebra with a graded derivation d^* of degree 1 and square 0. We emphasize again that the degree of an element here is shifted by one from the homological degree, so that f has degree $m-1$

when considering the Lie structure. Some authors choose notation to clarify this distinction, introducing a shift operator that shifts degree when needed.

Gerstenhaber [**Ger63**] more generally developed the notion of a pre-Lie algebra for handling the circle product and bracket operations and proving the Lie structure.

Let $\pi : A \otimes A \to A$ denote multiplication. The following lemma may be proven by tedious direct computation.

Lemma 1.4.3. *Let* $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ *and* $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. *Then*

(i) $f \smile g - (-1)^{mn} g \smile f$
$= (d^* g) \circ f + (-1)^m d^*(g \circ f) + (-1)^{m-1} g \circ (d^* f)$,

(ii) $[f, \pi] = -d^*(f)$.

As a consequence of Lemma 1.4.3(i), we have the following theorem.

Theorem 1.4.4. *Let A be an associative algebra over the commutative ring k. The cup product on* $\mathrm{HH}^*(A)$ *is graded commutative.*

As a consequence of Lemma 1.4.2(iii), the bracket $[\ ,\ ]$, as defined at the chain level, induces a well-defined operation on $\mathrm{HH}^*(A)$. Next we state a further property satisfied by this bracket on Hochschild cohomology. For a proof, see [**Ger63**, Corollary 1 of Theorem 5], where the difference of the left and right sides of the stated equation is shown to be a specific coboundary.

Lemma 1.4.5. *Let* $\alpha \in \mathrm{HH}^m(A)$, $\beta \in \mathrm{HH}^n(A)$, *and* $\gamma \in \mathrm{HH}^p(A)$. *Then*

$$[\alpha \smile \beta, \gamma] = [\alpha, \gamma] \smile \beta + (-1)^{m(p-1)} \alpha \smile [\beta, \gamma].$$

This property states that for each γ, the operation $[-, \gamma]$ is a graded derivation with respect to cup product. As a consequence of all the properties in the above lemmas, Hochschild cohomology is a Gerstenhaber algebra (sometimes also called a G-algebra), as we define next. Recall that we consider 0 to be an element of $\mathbb{N}$.

Definition 1.4.6. A *Gerstenhaber algebra* $(H, \smile, [\ ,\ ])$ is a free $\mathbb{N}$-graded k-module H for which $(H, \smile)$ is a graded commutative associative algebra, $(H, [\ ,\ ])$ is a graded Lie algebra with bracket $[\ ,\ ]$ of degree -1 and corresponding degree shift by -1 on elements, and

$$[\alpha \smile \beta, \gamma] = [\alpha, \gamma] \smile \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \smile [\beta, \gamma]$$

for all homogeneous α, β, γ in H.

Theorem 1.4.7. *Hochschild cohomology* $\mathrm{HH}^*(A)$ *is a Gerstenhaber algebra.*

Proof. The main properties to prove are dealt with in Theorem 1.4.4 and Lemmas 1.4.2 and 1.4.5. ☐

In Chapter 5 we will examine the Gerstenhaber bracket in more detail, including ways to define it on an arbitrary resolution, independent of the bar resolution, or on exact sequences.

1.5. Cap product, shuffle product, Hodge decomposition

The cap product is a pairing between Hochschild cohomology and homology

$$\mathrm{HH}^m(A) \otimes \mathrm{HH}_n(A) \xrightarrow{\frown} \mathrm{HH}_{n-m}(A),$$

defined as follows, where we consider $\mathrm{HH}_i(A)$ to be 0 for all $i < 0$. Let $f \in \mathrm{Hom}_{A^e}(A^{\otimes(m+2)}, A)$ be a function representing an element of $\mathrm{HH}^m(A)$. Let $a_0, \ldots, a_n \in A$ and consider $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ be an element of $A \otimes_{A^e} A^{\otimes(n+2)}$ symbolically representing an element of $\mathrm{HH}_n(A)$. The *cap product* is

$$f \frown (a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^m a_0 f(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes \cdots \otimes a_n.$$

This induces a well-defined pairing as claimed: Assuming $\sum_i a_0^i \otimes \cdots \otimes a_n^i$ is a finite sum with $a_j^i \in A$ that is a cycle, and that $f \in \mathrm{Hom}_{A^e}(A^{\otimes(m+2)}, A)$ is a cocycle, one sees that their cup product is a cycle by rewriting the image of the differential on $\sum_i a_0^i \otimes \cdots \otimes a_n^i$ in such a way as to take advantage of the relation $d_{m+1}^*(f) = 0$. Thus the cap product of a cocycle with a cycle is a cycle. Similarly one sees that the cap product of a coboundary with a cycle, or of a cocycle with a boundary, is a boundary. By its definition, the cap product gives $\mathrm{HH}_*(A)$ the structure of an $\mathrm{HH}^*(A)$-module.

Example 1.5.1. Let $A = k[x]$, as in Example 1.1.14. By our work in that example, we see that $\mathrm{HH}_*(A)$ is a free A-module and $\mathrm{HH}^0(A) \cong A$ acts accordingly. We identify $\mathrm{HH}^1(A)$ with $A \otimes V^*$ and $\mathrm{HH}_1(A)$ with $A \otimes V$ where V is a vector space of dimension 1. The action of $\mathrm{HH}^1(A)$ on $\mathrm{HH}_1(A)$ is multiplication in the first tensor factor and evaluation in the second.

For the rest of this section, assume that A is a commutative algebra. The shuffle product, defined next, is a product on Hochschild homology $\mathrm{HH}_*(A)$ in this commutative case. Recall the subset $S_{p,q}$ of (p,q)-shuffles of the symmetric group S_{p+q}, as in Definition 1.1.18. The *shuffle product* on $\mathrm{HH}_*(A)$ is defined at the chain level on the complex (1.1.7) with $M = A$ by

$$\begin{aligned}&(a_0 \otimes a_1 \otimes \cdots \otimes a_p) \cdot (a_0' \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}) \\ &= \sum_{\sigma \in S_{p,q}} (\operatorname{sgn} \sigma) a_0 a_0' \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)}.\end{aligned}$$

for all $a_0, \ldots, a_{p+q} \in A$.

Theorem 1.5.2. *Let A be a commutative algebra. Then $\mathrm{HH}_*(A)$ is a graded commutative algebra under the shuffle product.*

For a proof, see [**Wei94**, Proposition 9.4.2].

Example 1.5.3. Let $A = k[x]$, as in Example 1.1.14. Then $\mathrm{HH}_*(A) \cong k[x,y]/(y^2)$ where $|x| = 0$, $|y| = 1$.

We next define the Hodge decomposition of Hochschild cohomology of a commutative algebra A, when k has characteristic 0, due to Gerstenhaber and Schack [**GS87**] and inherent in work of Quillen [**Qui70**]: For each r, let

$$s_{r,n-r} = \sum_{\sigma \in S_{r,n-r}} (\operatorname{sgn} \sigma)\sigma,$$

where the sum is taken over all $(r, n-r)$-shuffles as in Definition 1.1.18. Let

$$s_n = \sum_{r=1}^{n-1} s_{r,n-r}.$$

By [**GS87**, Theorem 1.2], $s_n = \lambda_1 e_n(1) + \cdots \lambda_n e_n(n)$ where $\lambda_i = 2^i - 2$ and for each j,

$$e_n(j) = \prod_{i \neq j} (\lambda_j - \lambda_i)^{-1} \prod_{i \neq j} (s_n - \lambda_i)$$

in the group algebra kS_n. The elements $e_n(j)$ are the *Eulerian idempotents*; for each n, the set $\{e_n(1), \ldots, e_n(n)\}$ is a set of orthogonal idempotents whose sum is 1. As operators on the bar complex, they commute with the differentials in the sense that $d_n e_n(j) = e_{n-1}(j) d_n$. Thus $C^*(A, A)$ has an induced direct sum decomposition, and so does $\mathrm{HH}^*(A)$: For each n,

$$\mathrm{HH}^n(A) = \oplus_{j \geq 0} \mathrm{HH}^n(A) e_n(j),$$

called the *Hodge decomposition.* The component $\mathrm{HH}^n(A)e_n(1)$ is precisely Harrison cohomology in degree n.

1.6. Actions of Hochschild cohomology

Let k be a field. For any two (left) A-modules M and N, the Hochschild cohomology ring $\mathrm{HH}^*(A)$ acts on $\mathrm{Ext}^*_A(M, N)$, in such a way that $\mathrm{Ext}^*_A(M, M)$ is an $\mathrm{HH}^*(A)$-module. (There is also an action for any two right modules.) Similarly, for any A-bimodule M, Hochschild cohomology $\mathrm{HH}^*(A)$ acts on $\mathrm{HH}^*(A, M)$. We describe these actions next.

Let M, N be A-modules. Choose an A^e-projective resolution of A as an A^e-module,

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0, \tag{1.6.1}$$

for example, we could take the bar resolution (1.1.4). Apply $- \otimes_A M$ and the isomorphism $A \otimes_A M \cong M$ to obtain the sequence:

$$\cdots \xrightarrow{d_2 \otimes 1_M} P_1 \otimes_A M \xrightarrow{d_1 \otimes 1_M} P_0 \otimes_A M \xrightarrow{\varepsilon \otimes 1_M} M \to 0.$$

Since each term of the sequence (1.6.1) is projective as a right A-module, and therefore a direct sum of a free A-module, each $P_i \otimes_A M$ is projective as a left A-module, where the action is on the left tensor factor only. As tensor product is right exact, the map $\varepsilon \otimes 1_M$ is surjective. Since all P_i and $\mathrm{Im}(d_i)$ are projective as right A-modules, the sequence above is exact. (One sees this for example in Theorem A.3.6, the first long exact sequence for Tor, applied to the short exact sequences $0 \to \mathrm{Im}(d_{i-1}) \to P_i \to \mathrm{Im}(d_i) \to 0$ of which sequence (1.6.1) may be built.) Therefore it is a projective resolution of M as a (left) A-module.

Let $f \in \mathrm{Hom}_{A^e}(P_i, A)$ represent an element of $\mathrm{HH}^i(A)$. Let

$$\phi_M(f) = f \otimes_A 1_M \tag{1.6.2}$$

in $\mathrm{Hom}_A(P_i \otimes_A M, M)$, representing an element of $\mathrm{Ext}^i_A(M, M)$. By the Comparison Theorem (Theorem A.1.6), in light of the isomorphism of A-modules $A \otimes_A M \xrightarrow{\sim} M$, one may lift $\phi_M(f)$ to a chain map, that is to maps $P_{i+l} \otimes_A M \to P_l \otimes_A M$ for all $l \geq 0$:

$$\begin{array}{ccccccc}
P_{i+l} \otimes_A M & \longrightarrow \cdots \longrightarrow & P_{i+1} \otimes_A M & \longrightarrow & P_i \otimes_A M & & \\
\downarrow{\scriptstyle \phi_M(f)_l} & & \downarrow{\scriptstyle \phi_M(f)_1} & & \downarrow{\scriptstyle \phi_M(f)_0} & \searrow^{\phi_M(f)} & \\
P_l \otimes_A M & \longrightarrow \cdots \longrightarrow & P_1 \otimes_A M & \longrightarrow & P_0 \otimes_A M & \longrightarrow & M
\end{array}$$

Compose with any function $g \in \mathrm{Hom}_A(P_j \otimes_A M, N)$ to obtain the element $g\phi_M(f)_j$ of $\mathrm{Hom}_A(P_{i+j} \otimes_A M, N)$. One may check that this induces a well-defined map

$$\mathrm{Ext}^j_A(M, N) \otimes \mathrm{HH}^i(A) \to \mathrm{Ext}^{i+j}_A(M, N),$$

that is a right module action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M, N)$. Similarly, there is a left action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M, N)$: First apply $-\otimes_A N$ to the resolution $P_\bullet$ to obtain a map ϕ_N from $\mathrm{HH}^*(A)$ to $\mathrm{Ext}^*_A(N, N)$. Then compose chain maps corresponding to elements of $\mathrm{Ext}^*_A(N, N)$ and $\mathrm{Ext}^*_A(M, N)$. The right and left actions are the same up to a sign, as the next theorem shows. The theorem is a special case of [**SS04**, Theorem 1.1].

Theorem 1.6.3. *Let $\alpha \in \mathrm{HH}^i(A)$ and $\beta \in \mathrm{Ext}^j_A(M, N)$. Then*

$$\alpha \cdot \beta = (-1)^{ij} \beta \cdot \alpha.$$

That is, the left and right actions of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M, N)$ agree up to a sign.

Proof. The element α may be represented by an i-extension of A by A:

$$0 \to A \xrightarrow{\iota} E \to P_{i-2} \to \cdots \to P_1 \to P_0 \to A \to 0 \tag{1.6.4}$$

for an A^e-module E and projective A^e-modules P_i. We may assume that E and all P_i are projective as right A-modules. Suppose $\beta \in \mathrm{Ext}^0_A(M, N) \cong$

$\mathrm{Hom}_A(M,N)$. If $\alpha \in \mathrm{HH}^0(A) \cong Z(A)$, the action is by multiplication, and so the equation holds. If $\alpha \in \mathrm{HH}^1(A)$, then the 1-extension (1.6.4) is simply $0 \to A \xrightarrow{\iota} E \xrightarrow{\varepsilon} A \to 0$. We let

$$X = (N \oplus (E \otimes_A M))/\{(a\beta(m),\ -\iota(a) \otimes m) \mid a \in A,\ m \in M\}.$$

Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes_A M & \xrightarrow{\iota\otimes 1} & E \otimes_A M & \xrightarrow{\varepsilon\otimes 1} & A \otimes_A M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\beta} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & A \otimes_A N & \xrightarrow{\binom{1}{0}} & X & \xrightarrow{(0,\varepsilon\otimes 1)} & A \otimes_A M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle (\iota\otimes 1, 1\otimes\beta)} & & \downarrow{\scriptstyle 1\otimes\beta} & & \\
0 & \longrightarrow & A \otimes_A N & \xrightarrow{\iota\otimes 1} & E \otimes_A N & \xrightarrow{\varepsilon\otimes 1} & A \otimes_A N & \longrightarrow & 0
\end{array}$$

Note that X is the pushout (as defined in Section A.1) of the upper left corner, and that the diagram commutes. Therefore $\alpha \cdot \beta = \beta \cdot \alpha$. A similar argument applies for any $\alpha \in \mathrm{HH}^i(A)$.

Next suppose $\beta \in \mathrm{Ext}^1_A(M,N)$ and that $0 \to N \to X \to M \to 0$ is a corresponding 1-extension. For each l, consider the short exact sequence

$$0 \to K_l \to P_{l-1} \to K_{l-1} \to 0 \tag{1.6.5}$$

where K_l, K_{l-1} are the lth and $(l-1)$st syzygies in the sequence (1.6.4). (Take $K_0 = A$.) We may assume that each K_l is projective as a right A-module, since (1.6.4) consists entirely of modules that are projective as right A-modules. Tensor this sequence with $0 \to N \to X \to M \to 0$ over A to obtain the following commuting diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_l \otimes_A N & \longrightarrow & P_{l-1} \otimes_A N & \longrightarrow & K_{l-1} \otimes_A N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_l \otimes_A X & \longrightarrow & P_{l-1} \otimes_A X & \longrightarrow & K_{l-1} \otimes_A X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_l \otimes_A M & \longrightarrow & P_{l-1} \otimes_A M & \longrightarrow & K_{l-1} \otimes_A M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then by [**Lan95**, Lemma VIII.3.1], the 2-extensions obtained from the edges,

$$0 \to K_l \otimes_A N \longrightarrow P_{l-1} \otimes_A N \longrightarrow K_{l-1} \otimes_A X \longrightarrow K_{l-1} \otimes_A M \to 0,$$

$$0 \to K_l \otimes_A N \longrightarrow K_l \otimes_A X \longrightarrow P_{l-1} \otimes_A M \longrightarrow K_{l-1} \otimes_A M \to 0,$$

are equivalent up to a sign. Considering α to correspond to the Yoneda splice of all the short exact sequences (1.6.5), we find by induction on i that $\alpha \cdot \beta = (-1)^{ij} \beta \cdot \alpha$.

Finally, if $\beta \in \mathrm{Ext}^j_A(M, N)$ for $j > 1$, view β as a Yoneda splice of j short exact sequences. By induction on j, we have $\alpha \cdot \beta = (-1)^{ij} \beta \cdot \alpha$. □

Note that $\mathrm{Ext}^*_A(M, M)$ is itself an algebra with product given by Yoneda product or equivalently Yoneda composition. The *graded center* is the subalgebra $Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(M, M))$ generated by all homogeneous elements $\alpha \in \mathrm{Ext}^*_A(M, M)$ such that $\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta \cdot \alpha$ for all homogeneous $\beta \in \mathrm{Ext}^*_A(M, M)$.

Theorem 1.6.3 has the following corollary:

Corollary 1.6.6. *Let M be an A-module. The map*

$$\phi_M : \mathrm{HH}^*(A) \to \mathrm{Ext}^*_A(M, M),$$

defined at the chain level by $\phi_M(f) = f \otimes \mathrm{id}_M$ as in (1.6.2), is a ring homomorphism whose image is contained in the graded center $Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(M, M))$.

For some algebras and modules, there are general results stating precisely the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(M, M)$. For Koszul algebras and a canonical choice of M, the map is surjective. (See [**BGMS05**, Theorem 4.1] for details; some discussion is in Section 2.3.) For some more general algebras defined by quivers and relations, the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(M, M)$, for a canonical choice of M, is the A_∞-center. (Some discussion is in Section 6.4; see [**BG**] for details.)

Example 1.6.7. Let $A = k[x]/(x^n)$ as in Examples 1.1.16 and 1.3.7. Let $M = k$, an A-module via the augmentation map $\varepsilon : A \to k$ given by $\varepsilon(x) = 0$. The following is a free resolution of k as an A-module:

$$\cdots \longrightarrow A \xrightarrow{x^{n-1}\cdot} A \xrightarrow{x\cdot} A \xrightarrow{x^{n-1}\cdot} A \xrightarrow{x\cdot} A \xrightarrow{\varepsilon} k \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(-, k)$, dropping the term $\mathrm{Hom}_A(k, k)$, and identifying the space $\mathrm{Hom}_A(A, k)$ with k, we have:

$$\cdots \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k \longleftarrow 0$$

The maps are indeed all zero maps since $\varepsilon(x) = 0$ and $\varepsilon(x^{n-1}) = 0$. Therefore $\mathrm{Ext}^i_A(k, k) \cong k$ for all i. Under a Yoneda/cup product defined in the same way as that for Hochschild cohomology, similar calculations to those in Example 1.3.7 show that

$$\mathrm{Ext}^*_A(k, k) \cong \begin{cases} k[y], & \text{if } n = 2, \\ k[y, z]/(y^2), & \text{otherwise,} \end{cases}$$

where $|y| = 1$ and $|z| = 2$. We will determine the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(k, k)$ under the map ϕ_k defined by (1.6.2).

First assume that n is not divisible by $p = \mathrm{char}(k)$ and note that by the definition of g in Example 1.3.7, $\phi_k(g) = 0$. If f is the function defined in Example 1.3.7 and $n = 2$, then $\phi_k(f) = y^2$, and if $n > 2$ then $\phi_k(f) = z$. Next assume that p divides n. If $n = 2$, then $\phi_k(g) = y$ and $\phi_k(f) = y^2$. If $n > 2$, then $\phi_k(g) = y$ and $\phi_k(f) = z$. In particular, note that in case $n = 2$, independently of the characteristic p of k, the image of $\mathrm{HH}^*(A)$ under ϕ_k is the full graded center of $\mathrm{Ext}^*_A(k, k)$. In case $n > 2$ and n is not divisible by $\mathrm{char}(k)$, the image is not the full graded center. We will return to this example in Section 6.4.

If M is an A-bimodule, then $\mathrm{HH}^*(A)$ acts on $\mathrm{HH}^*(A, M)$ similarly, by taking a Yoneda product, or by first tensoring with M and then taking a Yoneda product. Again these two actions agree up to a sign by a proof similar to that of Theorem 1.6.3. See [**SS04**, Theorem 1.1] for a more general setting that includes as special cases both this statement and Theorem 1.6.3 (by taking one of the two rings to be the field in [**SS04**]).

Chapter 2

Examples

In this chapter, we look at some algebras for which there are elementary techniques for computing and understanding Hochschild (co)homology. These algebras will provide a rich assortment of examples on which to draw in later chapters. In particular, we will consider tensor products and twisted tensor products of algebras, Koszul algebras, monomial algebras, and skew group algebras.

From now on, we let k be a field. Generally A will denote an algebra over k.

2.1. Tensor product of algebras

Let A and B be k-algebras. Their *tensor product algebra* is $A \otimes B$ as a vector space, with multiplication determined by

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$$

for all $a, a' \in A$ and $b, b' \in B$. If A and B are $\mathbb{N}$-graded algebras, their *graded tensor product algebra* is $A \otimes B$ as a vector space, with

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$$

for all homogeneous $a, a' \in A$ and $b, b' \in B$, where $|a'|$ denotes the degree of a' in $\mathbb{N}$, and similarly $|b|$. (We have used the same notation for homological degree, but this should cause no confusion, as it will be clear in each context which degree is meant. In Theorem 2.1.2 below, they are the same, since Hochschild cohomology is graded by its homological degree.)

Under some finiteness conditions, the Hochschild cohomology ring of a tensor product of two algebras is the graded tensor product of the Hochschild cohomology rings of the two algebras, as we will see in Theorem 2.1.2 below.

This will allow us to understand many algebras that are tensor products of simpler algebras. First we will construct a needed resolution.

Let $P_{\bullet}$ be a projective A^e-resolution of A, and $Q_{\bullet}$ a projective B^e-resolution of B. Taking their tensor product over k, consider the complex

$$P_{\bullet} \otimes Q_{\bullet} \tag{2.1.1}$$

and its corresponding total complex $\mathrm{Tot}(P_{\bullet} \otimes Q_{\bullet})$ to which we sometimes refer as simply $P_{\bullet} \otimes Q_{\bullet}$ when no confusion will arise. We claim that for each i, j, the $(A \otimes B)^e$-module $P_i \otimes Q_j$ is projective: Since P_i is a projective A^e-module, there is an A^e-module P_i' such that $P_i \oplus P_i' \cong (A^e)^{\oplus I}$ for some indexing set I. Similarly, there is a B^e-module Q_j' such that $Q_j \oplus Q_j' \cong (B^e)^{\oplus J}$ for some indexing set J. Since tensor product distributes over direct sum, $P_i \otimes Q_j$ is a direct summand of

$$(A^e)^{\oplus I} \otimes (B^e)^{\oplus J} \cong (A^e \otimes B^e)^{\oplus (I \times J)} \cong ((A \otimes B)^e)^{\oplus (I \times J)}$$

as vector spaces. The module actions of A and B commute, and so $P_i \otimes Q_j$ is in fact a direct summand of $((A \otimes B)^e)^{\oplus (I \times J)}$ as an $(A \otimes B)^e$-module. By the Künneth Theorem (Theorem A.4.1), since the tensor product is over the field k, the total complex of $P_{\bullet} \otimes Q_{\bullet}$ has homology concentrated in degree 0, where it is $\mathrm{H}_0(P_{\bullet} \otimes Q_{\bullet}) \cong \mathrm{H}_0(P_{\bullet}) \otimes \mathrm{H}_0(Q_{\bullet}) \cong A \otimes B$. Thus $P_{\bullet} \otimes Q_{\bullet}$ is a projective resolution of $A \otimes B$ as an $(A \otimes B)^e$-module.

Theorem 2.1.2. *Let A and B be finite dimensional k-algebras. Then*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B)$$

as algebras, where the algebra on the right side is a graded tensor product algebra.

If A and B are themselves $\mathbb{N}$-graded algebras, we may weaken the hypothesis in the theorem to assume instead that they are locally finite dimensional, that is, each graded component is finite dimensional. The isomorphism in the theorem is in fact an isomorphism of Gerstenhaber algebras; see [**LZ14**] for the definition of Gerstenhaber bracket on a graded tensor product of Gerstenhaber algebras, and a proof of this statement.

Proof of Theorem 2.1.2. Let $P_{\bullet} \otimes Q_{\bullet}$ be the resolution (2.1.1) defined above. The Hochschild cohomology $\mathrm{HH}^*(A \otimes B)$ is the cohomology of the total complex of $\mathrm{Hom}_{(A\otimes B)^e}(P_{\bullet} \otimes Q_{\bullet}, A \otimes B)$. Without loss of generality we may assume that $P_{\bullet}$ and $Q_{\bullet}$ are free resolutions, so that $P_i \otimes Q_j \cong (A^e)^{\oplus I} \otimes (B^e)^{\oplus J} \cong ((A \otimes B)^e)^{\oplus (I \times J)}$, for some finite indexing sets I, J. Thus there are finite dimensional vector spaces P_i', Q_j' such that $P_i \cong A \otimes P_i' \otimes A$, $Q_j \cong B \otimes Q_j' \otimes B$, and so

$$\mathrm{Hom}_{(A\otimes B)^e}(P_i \otimes Q_j, A \otimes B) \cong \mathrm{Hom}_k(P_i' \otimes Q_j', A \otimes B).$$

Consider embedding $\mathrm{Hom}_k(P'_i, A)\otimes\mathrm{Hom}_k(Q'_j, B)\hookrightarrow\mathrm{Hom}_k(P'_i\otimes Q'_j, A\otimes B)$ via the tensor product of functions. Since P'_i, Q'_j, A, and B are finite dimensional vector spaces, this is an isomorphism, and moreover the differentials correspond under this isomorphism. By the Künneth Theorem (Theorem A.4.1), since the tensor product is taken over the field k, the cohomology is $\mathrm{HH}^*(A)\otimes\mathrm{HH}^*(B)$ as a vector space.

Next we determine the cup product structure. Define a diagonal map $\Delta : P_\bullet\otimes Q_\bullet\to(P_\bullet\otimes Q_\bullet)\otimes_{A\otimes B}(P_\bullet\otimes Q_\bullet)$ on the total complex of $P_\bullet\otimes Q_\bullet$ by

$$P_\bullet\otimes Q_\bullet\xrightarrow{\Delta_A\otimes\Delta_B}(P_\bullet\otimes_A P_\bullet)\otimes(Q_\bullet\otimes_B Q_\bullet)\xrightarrow{\sim}(P_\bullet\otimes Q_\bullet)\otimes_{A\otimes B}(P_\bullet\otimes Q_\bullet),$$

where Δ_A, Δ_B are diagonal maps for $P_\bullet$, $Q_\bullet$, respectively. The second map is the isomorphism given by interchanging factors, with a sign, that is,

$$(x\otimes_A x')\otimes(y\otimes_B y')\mapsto(-1)^{|x'||y|}(x\otimes y)\otimes_{A\otimes B}(x'\otimes y')$$

for all homogeneous $x, x'\in P_\bullet$ and $y, y'\in Q_\bullet$. Then Δ is a chain map by its definition and by the definition of the differential of a tensor product complex. Now let f, g be homogeneous elements of degrees m, n in $\mathrm{Hom}_{A^e}(P_\bullet, A)$ and f', g' be homogeneous elements of degrees m', n' in $\mathrm{Hom}_{B^e}(Q_\bullet, B)$. Then $f\otimes f'$ is an element in

$$\begin{aligned}\mathrm{Hom}_{A^e}(P_m, A)\otimes\mathrm{Hom}_{B^e}(Q_{m'}, B) &\cong \mathrm{Hom}_k(P_m, A)\otimes\mathrm{Hom}_k(Q_{m'}, B)\\ &\cong \mathrm{Hom}_k(P_m\otimes Q_{m'}, A\otimes B)\\ &\cong \mathrm{Hom}_{(A\otimes B)^e}(P_m\otimes Q_{m'}, A\otimes B),\end{aligned}$$

and $g\otimes g'$ may be identified with an element in $\mathrm{Hom}_{(A\otimes B)^e}(P_n\otimes Q_{n'}, A\otimes B)$. The cup product $(f\otimes f')\smile(g\otimes g')$ may be determined using the diagonal map Δ and formula (1.3.9): Let $x\in P_r$, $y\in Q_s$ for some r, s with $r+s = m+n+m'+n'$, and suppose $\Delta_A(x)$ has component in $P_m\otimes P_n$ given by $\sum_s x'_s\otimes x''_s$ and $\Delta_B(y)$ has component in $Q_{m'}\otimes Q_{n'}$ given by $\sum_t y'_t\otimes y''_t$. (This is the only component on which the cup product potentially takes a nonzero value.) Letting π_A, π_B, and $\pi_{A\otimes B}$ denote multiplication on A, B, and $A\otimes B$, respectively,

$$\begin{aligned}&((f\otimes f')\smile(g\otimes g'))(x\otimes y)\\ &=\pi_{A\otimes B}(f\otimes f'\otimes g\otimes g')(\Delta(x\otimes y))\\ &=\pi_{A\otimes B}\left(\sum_{s,t}(-1)^{|x''_s||y'_t|}f(x'_s)\otimes f'(y'_t)\otimes g(x''_s)\otimes g'(y''_t)\right)\\ &=(-1)^{m'n}(\sum_s f(x'_s)g(x''_s))\otimes(\sum_t f'(y'_t)g'(y''_t))\\ &=(-1)^{m'n}\pi_A\big((f\otimes g)\Delta_A(x)\big)\otimes\pi_B\big((f'\otimes g')\Delta_B(y)\big)\\ &=(-1)^{m'n}((f\smile g)(x))\otimes((f'\smile g')(y)),\end{aligned}$$

and so $(f \otimes f') \smile (g \otimes g') = (-1)^{m'n}(f \smile g) \otimes (f' \smile g')$. □

The theorem allows us to compute Hochschild cohomology for many more algebras from those we already have, as the following examples show. Our first examples are polynomial rings.

Example 2.1.3. Let $A_1 = k[x_1]$, $A_2 = k[x_2]$, and $A = A_1 \otimes A_2$. Then $A \cong k[x_1, x_2]$, a polynomial ring in two indeterminates. In Example 1.3.5, we found that $\mathrm{HH}^*(A_i) \cong k[x_i, y_i]/(y_i^2)$, where $|x_i| = 0$, $|y_i| = 1$, for $i = 1, 2$. By Theorem 2.1.2, the Hochschild cohomology $\mathrm{HH}^*(k[x_1, x_2])$ is the graded tensor product $\mathrm{HH}^*(A_1) \otimes \mathrm{HH}^*(A_2)$. We thus see that

$$\mathrm{HH}^*(k[x_1, x_2]) \cong k[x_1, x_2] \otimes \bigwedge(y_1, y_2)$$

as a graded algebra, where by $\bigwedge(y_1, y_2)$ we mean the exterior algebra on a vector space W with basis y_1, y_2 in degree 1 (and x_1, x_2 have degree 0).

More generally, by induction on the number m of indeterminates,

$$\mathrm{HH}^*(k[x_1, \dots, x_m]) \cong k[x_1, \dots, x_m] \otimes \bigwedge(y_1, \dots, y_m).$$

The vector space W with basis $y_1, \dots, y_m$ identifies naturally with the dual space V^* of $V = \mathrm{Span}_k\{x_1, \dots, x_m\}$, and we may view the Hochschild cohomology of $A = k[x_1, \dots, x_m]$ as simply $A \otimes \bigwedge(V^*)$. We explain this in more detail next.

The A^e-projective resolution of $A = k[x_1, \dots, x_m]$ obtained by the iterated tensor product construction mentioned above may be described as follows. As a graded vector space, let

$$P_{\bullet} = A \otimes \textstyle\bigwedge^{\bullet} V \otimes A, \tag{2.1.4}$$

where $V = \mathrm{Span}_k\{x_1, \dots, x_m\}$. As a free A^e-module, we may identify each P_n with the degree n component of the tensor product of n copies of resolution (1.1.15), one for each of $x_1, \dots, x_m$. Under this identification, the differential on $P_{\bullet}$ is given by

$$\begin{aligned} d_n(1 \otimes v_1 \wedge \cdots \wedge v_n \otimes 1) = \sum_{i=1}^{n} (-1)^{i-1}\big(& v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_n \otimes 1 \\ & - 1 \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_n \otimes v_i\big) \end{aligned}$$

for all $v_1, \dots, v_n \in V$, where the notation $\hat{v}_i$ indicates a wedge factor that is missing. Applying $\mathrm{Hom}_{A^e}(-, A)$, since A is commutative, we see that the induced differentials are all 0. Thus the Hochschild cohomology can be identified with $A \otimes \bigwedge(V^*)$ as stated above.

Our next examples are truncated polynomial rings.

Example 2.1.5. Let $n_1, n_2 \geq 2$. Let $A_1 = k[x_1]/(x_1^{n_1})$, $A_2 = k[x_2]/(x_2^{n_2})$, and $A = A_1 \otimes A_2$. Then $A \cong k[x_1, x_2]/(x_1^{n_1}, x_2^{n_2})$. In Example 1.3.7, we found that if n_i is not divisible by $\mathrm{char}(k)$, then $\mathrm{HH}^*(A_i) \cong k[x_i, y_i, z_i]/(x_i^{n_i-1}y_i,\ y_i^2)$ where $|x_i| = 0$, $|y_i| = 1$, $|z_i| = 2$. If neither n_1 nor n_2 is divisible by $\mathrm{char}(k)$, then by Theorem 2.1.2,

$$\mathrm{HH}^*(k[x_1, x_2]/(x_1^{n_1}, x_2^{n_2}) \cong k[x_1, x_2, z_1, z_2] \otimes \bigwedge(y_1, y_2)/(x_1^{n_1-1} \otimes y_1,\ x_2^{n_2-1} \otimes y_2).$$

The cases where one or both of n_1, n_2 is divisible by $\mathrm{char}(k)$ may be treated similarly. More generally, we see by induction on the number m of generators that if none of $n_1, \ldots, n_m$ is divisible by $\mathrm{char}(k)$, then

$$\begin{aligned}&\mathrm{HH}^*(k[x_1, \ldots, x_m]/(x_1^{n_1}, \ldots, x_m^{n_m}))\\ &\cong k[x_1, \ldots, x_m, z_1, \ldots, z_m] \otimes \bigwedge(y_1, \ldots, y_m)/(x_1^{n_1-1} \otimes y_1, \ldots, x_m^{n_m-1} \otimes y_m)\end{aligned}$$

where $|x_i| = 0$, $|y_i| = 1$, $|z_i| = 2$. If one or more of $n_1, \ldots, n_m$ is divisible by $\mathrm{char}(k)$, then we obtain the Hochschild cohomology by applying Theorem 2.1.2 repeatedly, using the results of Example 1.3.7, for a similar expression.

It is useful to record the resolution of $A = k[x_1, \ldots, x_m]/(x_1^{n_1}, \ldots, x_m^{n_m})$ obtained as an iterated tensor product of those of factors $A_i = k[x_i]/(x_i^{n_i})$: Let V be the vector space with basis $\xi_1, \ldots, \xi_m$ and let $S(V)$ be the symmetric algebra on the symbols $\xi_1, \ldots, \xi_m$ (polynomials in these indeterminates). Let

$$P_\bullet = A \otimes S(V) \otimes A,$$

where $|\xi_i| = 1$ for each i, that is, for each j, P_j is the free A^e-module on monomials in $S(V)$ of degree j. Then $P_\bullet$ may be identified with the total complex of the tensor product of m copies of resolution (1.1.17), one for each x_i: The monomial $1 \otimes \xi_1^{i_1} \cdots \xi_m^{i_m} \otimes 1$ is identified with the tensor product of m copies of $1 \otimes 1$, one in each $A_i \otimes A_i$, where the exponents $i_1, \ldots, i_m$ indicate the homological degree of the corresponding factor. We see then that the differential is given by $d = \sum_{j=1}^m d_j$ where

(2.1.6)

$$\begin{aligned}&d_j(1 \otimes \xi_1^{i_1} \cdots \xi_m^{i_m} \otimes 1)\\ &= (-1)^{i_1 + \cdots + i_{j-1}} \begin{cases} 0, & \text{if } i_j = 0,\\ u_j \cdot (1 \otimes \xi_1^{i_1} \cdots \xi_j^{i_j - 1} \cdots \xi_m^{i_m} \otimes 1), & \text{if } i_j \text{ is odd},\\ v_j \cdot (1 \otimes \xi_1^{i_1} \cdots \xi_j^{i_j - 1} \cdots \xi_m^{i_m} \otimes 1), & \text{if } i_j \text{ is even and } i_j > 0\end{cases}\end{aligned}$$

with $u_j = x_j \otimes 1 - 1 \otimes x_j$ and $v_j = x_j^{n_j-1} \otimes 1 + x_j^{n_j-2} \otimes x_j + \cdots + 1 \otimes x_j^{n_j-1}$. Applying $\mathrm{Hom}_{A^e}(-, A)$, we find that since A is commutative, the induced maps in odd degrees are all 0 while in even degrees they are multiplication by $n_j x_j^{n_j-1}$ (which is 0 if and only if n_j is divisible by $\mathrm{char}(k)$). This results in the graded vector space structure of $\mathrm{HH}^*(A)$ as claimed.

2.2. Twisted tensor product of algebras

Twisted tensor products generalize the graded tensor product of ($\mathbb{N}$-)graded algebras discussed above. Here we present the work of Bergh and Oppermann [**BO08**] on twisting resolutions when the twisting arises from a bicharacter on grading groups. In the case of an $\mathbb{N}$-graded algebra, the grading group may be considered to be $\mathbb{Z}$ via the canonical embedding $\mathbb{N} \subset \mathbb{Z}$. More general versions of twisting are dealt with in the literature.

Let A_1, A_2 be k-algebras that are graded by abelian groups Γ_1, Γ_2, respectively. That is,

$$A_i = \bigoplus_{\gamma \in \Gamma_i} (A_i)_\gamma,$$

a direct sum of vector spaces, for $i = 1, 2$, such that $(A_i)_\gamma (A_i)_{\gamma'} \subset (A_i)_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma_i$, where we write the groups Γ_i multiplicatively. Let $t : \Gamma_1 \times \Gamma_2 \to k^\times$ be a *bicharacter*, that is,

$$\begin{aligned} t(1_{\Gamma_1}, \gamma_2) &= t(\gamma_1, 1_{\Gamma_2}) = 1, \\ t(\gamma_1\gamma_1', \gamma_2) &= t(\gamma_1, \gamma_2)t(\gamma_1', \gamma_2), \\ t(\gamma_1, \gamma_2\gamma_2') &= t(\gamma_1, \gamma_2)t(\gamma_1, \gamma_2') \end{aligned}$$

for all $\gamma_i, \gamma_i' \in \Gamma_i$, in which 1_{Γ_i} denotes the identity element of Γ_i $(i = 1, 2)$. The *twisted tensor product algebra*

$$A = A_1 \otimes^t A_2$$

is $A_1 \otimes A_2$ as a vector space, and multiplication is determined by

$$(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = t(|a_1'|, |a_2|) a_1 a_1' \otimes a_2 a_2'$$

for all homogeneous elements $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$, where $|a_1'|$ denotes the degree of a_1' in Γ_1 and similarly $|a_2|$ in Γ_2. By its definition, $A_1 \otimes^t A_2$ is graded by the group $\Gamma_1 \times \Gamma_2$: $(A_1 \otimes^t A_2)_{(\gamma_1, \gamma_2)} = (A_1)_{\gamma_1} \otimes (A_2)_{\gamma_2}$ for all $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$.

Our first example is the quantum plane.

Example 2.2.1. Let $A_1 = k[x_1]$ and $A_2 = k[x_2]$, each graded by $\mathbb{Z}$ in the usual way: $|x_1| = |x_2| = 1$, where 1 denotes the generator of the abelian group $\mathbb{Z}$ (to mix multiplicative and additive notation). Let q be any nonzero scalar. Define $t : \mathbb{Z} \times \mathbb{Z} \to k^\times$ by $t(1, 1) = q^{-1}$ (this extends uniquely to a bicharacter since each factor of $\mathbb{Z}$ in $\mathbb{Z} \times \mathbb{Z}$ is a free abelian group with generator 1). So $t(m, n) = q^{mn}$ for all $m, n \in \mathbb{Z}$. Denote by $k\langle x_1, x_2 \rangle$ the free k-algebra on x_1, x_2, that is, it is the tensor algebra $T(V) = T_k(V)$ where V is a vector space with basis x_1, x_2. The twisted tensor product $A_1 \otimes^t A_2$ of A_1 and A_2 is given by

$$A_1 \otimes^t A_2 \cong k\langle x_1, x_2 \rangle / (x_1 x_2 - q x_2 x_1),$$

called a *skew polynomial ring* or a *quantum plane*, and is denoted $k_q[x_1, x_2]$. The latter terminology recalls the Cartesian plane, which may be identified with the set of maximal ideals of the commutative polynomial ring $k[x_1, x_2]$ (the case $q = 1$). More generally we iterate this construction to obtain a *skew polynomial ring* or *quantum affine space*

$$k_{\mathbf{q}}[x_1, \dots, x_m] = k\langle x_1, \dots, x_m\rangle/(x_i x_j - q_{ij} x_j x_i \mid 1 \le i, j \le m)$$

determined by a set $\mathbf{q} = \{q_{ij}\}$ of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j $(1 \le i, j \le m)$. We call such a set $\mathbf{q}$ a *quantum system of parameters*. Sometimes $k_{\mathbf{q}}[x_1, \dots, x_m]$ is also called a *quantum symmetric algebra*, however this term is also used both more generally and slightly differently in the context of Nichols algebras, so we will not use it here.

Our next example is a noncommutative version of a truncated polynomial ring.

Example 2.2.2. Let n_1, n_2 be positive integers, $n_1, n_2 \ge 2$, and let $A_1 = k[x_1]/(x_1^{n_1})$ and $A_2 = k[x_2]/(x_2^{n_2})$. Each of A_1, A_2 is $\mathbb{Z}$-graded just as in Example 2.2.1. Alternatively, A_i may be viewed as being graded by $\mathbb{Z}/n_i\mathbb{Z}$ for $i = 1, 2$. Define a bicharacter t just as in Example 2.2.1; if we view each A_i as being graded by a finite quotient of $\mathbb{Z}$ instead of by $\mathbb{Z}$ itself, we must choose q to be a suitable root of unity. The twisted tensor product algebra $A_1 \otimes^t A_2$ is isomorphic to $k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$. More generally we iterate this construction to obtain a *truncated skew polynomial ring*,

$$k\langle x_1, \dots, x_m\rangle/(x_i x_j - q_{ij} x_j x_i,\ x_i^{n_i} \mid 1 \le i, j \le m)$$

determined by a quantum system of parameters $\mathbf{q} = \{q_{ij}\}$ as defined in Example 2.2.1, and positive integers $n_1, \dots, n_m$. These algebras are also called *quantum complete intersections*, recalling the special case in which all $q_{ij} = 1$ (a commutative algebra that is an example of a complete intersection).

We return to the general case of a twisted tensor product $A_1 \otimes^t A_2$ of algebras A_1, A_2 determined by abelian grading groups Γ_1, Γ_2 and a bicharacter $t : \Gamma_1 \times \Gamma_2 \to k^\times$, and construct a projective resolution of the $(A_1 \otimes^t A_2)^e$-module $A_1 \otimes^t A_2$ from those of A_1 and A_2. Let $P_\bullet$ be a graded A_1^e-projective resolution of A_1, that is, each A_1^e-module P_i is graded by Γ_1 in such a way that $(A_1)_\gamma (P_i)_{\gamma'} (A_1)_{\gamma''} \subset (P_i)_{\gamma\gamma'\gamma''}$ for all $\gamma, \gamma', \gamma'' \in \Gamma_1$, the differentials preserve the grading, and P_i embeds as a direct summand of a free module via a graded map (that is, that preserves Γ_1-degree). For example, the bar resolution (1.1.4) is graded (by grading a tensor product in the usual way, $(A \otimes A)_\gamma = \oplus_{\gamma'\gamma''=\gamma}(A_{\gamma'} \otimes A_{\gamma''})$). Similarly, let $Q_\bullet$ be a graded projective resolution of the A_2^e-module A_2.

Consider the tensor product complex

$$P_{\bullet} \otimes Q_{\bullet}$$

as a complex of vector spaces, as well as its total complex, for which we use the same notation. By the Künneth Theorem (Theorem A.4.1), since the tensor product is over the field k, the total complex has homology 0 in all positive degrees and in degree 0 its homology is the vector space $A_1 \otimes A_2$. We will put the structure of an $(A_1 \otimes^t A_2)^e$-module on each $P_i \otimes Q_j$ in such a way that it is projective, and it will follow that the total complex of $P_{\bullet} \otimes Q_{\bullet}$ is an $(A_1 \otimes^t A_2)^e$-projective resolution of $A_1 \otimes^t A_2$. For all homogeneous $x \in P_i$, $y \in Q_j$, $a_1, a_1' \in A_1$, and $a_2, a_2' \in A_2$, set

$$(a_1 \otimes a_2)(x \otimes y)(a_1' \otimes a_2') = t(|x||a_1'|, |a_2|)t(|a_1'|, |y|)a_1 x a_1' \otimes a_2 y a_2'.$$

A calculation shows that this gives $P_i \otimes Q_j$ the structure of an $(A_1 \otimes^t A_2)^e$-module. Moreover, it is projective: Since P_i is a graded projective A_1^e-module, it is a direct summand of a direct sum of copies of A_1^e, say $(A_1^e)^{\oplus I}$ where I is some indexing set, and the embedding is a graded map. Similarly, Q_j is a direct summand of $(A_2^e)^{\oplus J}$ where J is some indexing set. We show that $P_i \otimes Q_j$ is a direct summand of a direct sum of copies of $(A_1 \otimes^t A_2)^e$ as an $(A_1 \otimes^t A_2)^e$-module: Since tensor product distributes over direct sum, as a vector space, $P_i \otimes Q_j$ is a direct summand of $(A_1^e)^{\oplus I} \otimes (A_2^e)^{\oplus J} \cong (A_1^e \otimes A_2^e)^{\oplus (I \times J)}$. Now, $A_1^e \otimes A_2^e \cong (A_1 \otimes^t A_2)^e$ as $(A_1 \otimes^t A_2)^e$-modules by definition of the module structure, and the embeddings of P_i, Q_j into free modules are graded maps and so $P_i \otimes Q_j$ embeds into this direct sum as an $(A_1 \otimes^t A_2)^e$-module.

Remark 2.2.3. Suppose that A_1, A_2 are augmented algebras, that is, for each i, there is an algebra homomorphism $\varepsilon_i : A_i \to k$, called an *augmentation map*. If these augmentation maps ε_1, ε_2 are graded maps, then the twisted tensor product $A_1 \otimes^t A_2$ is augmented by $\varepsilon = \varepsilon_1 \otimes \varepsilon_2$. Consider k itself to be a module for each of $A_1, A_2, A_1 \otimes^t A_2$ via the augmentation maps $\varepsilon_1, \varepsilon_2, \varepsilon$, respectively. A construction similar to that above leads to a projective resolution of k as $A_1 \otimes^t A_2$-module from projective resolutions of k as A_1-module and as A_2-module. See [**BO08**] for details.

The resolution given by the total complex of $P_{\bullet} \otimes Q_{\bullet}$ constructed above may be used to compute Hochschild cohomology of the twisted tensor product algebra $A_1 \otimes^t A_2$. There is however not a result such as Theorem 2.1.2 that describes the full Hochschild cohomology ring of $A_1 \otimes^t A_2$ in terms of that of A_1 and A_2. (For one thing, Hochschild cohomology is graded commutative, while the twisted tensor product typically is not.) The grading on $P_{\bullet}$, $Q_{\bullet}$ by Γ_1, Γ_2 does impart some structure to the Hochschild cohomology of $A_1 \otimes^t A_2$, and a consequence is that there is a result analogous to Theorem 2.1.2 for the part of Hochschild cohomology graded by the bikernel of the bicharacter t. One proves this by retracing the steps of the proof of Theorem 2.1.2, noting that the sequence of isomorphisms of Hom spaces given there works for subspaces graded by the bikernel of the bicharacter here. See [**BO08**, Theorem 4.7].

We give details of the resolution of the twisted tensor product $A_1 \otimes^t A_2$ constructed above for our Examples 2.2.1 and 2.2.2.

Example 2.2.4. Let $A = k_q[x_1, x_2] \cong A_1 \otimes^t A_2$ as in Example 2.2.1. Let $P_{\bullet}$ be the resolution (1.1.15) of Example 1.1.14 for A_1, where we shift the $\mathbb{Z}$-grading of A_1^e in the homological degree 1 component so that $1 \otimes 1$ there has graded degree 1. Then the differential is a graded map. Let $Q_{\bullet}$ be a similar resolution for A_2. Then $P_{\bullet} \otimes Q_{\bullet}$ is an A^e-projective resolution of A. Apply $\mathrm{Hom}_{A^e}(-, A)$ to obtain

$$0 \longleftarrow \mathrm{Hom}_{A^e}(P_1 \otimes Q_1, A) \longleftarrow \mathrm{Hom}_{A^e}\big((P_0 \otimes Q_1) \oplus (P_1 \otimes Q_0), A\big) \\ \longleftarrow \mathrm{Hom}_{A^e}(A^e, A) \longleftarrow 0,$$

which is equivalent, under standard isomorphisms, to

$$0 \longleftarrow A \xleftarrow{d_2^*} A \oplus A \xleftarrow{d_1^*} A \longleftarrow 0.$$

Under our identifications and degree shifts, a calculation shows that the maps d_1^*, d_2^* are given by

$$\begin{aligned} d_1^*(a \otimes b) &= \big((q^{|a|} - 1)a \otimes bx_2,\ (1 - q^{|b|})ax_1 \otimes b\big), \\ d_2^*(a \otimes b,\ a' \otimes b') &= (1 - q^{|b|+1})ax_1 \otimes b + (1 - q^{|a'|+1})a' \otimes b'x_2 \end{aligned}$$

for all homogeneous $a, a' \in A_1$ and $b, b' \in A_2$. (Note that in case $q = 1$, these differentials are indeed 0, in accordance with Example 2.1.3.) We may iterate this construction to obtain a free resolution of A as an A^e-module for $A = k_{\mathbf{q}}[x_1, \ldots, x_m]$.

Example 2.2.5. Another important family of algebras that may be constructed as twisted tensor products are the quantum complete intersections of Example 2.2.2. For simplicity, we focus here on the case $m = 2$ and $n_1 = n_2 = 2$, but similar techniques yield information about the more general case. Buchweitz, Green, Madsen, and Solberg [**BGMS05**] used precisely these examples to answer a question of Happel [**Hap89**], showing that these are finite dimensional algebras that have finite dimensional Hochschild cohomology, and yet have infinite global dimension. We give some details next.

Let $P_{\bullet}$ be the resolution (1.1.17) for $A_1 = k[x_1]/(x_1^{n_1})$ and let $Q_{\bullet}$ be the corresponding resolution for $A_2 = k[x_2]/(x_2^{n_2})$. As we have seen, $P_{\bullet} \otimes Q_{\bullet}$ can be given the structure of an A^e-projective resolution of A where $A = A_1 \otimes^t A_2$. This is equivalent to the minimal projective resolution constructed in [**BGMS05**]. Consider k to be an A-module on which x_1, x_2 each act as 0, that is, A is augmented with augmentation map $\varepsilon : A \to k$ given by the algebra homomorphism sending each of x_1, x_2 to 0. Applying $- \otimes_A k$ to $P_\bullet \otimes Q_\bullet$, we obtain the minimal projective resolution of k as an A-module, which is not bounded, and it follows that A has infinite global dimension.

If q is not a root of unity and $\mathrm{char}(k) \neq 2$, then $\mathrm{HH}^*(A)$ is 5-dimensional as a vector space: In this case, computations using our resolution $P_\bullet \otimes Q_\bullet$ show that $\mathrm{HH}^0(A)$ is a vector space spanned by 1 and x_1x_2 (the center of A), $\mathrm{HH}^1(A)$ is a vector space spanned by elements y_1 and y_2 that arise from functions at the chain level taking $(1 \otimes 1) \otimes (1 \otimes 1)$ in degree $(1, 0)$ to $x_1 \otimes 1$ and $(1 \otimes 1) \otimes (1 \otimes 1)$ in degree $(0, 1)$ to $1 \otimes x_2$, respectively, and $\mathrm{HH}^2(A)$ is a vector space spanned by $y_1 \smile y_2$. The cup product of the degree 0 element x_1x_2 with y_i is 0 for $i = 1, 2$. Thus there is an isomorphism of algebras

$$\mathrm{HH}^*(A) \cong k[x_1x_2]/((x_1x_2)^2) \times_k \bigwedge(y_1, y_2),$$

where the latter is a fiber product. (The *fiber product* $R_1 \times_k R_2$ of two augmented k-algebras R_1, R_2 is the subring of $R_1 \oplus R_2$ consisting of pairs (r_1, r_2) such that the images of r_1 and r_2 under the respective augmentation maps are equal, i.e., it is the pullback of the two augmentation maps.) See [**BGMS05**] for details, as well as the cases where $\mathrm{char}(k) = 2$ or where $q = 0$.

The case that q is a root of unity is also very interesting. In this case, Hochschild cohomology is infinite dimensional, yet there are large gaps, that is, it is 0 in infinitely many degrees. See [**BGMS05**].

2.3. Koszul algebras and Koszul complexes

Some of the examples we have seen in this chapter are in fact Koszul algebras. We will define them next, revisiting our earlier examples in this light. Koszul algebras were introduced by Priddy [**Pri70**]. We will assume here that our algebra A is graded (by $\mathbb{N}$) and *connected*, that is $A_0 = k$. More general Koszul algebras are defined in the literature.

Let V be a finite dimensional vector space and let $T(V) = T_k(V)$ denote the *tensor algebra* of V:

$$T(V) = \oplus_{n \geq 0} T^n(V)$$

where $T^0(V) = k$, $T^1(V) = V$, and $T^n(V) = V \otimes \cdots \otimes V$ (n tensor factors). Multiplication is simply $\otimes$, that is,

$$(v_1 \otimes \cdots \otimes v_m) \cdot (v_1' \otimes \cdots \otimes v_n') = v_1 \otimes \cdots \otimes v_m \otimes v_1' \otimes \cdots \otimes v_n'$$

for all $v_1, \ldots, v_m, v_1', \ldots, v_n' \in V$. Then $T(V)$ is a graded algebra with $|v| = 1$ for all $v \in V$. Sometimes we write $v_1 \cdots v_m$ in place of $v_1 \otimes \cdots \otimes v_m$, for simplicity of notation. Let R be a subspace of $T^2(V)$, that is

$$R \subset V \otimes V,$$

and let

$$A = T(V)/(R),$$

where (R) denotes the (two-sided) ideal generated by R in $T(V)$. We call R the space of *relations* for A. By definition, A is a *quadratic algebra*, that is A is a graded algebra generated by elements in degree 1 and with relations in degree 2.

Let $V^* = \mathrm{Hom}_k(V, k)$ be the dual vector space to V. Since V is finite dimensional, we may identify $(V \otimes V)^*$ with $V^* \otimes V^*$. Let

$$R^\perp = \{u \in V^* \otimes V^* \mid u(r) = 0 \ \text{ for all } \ r \in R\}.$$

The *quadratic dual* (or *Koszul dual*) of A is the quadratic algebra

$$A^! = T(V^*)/(R^\perp).$$

Example 2.3.1. Let V be a vector space with basis $x_1, \ldots, x_m$. Let $\mathbf{q} = \{q_{ij}\}$ be a quantum system of parameters as defined in Example 2.2.1. Let

$$R = \mathrm{Span}_k\{x_i \otimes x_j - q_{ij} x_j \otimes x_i \mid 1 \le i, j \le m\}.$$

Then $A = T(V)/(R) \cong k_{\mathbf{q}}[x_1, \ldots, x_m]$, the skew polynomial ring of Example 2.2.4. Let $x_1^*, \ldots, x_m^*$ denote the dual basis to the basis $x_1, \ldots, x_m$ of V. The subspace $R^\perp$ of $V^* \otimes V^*$ is

$$R^\perp = \mathrm{Span}_k\{x_i^* \otimes x_j^* + q_{ij}^{-1} x_j^* \otimes x_i^*, \ (x_i^*)^2 \mid 1 \le i, j \le m\}.$$

Setting $y_i = x_i^*$ for each i, the Koszul dual of A is thus

$$A^! \cong k\langle y_1, \ldots, y_m\rangle/(y_i y_j + q_{ij}^{-1} y_j y_i, \ y_i^2 \mid 1 \le i, j \le m),$$

sometimes denoted $\bigwedge_{\mathbf{q}^{-1}}(V^*)$, a *quantum exterior algebra*. As a special case, if $q_{ij} = 1$ for all i, j, then $A \cong k[x_1, \ldots, x_m]$ and $A^! \cong \bigwedge(V^*)$.

Example 2.3.2. Let V be a finite dimensional vector space and let $R = 0$, the zero subspace of $V \otimes V$. Then $A = T(V)/(R) \cong T(V)$ and $A^! \cong k \oplus V^*$ since $R^\perp = V^* \otimes V^*$.

In our general setting of a quadratic algebra A, view the field k as the quotient A/A_+ where $A_+ = \oplus_{n>0} A_n$. Let $\varepsilon : A \to k$ be the quotient map. Then A is an augmented algebra via this augmentation map ε. Consider a graded free resolution $P_{\bullet}$ of k, that is, each module is free and so graded via the grading on A, and the differentials are graded maps. For each P_i, choose a free basis $\{p_l^i \mid l \in L_i\}$, for L_i some indexing set, in order to identify it with the free module $A^{\oplus L_i} \cong \oplus_{l \in L_i} A p_l^i$. The differentials may then be viewed as matrices (locally finite) with entries in A. We say that $P_{\bullet}$ is *minimal* if the matrix entries are all in A_+, and *linear* if the matrix entries are all in A_1.

Definition 2.3.3. A graded connected quadratic algebra $A = T(V)/(R)$ is a *Koszul algebra* if k has a linear minimal graded free resolution as an A-module.

There are many equivalent definitions of Koszul algebras, as we will see in the next theorem below. In fact, for any graded connected locally finite dimensional algebra, existence of a linear minimal graded free resolution implies it is quadratic, so the assumption that the algebra is quadratic need not be part of the definition. Some of the equivalent definitions of Koszul algebras involve specific complexes, which we will construct next.

Consider the following sequence.

(2.3.4)

$$K_\bullet(A): \qquad \cdots \xrightarrow{d_4} K_3(A) \xrightarrow{d_3} A\otimes R\otimes A \xrightarrow{d_2} A\otimes V\otimes A \xrightarrow{d_1}$$

$$A\otimes A \xrightarrow{\quad\pi\quad} A \longrightarrow 0$$

where for each $n \geq 2$, $K_n(A) = A \otimes K_n'(A) \otimes A$ with

$$K_n'(A) = \bigcap_{i+j=n-2} (V^{\otimes i} \otimes R \otimes V^{\otimes j}),$$

and $K_0(A) = A\otimes A$, $K_1(A) = A\otimes V\otimes A$. The differentials d_n are those of the bar resolution $B_\bullet(A)$ defined in (1.1.4) under the canonical embedding of $K_\bullet(A)$ into $B_\bullet(A)$; a calculation shows that $K_\bullet(A)$ is closed under this differential. Then $K_\bullet(A)$ is a chain complex.

For each $n \geq 0$, let $\widetilde{K}_n(A) = A \otimes K_n'(A)$, a left A-module by multiplication by the leftmost factor. Note that $\widetilde{K}_n \cong K_n \otimes_A k$. We will be interested in the resulting sequence

(2.3.5)

$$\widetilde{K}_\bullet(A): \qquad \cdots \longrightarrow \widetilde{K}_2(A) \longrightarrow \widetilde{K}_1(A) \longrightarrow \widetilde{K}_0(A) \longrightarrow k \longrightarrow 0.$$

Theorem 2.3.6. *Let $A = T(V)/(R)$ be a finitely generated graded connected quadratic algebra. The following are equivalent:*

(i) *A is a Koszul algebra.*

(ii) *$K_\bullet(A)$ is a resolution of A as an A^e-module.*

(iii) *$\widetilde{K}_\bullet(A)$ is a resolution of k as an A-module.*

(iv) *$\mathrm{Ext}^*_A(k,k)$ is generated by $\mathrm{Ext}^1_A(k,k)$ as an algebra.*

(v) *$\mathrm{Ext}^*_A(k,k) \cong A^!$ as graded algebras.*

For a proof, see, for example, [**Krä**].

Example 2.3.7. Let $A = k[x_1, \dots, x_n] \cong T(V)/(R)$ where V is the vector space with basis $x_1, \dots, x_n$ and

$$R = \mathrm{Span}_k\{v \otimes w - w \otimes v \mid v, w \in V\}.$$

We claim that $K_\bullet(A)$, for this algebra A, is equivalent to the resolution (2.1.4) given in Example 2.1.3. To see this, let $\phi : A \otimes \bigwedge^\bullet V \otimes A \to B_\bullet(A)$ be defined by

$$\phi(1 \otimes v_1 \wedge \cdots \wedge v_n \otimes 1) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \otimes 1$$

for all $v_1, \dots, v_n \in V$, where S_n is the symmetric group on n letters. One checks that ϕ is a chain map, and that the image of ϕ is precisely $K_\bullet(A)$. By Theorem 2.3.6(ii), A is a Koszul algebra.

Similarly, $A = k_{\mathbf{q}}[x_1, \dots, x_m]$ is a Koszul algebra and the resolution discussed in Example 2.2.4 is equivalent to $K_\bullet(A)$ for this algebra.

Example 2.3.8. Let $\mathbf{q}$ be a quantum system of parameters, as in Example 2.2.1. Let $A = k_{\mathbf{q}}[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2) \cong T(V)/(R)$, where V is the vector space with basis $x_1, \dots, x_m$ and

$$R = \mathrm{Span}_k\{x_i \otimes x_j - q_{ij} x_j \otimes x_i \mid 1 \le i, j \le m\} \oplus \mathrm{Span}_k\{x_1 \otimes x_1, \dots, x_m \otimes x_m\}.$$

The Koszul resolution $K_\bullet(A)$ is equivalent to that constructed in Example 2.2.5.

There is a close relationship between Hochschild cohomology and the Ext algebra $\mathrm{Ext}^*_A(k,k)$ when A is a Koszul algebra: Let $\phi_k : \mathrm{HH}^*(A) \to \mathrm{Ext}^*_A(k,k)$ be the map given by $- \otimes_A k$, as defined in (1.6.2). By Corollary 1.6.6, for any augmented algebra A, the image of ϕ_k is contained in the graded center $Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(k,k))$. In fact, for Koszul algebras, the image of ϕ_k is precisely $Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(k,k))$. For a proof, see [**BGSS08**, Theorem 4.1].

In some cases, the Koszul resolutions we have considered arise as Koszul complexes associated to regular sequences, as we explain next.

Definition 2.3.9. Let x be a central element of the algebra A. The *Koszul complex* associated to x is

$$K(x): \qquad 0 \longrightarrow A \xrightarrow{x\cdot} A \longrightarrow 0,$$

concentrated in degrees 0 and 1. More generally, let $\mathbf{x} = (x_1, \dots, x_m)$ be a sequence of central elements $x_1, \dots, x_m$ in A. The *Koszul complex* associated to $\mathbf{x}$ is the total complex

$$K(\mathbf{x}): \qquad \mathrm{Tot}(K(x_1) \otimes_A \cdots \otimes_A K(x_m)). \tag{2.3.10}$$

Koszul complexes will be most useful when associated to regular sequences, as we define next. Given an A-module M, we say that a nonzero element $x \in A$ is a *zero divisor* of M if $xm = 0$ for some $m \neq 0$.

Definition 2.3.11. A sequence $(x_1, \dots, x_n)$ of central elements in an algebra A is a *regular sequence* if x_1 is not a zero divisor of A and for each i, x_i is not a zero divisor of the A-module $A/(x_1, \dots, x_{i-1})$, where here the notation $(x_1, \dots, x_{i-1})$ refers to the ideal generated by these elements.

Theorem 2.3.12. *If $\mathbf{x} = (x_1, \dots, x_n)$ is a regular sequence in A, then the Koszul complex $K(\mathbf{x})$ is a free resolution of the A-module $A/(x_1, \dots, x_n)$.*

Proof. Suppose $n = 1$. Since x_1 is not a zero divisor of A, augmenting the sequence $K(x_1)$ by the term $A/(x_1)$ in degree -1 yields an exact sequence. Clearly the terms in nonnegative degrees are free as (left) A-modules. These terms are also free as right A-modules. By the Künneth Theorem (Theorem A.4.1) and induction on n, the result follows: The Tor terms are zero since $x_1, \dots, x_n$ are not zero divisors, and $A/(x_1, \dots, x_{i-1}) \otimes_A A/(x_i) \cong A/(x_1, \dots, x_i)$. $\square$

Example 2.3.13. Let $A = k[x_1, \dots, x_n]$. Then

$$(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$$

is a regular sequence of A^e. The associated Koszul complex

$$K(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$$

is precisely the Koszul resolution (2.1.4), also written $K_{\bullet}(k[x_1, \dots, x_n])$ as in Example 2.3.7, since $A^e/(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n) \cong A$. Similarly, $(x_1, \dots, x_n)$ is a regular sequence of A and $K(x_1, \dots, x_n)$ is the Koszul resolution of $k \cong A/(x_1, \dots, x_n)$ as an A-module.

2.4. Path algebras and monomial algebras

In this section we present Bardzell's construction [**Bar97**] of a minimal bimodule resolution of a monomial algebra. This has been generalized by Chouhy and Solotar [**CS15**] to algebras defined by quivers and relations.

First we define monomial algebras.

A *quiver* is a directed graph Q, that is, Q consists of a set Q_0 of *vertices*, a set Q_1 of *arrows*, and two maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$, associating to each arrow α its *source* $s(\alpha)$ and *target* $t(\alpha)$. A *path* in Q is a sequence of arrows $(\alpha_1, \dots, \alpha_l)$ for which $t(\alpha_i) = s(\alpha_{i+1})$ for all i. We denote the associated path by $\alpha_1 \cdots \alpha_l$. Its *length* is l. There is a path of length 0 associated to each vertex a of Q_0, denoted e_a. The *path algebra* kQ of Q is the k-algebra whose underlying vector space is the set of all paths $\alpha_1 \cdots \alpha_l$ of length $l \geq 0$ with multiplication determined by

$$(\alpha_1 \cdots \alpha_l) \cdot (\beta_1 \cdots \beta_m) = \begin{cases} \alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_m, & \text{if } t(\alpha_l) = s(\beta_1), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.4.1. Let A be any finite dimensional algera. Then A is Morita equivalent to a quotient kQ/I for some quiver Q and ideal I. That is, the category of A-modules is equivalent to the category of kQ/I-modules. Thus quiver techniques are very important in the representation theory of finite dimensional algebras.

Definition 2.4.2. A *monomial algebra* is an algebra of the form $A = kQ/I$ where Q is a finite quiver and I is the ideal generated by a finite set of paths of length at least 2.

Let $A = kQ/I$ be a monomial algebra. Let R be a minimal set of paths, of minimal length, for which $I = (R)$.

Let u, w be paths in Q. We say that y is a *(proper) subpath* of w if $w = xyz$ for some paths x, z, not both of length 0. We write $x = l(y)$, $z = r(y)$, where dependence on w is suppressed in the notation. Denote by $\mathrm{Sub}(w)$ the set of subpaths of w.

We define the *Bardzell resolution* $P_\bullet$ of A, following the presentation of Redondo and Róman [**RR18**]. Let

$$\begin{aligned} P_0 &= A \otimes Q_0 \otimes A, \\ P_1 &= A \otimes Q_1 \otimes A, \\ P_2 &= A \otimes R \otimes A, \end{aligned}$$

and P_n for $n > 2$ is defined as follows. Let $w = \alpha_1 \cdots \alpha_l$ be a path in Q and order the vertices occurring in the path according to its layout, that is,

$$s(\alpha_1) < s(\alpha_2) < \cdots < s(\alpha_l) < t(\alpha_l).$$

Let $R(w) = R \cap \mathrm{Sub}(w)$, that is, the set of paths in R that are subpaths of w. Choose $p_1 \in R(w)$. We construct sets $L_1, L_2, \ldots$ recursively: Let

$$L_1 = \{p \in R(w) \mid s(p_1) < s(p) < t(p_1)\}.$$

If $L_1 \neq \varnothing$, let $p_2 \in R(w)$ be a path for which $s(p_2)$ is minimal for paths in L_1. Next assume $L_1, \ldots, L_j$ have been defined and let p_j be such that $s(p_j)$ is minimal for paths in L_{j-1}. Let

$$L_{j+1} = \{p \in R(w) \mid t(p_{j-1}) \leq s(p) < t(p_j)\}.$$

Now write $w(p_1, \ldots, p_{n-1})$ for the *support* of the sequence $p_1, \ldots, p_{n-1}$, that is the path from $s(p_1)$ to $t(p_{n-1})$ that contains $p_1, \ldots, p_{n-1}$ as subpaths (and this is a subpath of w by construction). Let P_n be the set of all supports of such sequences $p_1, \ldots, p_{n-1}$, called *n-concatenations*.

Next we describe the differentials on the Bardzell resolution $P_\bullet$. The map $\pi : P_0 \to A$ is given by multiplication on $A \otimes Q_0 \otimes A$. The map $d_1 : P_1 \to P_0$ is defined by

$$d_1(1 \otimes \alpha \otimes 1) = \alpha \otimes e_{t(\alpha)} \otimes 1 - 1 \otimes e_{s(\alpha)} \otimes \alpha$$

for all $\alpha \in Q_1$. The map $d_2 : P_2 \to P_1$ is given by

$$d_2(1 \otimes r \otimes 1) = \sum_{\psi \in \mathrm{Sub}(r)} l(\psi) \otimes \psi \otimes r(\psi).$$

More generally, in even degrees, the differential is similar:

$$d_{2m}(1 \otimes w \otimes 1) = \sum_{\psi \in \mathrm{Sub}(w)} l(\psi) \otimes \psi \otimes r(\psi),$$

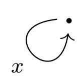
while in odd degrees we set

$$d_{2m+1}(1 \otimes w \otimes 1) = l(\psi_2) \otimes \psi_2 \otimes 1 - 1 \otimes \psi_1 \otimes r(\psi_1),$$

where $w = l(\psi_2)\psi_2 = \psi_1 r(\psi_1)$ with ψ_1, ψ_2 the supports of $2m$-concatenations.

Exactness and minimality of $P_\bullet \to A$ are proven in [**Bar97**]. For a different proof, see Skoldberg [**Skö08**]. Some examples are given in [**Bar97**, Section 7]. Redondo and Román [**RR18**] provided chain maps between the Bardzell resolution and the bar resolution with the goal of computing some of the structure of Hochschild cohomology of monomial algebras.

Example 2.4.3. Let $A = k[x]/(x^n)$ as in Example 1.1.16. We show that the resolution (1.1.17) is equivalent to a Bardzell resolution. Let Q be the quiver with one vertex and one arrow (labeled x):



Let $I = (x^n)$. Then $A \cong kQ/I$. The paths in Q are all nonnegative integer powers of x, and we may take $R = \{x^n\}$. For each m, there is a unique m-concatenation: In even degrees m, it is $x^{\frac{mn}{2}}$, and in odd degrees m it is $x^{\frac{(m-1)n}{2}+1}$. The above formula for the differentials on the Bardzell resolution agrees with the differentials in the sequence (1.1.17) once we identify $1 \otimes 1$ in each degree in (1.1.17) with these free generators in the Bardzell resolution.

2.5. Skew group algebras

Skew group algebras arise by considering an algebra together with a group of algebra automorphisms, forming a larger algebra encoding both structures. When a group acts on a geometric space such as a manifold or algebraic variety, it correspondingly acts on a suitable ring of functions on the space. The associated skew group algebra is studied in noncommutative geometry.

Let G be a finite group acting by automorphisms on an algebra A. We use left superscript to denote the action in order to distinguish it from multiplication in an algebra, that is, ${}^{g}a$ is the result of applying $g \in G$ to $a \in A$. The *skew group algebra* $A \rtimes G$ (also denoted $A\#G$ or $A\#kG$ or $A * G$) is $A \otimes kG$ as a vector space, with multiplication given by

$$(a \otimes g)(b \otimes h) = a({}^{g}b) \otimes gh$$

for all $a, b \in A$ and $g, h \in G$. Note that A is isomorphic to the subalgebra $A \otimes k$ of $A \rtimes G$ and that kG is isomorphic to the subalgebra $k \otimes kG$ of $A \rtimes G$. For simplicity of notation, we will abbreviate $a \otimes g$ by ag when it will cause no confusion. In this notation, the action of G on A is by conjugation in $A \rtimes G$: $gag^{-1} = {}^{g}a$.

There is a spectral sequence describing the (co)homology of $A \rtimes G$ in terms of that of A and of G. This is a special case of a construction in Section 8.6 for smash products with Hopf algebras. Here we will assume the characteristic of k does not divide the order of G, so that the group algebra kG is semisimple by Maschke's Theorem. In this case, there is a more elementary description of the Hochschild (co)homology of $A \rtimes G$ as we see next.

For any set X on which G acts, we use a superscript G to denote invariants, that is,

$$X^G = \{x \in X \mid {}^{g}x = x \text{ for all } g \in G\}.$$

In the following, the algebra structure on $\mathrm{HH}^n(A, A \rtimes G)$ is as described in Remark 1.3.4.

Theorem 2.5.1. *Assume the characteristic of k does not divide the order of the finite group G. There are actions of G for which*

$$\mathrm{HH}^n(A \rtimes G) \cong \mathrm{HH}^n(A, A \rtimes G)^G \quad \textit{and} \quad \mathrm{HH}_n(A \rtimes G) \cong \mathrm{HH}_n(A, A \rtimes G)^G$$

as graded algebras and graded vector spaces, respectively.

Proof. We will see in the course of the proof that the action of G is that induced by its action on the bar resolution of A (diagonal on each tensor factor) and by conjugation on $A \rtimes G$. Let

$$\mathcal{D} = \bigoplus_{g \in G} (Ag) \otimes (A^{\mathrm{op}} g^{-1}),$$

a subalgebra of $(A \rtimes G)^e$. We claim that $(A \rtimes G)^e$ is free as a right $\mathcal{D}$-module under multiplication: Take the set $\{1 \otimes g \mid g \in G\}$ as a free basis.

We claim that there is an isomorphism of $(A \rtimes G)$-bimodules,

$$A \rtimes G \xrightarrow{\sim} A \uparrow_{\mathcal{D}}^{(A \rtimes G)^e}, \tag{2.5.2}$$

where uparrow denotes tensor induction: $A\uparrow_{\mathcal{D}}^{(A\rtimes G)^e} = (A\rtimes G)^e\otimes_{\mathcal{D}} A$, on which $(A\rtimes G)^e$ acts by multiplication on the leftmost factor. This isomorphism is given by sending $a\otimes g$ to $(1\otimes g)\otimes a$ for all $a\in A$ and $g\in G$. Next note that $(A\rtimes G)^e$ is free as a right $\mathcal{D}$-module under multiplication (a free basis is given by the set of all $g\otimes 1$ for $g\in G$). In light of the isomorphism (2.5.2), by the Eckmann-Shapiro Lemma (Lemma A.5.2),

$$\mathrm{Ext}^*_{(A\rtimes G)^e}(A\rtimes G, A\rtimes G)\cong \mathrm{Ext}^*_{\mathcal{D}}(A, A\rtimes G).$$

Now, any $\mathcal{D}$-projective resolution of A may be viewed as an A^e-projective resolution with an action of G that commutes with the differentials. For any pair of $\mathcal{D}$-modules U, V, note that $\mathrm{Hom}_{\mathcal{D}}(U,V)\cong \mathrm{Hom}_{A^e}(U,V)^G$, where the action of G on such functions is given by $({}^g f)(u) = {}^g(f({}^{g^{-1}}u))$ for $g\in G$, $f\in \mathrm{Hom}_{A^e}(U,V)$, $u\in U$. Since the characteristic of k does not divide the order of G, the space of G-invariants of any kG-module V is the image of $\frac{1}{|G|}\sum_{g\in G} g$ as an operator on V. It follows that taking G-invariants commutes with taking (co)homology, and so $\mathrm{Ext}^*_{\mathcal{D}}(A, A\rtimes G)\cong (\mathrm{Ext}^*_{A^e}(A, A\rtimes G))^G$. One checks that this is in fact an algebra isomorphism.

There is an isomorphism similar to (2.5.2), $A\rtimes G\cong A\otimes_{\mathcal{D}}(A\rtimes G)^e$ of *right* $(A\rtimes G)^e$-modules, which we use to see similarly that

$$\mathrm{HH}_n(A\rtimes G)\cong \mathrm{Tor}_n^{\mathcal{D}}(A, A\rtimes G),$$

as follows. A $\mathcal{D}$-projective resolution of A is given for example by the bar resolution $P_{\bullet}$ of A as an A^e-module. It admits an action of G commuting with the differentials and thus an action of $\mathcal{D}$. Tensoring with $(A\rtimes G)^e$ over $\mathcal{D}$, we obtain $P_{\bullet}\otimes_{\mathcal{D}}(A\rtimes G)^e$. Let k be the trivial kG-module, on which each group element acts as the identity. We claim that for each i, there is an isomorphism $P_i\otimes_{\mathcal{D}}(A\rtimes G)\xrightarrow{\sim} k\otimes_{kG}(P_i\otimes_{A^e}(A\rtimes G))$, given by sending $x\otimes b$ to $1\otimes x\otimes b$ for $x\in P_i$, $b\in A\rtimes G$. The inverse map is given by $1\otimes x\otimes b\mapsto x\otimes b$. The action of kG on the tensor product of two modules is given as usual: g acts as $g\otimes g$ for all $g\in G$. One checks directly that these maps are well-defined. Finally, we see that $k\otimes_{kG}(P_i\otimes_{A^e}(A\rtimes G))\xrightarrow{\sim}(P_i\otimes_{A^e}(A\rtimes G))^G$ by sending $1\otimes x\otimes b$ to $\frac{1}{|G|}\sum_{g\in G}{}^g x\otimes {}^g b$. The inverse map sends $x\otimes b$ to $1\otimes x\otimes b$. □

We may further rewrite the expressions in the theorem: As an A^e-module, $A\rtimes G\cong \oplus_{g\in G} Ag$, which yields an isomorphism of graded vector spaces,

$$\mathrm{HH}^*(A, A\rtimes G)\cong \bigoplus_{g\in G}\mathrm{HH}^*(A, Ag).$$

The action of G permutes these components via conjugation: Letting $h\in G$, we have ${}^h(ag) = {}^h a hgh^{-1}$ for all $a\in A$ and $g\in G$, and so h takes $\mathrm{HH}^*(A, Ag)$ to $\mathrm{HH}^*(A, Ahgh^{-1})$. We may then apply h^{-1} to see that these two components are isomorphic as vector spaces. Since the G-invariant subspace is the image of the operator $\frac{1}{|G|}\sum_{g\in G} g$, we thus find that by Theorem 2.5.1,

$$\mathrm{HH}^*(A \rtimes G) \cong \bigoplus_{g\in\overline{G}} (\mathrm{HH}^*(A, Ag))^{C(g)},$$

where $\overline{G}$ is a set of conjugacy class representatives in G, and $C(g)$ is the centralizer in G of g. We will use this in the example of a polynomial ring next.

Example 2.5.3. Let $A = S(V)$ and $g \in G$. We wish to find an expression for $\mathrm{HH}^*(S(V), S(V)g)$, and use it to find $\mathrm{HH}^*(S(V) \rtimes G)$. We will use the Koszul resolution of $S(V)$ as an $S(V)$-bimodule, from Example 2.1.3 or 2.3.7, equivalently the Koszul complex (2.3.10). Since g has finite order, we may choose a basis $x_1, \ldots, x_n$ of V consisting of eigenvectors of g. Let $\lambda_1, \ldots, \lambda_n \in k$ be the corresponding eigenvalues. Assume they are ordered so that $\lambda_1 = 1, \ldots, \lambda_r = 1$ and $\lambda_{r+1} \neq 1, \ldots, \lambda_n \neq 1$. The invariant subspace V^g is then the k-linear span of $x_1, \ldots, x_r$. Let $V_g = \mathrm{Span}_k\{x_{r+1}, \ldots, x_n\} \cong V/\operatorname{Im}(1-g)$. Consider the Koszul complex $K(x_1, \ldots, x_n)$ defined in (2.3.10) to be the tensor product over $S(V)^e$ of the two Koszul complexes $K(x_1, \ldots, x_r)$ and $K(x_{r+1}, \ldots, x_n)$. Applying $\mathrm{Hom}_{S(V)^e}(-, S(V)g)$ and writing $S(V)g = S(V^g) \otimes S(V_g)g$, we obtain

$$\textstyle\bigwedge((V^g)^*) \otimes S(V^g) \otimes \bigwedge((V_g)^*) \otimes S(V_g)g.$$

To find the differentials, note that for each index i, for $a \in S(V)$ and $g \in G$,

$$(x_i \otimes 1 - 1 \otimes x_i) \cdot ag = x_i ag - agx_i = x_i ag - a^g x_i g = (1-\lambda_i)(x_i ag).$$

When $\lambda_i \neq 1$, the differential is thus just multiplication by a nonzero scalar multiple of x_i. It follows that the complex $\bigwedge((V_g)^*) \otimes S(V_g)g$ is exact other than in degree $n-r$, where it has homology $S(V_g)/(x_{r+1}, \ldots, x_n)S(V_g)g \cong k$ by Theorem 2.3.12 (applying Hom reverses the arrows). We will identify this with the top exterior power $\bigwedge^{n-r}((V_g)^*)$. The complex $\bigwedge((V^g)^*)\otimes S(V^g)$, by contrast, has differentials all 0. By the Künneth Theorem (Theorem A.4.1), since the tensor product is now over the field k, the homology of the complex is $\bigwedge((V^g)^*) \otimes S(V^g) \otimes \bigwedge^{n-r}((V_g)^*)g$. As $n - r = \operatorname{codim} V^g$, we may write

$$\mathrm{HH}^n(S(V) \rtimes G) \cong \bigoplus_{g\in\overline{G}} \Big(S(V^g)g \otimes \textstyle\bigwedge^{n-\operatorname{codim} V^g}((V^g)^*) \otimes \bigwedge^{\operatorname{codim} V^g}((V_g)^*)\Big)^{C(g)}.$$

In this expression, the factor $\bigwedge^{\operatorname{codim} V^g}((V_g)^*)$ is isomorphic to k as a vector space, but it potentially has a nontrivial $C(g)$-action, so we retain the factor in the notation.

Chapter 3

Smooth Algebras and Van den Bergh Duality

In this chapter we look at noncommutative analogs of some commutative notions such as dimension, smoothness, and Poincaré duality. These analogs can be defined in terms of Hochschild cohomology.

3.1. Dimension and smoothness

Let A be an associative k-algebra.

Definition 3.1.1. The *Hochschild dimension* of A is its projective dimension as an A^e-module:

$$\dim(A) = \operatorname{pdim}_{A^e}(A).$$

Some authors simply refer to this as the *dimension* of A, and we will as well when there is no confusion possible. However there are many other types of dimension for algebras, such as vector space dimension, Krull dimension, or global dimension, and we will use all of these in this book. See, for example, [**MR88**] for a discussion of dimension for noncommutative rings. Note that the global dimension of A is always less than or equal to its Hochschild dimension, since given an A^e-projective resolution of A, tensoring over A with any module M yields an A-projective resolution of M, as we saw in Section 1.6.

Example 3.1.2. By our work in Example 2.1.3, $\dim(k[x_1, \dots, x_m]) = m$. Specifically, we found there a projective resolution of A as an A^e-module of length m. There cannot exist such a resolution of shorter length since $\mathrm{HH}^m(A) \neq 0$.

Our first result describes a relationship among the Hochschild dimensions of two algebras and that of their tensor product algebra.

Theorem 3.1.3. *Let A and B be two algebras. Then*

$$\dim(A \otimes B) \leq \dim(A) + \dim(B).$$

Proof. Let $P_{\bullet}$ (respectively, $Q_{\bullet}$) be a projective resolution of A as an A^e-module (respectively, of B as a B^e-module). In Section 2.1 we showed that the total complex of the tensor product $P_{\bullet} \otimes Q_{\bullet}$ is a projective resolution of $A \otimes B$ as $(A \otimes B)^e$-module. The length of the tensor product of complexes is the sum of the lengths of the complexes. Therefore there is a projective resolution of $A \otimes B$ as $(A \otimes B)^e$-module of length $\dim(A) + \dim(B)$. It follows that $\dim(A \otimes B)$ is at most this number. □

The following definition is due to Van den Bergh [**dB98**].

Definition 3.1.4. The algebra A is *smooth* if its Hochschild dimension is finite and it has a finite projective resolution as an A^e-module by finitely generated projective modules.

Some authors use the term *homologically smooth* to distinguish this from other notions of smoothness. Note that if A and B are smooth, then so is $A \otimes B$: This follows from Theorem 3.1.3 and its proof, as the tensor product of finitely generated modules will also be finitely generated. If A is a finitely generated commutative algebra over k, this notion of smoothness is equivalent to the classical notion of smoothness [**dB98**].

Examples 3.1.5. A polynomial ring $A = k[x_1, \ldots, x_m]$ is smooth by our work in Example 2.1.3 (the resolution used there consists of finitely generated free modules). A skew polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$ is smooth by our work in Example 2.2.4. If G is a finite group acting on $k[x_1, \ldots, x_m]$ by degree-preserving automorphisms and $\mathrm{char}(k)$ does not divide the order of G, then the skew group algebra $k[x_1, \ldots, x_n] \rtimes G$ of Section 2.5 is smooth: The group G acts on the Koszul resolution of $k[x_1, \ldots, x_m]$, and our work in that section shows that it may be induced to a projective resolution of $k[x_1, \ldots, x_m] \rtimes G$ as a $(k[x_1, \ldots, x_m] \rtimes G)^e$-module. The quantum polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$ and the skew group algebras $k[x_1, \ldots, x_m] \rtimes G$ are smooth algebras that are generally not commutative.

We will look closely at some algebras with Hochschild dimension 0 or 1.

Definition 3.1.6. An algebra A is *separable* if $\dim(A) = 0$. It is *quasi-free* if $\dim(A) \leq 1$.

The notion of quasi-free algebras is due to Cuntz and Quillen [**CQ95**]. Quasi-free algebras have also been called *Cuntz-Quillen smooth* or *formally smooth*.

By definition, A is separable if and only if it is projective as an A^e-module. Another equivalent condition to separability is that any derivation from A to an A-bimodule is inner: Indeed, if A is separable, then $\mathrm{HH}^1(A, M) = 0$ for all A-bimodules M. By our work in Section 1.2, $\mathrm{HH}^1(A, M) = 0$ is equivalent to the statement that any derivation from A to M is inner. Conversely, let K_1 be the first syzygy of A in a given projective resolution $P_\bullet$ of A as an A^e-module. We claim that if A is not projective as an A^e-module, then $\mathrm{HH}^1(A, K_1) \neq 0$: By definition, $\mathrm{HH}^1(A, K_1) = \mathrm{Hom}_{A^e}(K_1, K_1)/\mathrm{Im}(d_1^*)$. This is nonzero, as if not, then the identity map from K_1 to K_1 is in the image of d_1^*, which implies that the short exact sequence $0 \to K_1 \to P_0 \to A \to 0$ splits and A is projective as an A^e-module, a contradiction.

We will look at some equivalent conditions to quasi-freeness in the next section. For now, we consider some examples and implications.

Example 3.1.7. $A = k[x]$ has Hochschild dimension 1 by our work in Example 2.1.3, and so is quasi-free. However, $k[x, y]$ has Hochschild dimension 2 and so is not quasi-free. Thus the tensor product of two quasi-free algebras is not always quasi-free. However, the free product of two quasi-free algebras is always quasi-free [**CQ95**, Proposition 5.3]. It follows that the tensor algebra $T(V)$ of a finite dimensional vector space V is quasi-free.

A quasi-free algebra is hereditary: If A is quasi-free, then there is a projective resolution $0 \to P_1 \to P_0 \to A \to 0$ of A as an A^e-module. For any A-module M, we may apply $- \otimes_A M$ to this sequence to obtain a projective resolution of M as an A-module, as explained in Section 1.6. Therefore the projective dimension $\mathrm{pdim}_A(M)$ of M is at most 1.

Example 3.1.8. Any semisimple algebra is separable, since if A is semisimple then so is A^e, and so all A^e-modules are projective. For example, if G is a finite group whose order is not divisible by the characteristic of k, then the group algebra kG is semisimple by Maschke's Theorem, and so kG is separable.

3.2. Noncommutative differential forms

We study quasi-free algebras in more detail in this section. We first introduce a version of Kähler differentials for noncommutative algebras. This material is from Cuntz and Quillen [**CQ95**] and Ginzburg [**Ginb**].

Let A be a k-algebra and for each $n \geq 0$, let

$$\Omega^n_{nc}A = A \otimes (\overline{A})^{\otimes n}$$

where $\overline{A} = A/k$ is the vector space quotient. We write elements of $\overline{A}$ via notation from A, viewing $\overline{A}$ noncanonically as a vector space direct summand of A, when this will not cause confusion. We will identify $\Omega^1_{nc}A$ with the kernel of the multiplication map $\pi : A \otimes A \to A$ as follows. This will give a connection to the Heller operator of similar notation Ω^1 and also a comparison to Kähler differentials [**Wei94**] in the special case that A is commutative.

Consider $\Omega^1_{nc}A$ to be an A-bimodule under the following actions:

$$c(a \otimes b) = ca \otimes b \quad \text{and} \quad (a \otimes b)c = a \otimes bc - ab \otimes c$$

for all $a, b, c \in A$. Let $j : A \otimes \overline{A} \to A \otimes A$ be given by $j(a \otimes b) = ab \otimes 1 - a \otimes b$ for $a, b \in A$. Then the sequence

$$0 \to \Omega^1_{nc}A \xrightarrow{j} A \otimes A \xrightarrow{\pi} A \to 0 \tag{3.2.1}$$

is an exact sequence of A-bimodules. To see this, note that j maps $\Omega^1_{nc}A$ isomorphically onto $\mathrm{Ker}(\pi)$: Due to exactness of the bar resolution $B(A)$ of the A^e-module A given in (1.1.4), $\mathrm{Ker}(\pi) = \mathrm{Im}(d_1)$, but this is precisely the image of j. Further, j is injective: Assume that $j(\sum_i a_i \otimes b_i) = 0$ for some elements a_i, b_i. Then, since $b_i \in \overline{A}$ and

$$j(\sum_i a_i \otimes b_i) = (\sum_i a_i b_i) \otimes 1 - \sum_i a_i \otimes b_i,$$

we have $\sum_i a_i b_i = 0$. It follows that $j(\sum_i a_i \otimes b_i) = \sum_i a_i \otimes b_i = 0$.

One important property of $\Omega^1_{nc}A$ is that for all A-bimodules M,

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_{A^e}(\Omega^1_{nc}A, M),$$

where $\mathrm{Der}(A, M)$ is the space of derivations from A to M. This follows immediately from the work in Section 1.2 interpreting Hochschild cohomology in degree 1, as this is precisely the space of Hochschild 1-cocycles. The isomorphism implies that $\Omega^1_{nc}A$ represents the functor $\mathrm{Der}(A, -)$ on the category of A-bimodules.

In comparison, for commutative algebras A, the Kähler differentials represent the functor $\mathrm{Der}(A, -)$ on the category of A-modules [**Wei94**]. The A-module of *Kähler differentials* $\Omega^1_{com}A$ is the A-module with one generator da for each $a \in A$ with $dc = 0$ for all $c \in k$. Relations are

$$d(a + b) = da + db \quad \text{and} \quad d(ab) = adb + bda$$

for all $a, b \in A$. We note that $\Omega^1_{com}A \cong \mathrm{Ker}\,\pi/(\mathrm{Ker}\,\pi)^2 \cong (\Omega^1_{nc}A)/(\mathrm{Ker}\,\pi)^2$ in case A is commutative.

Let

$$\Omega_{nc}A = \oplus_{n\geq 0}\Omega_{nc}^{n}A,$$

the algebra of *noncommutative differential forms* on A. This is a differential graded algebra with

$$\begin{aligned} d(a_0\otimes\cdots\otimes a_n) &= 1\otimes a_0\otimes\cdots\otimes a_n, \\ (a_0\otimes\cdots\otimes a_n)(a_{n+1}\otimes\cdots\otimes a_r) &= \sum_{i=0}^{n}(-1)^{n-i}a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_r \end{aligned}$$

for all $a_0,\ldots,a_r\in A$, as the following theorem shows. Moreover, the theorem states that $\Omega_{nc}A$ is universal with respect to differential graded algebras whose degree 0 term is the target of an algebra homomorphism from A.

Theorem 3.2.2. [**CQ95**, Proposition 1.1] *The differential and multiplication given above provide a differential graded algebra structure on $\Omega_{nc}A$, unique such that*

$$a_0(da_1)\cdots(da_n) = a_0\otimes\cdots\otimes a_n$$

for all $a_0,\ldots,a_n\in A$. Moreover, for any differential graded algebra Γ and algebra homomorphism $u : A\to\Gamma^0$, there is a unique differential graded algebra homomorphism $u_* : \Omega A\to\Gamma$ that extends u.

Proof. By properties of differential graded algebras, the differential and multiplication are those uniquely determined by the structure of the algebra A. One checks that $\Omega_{nc}A$ is indeed a differential graded algebra, or see [**CQ95**, Proposition 1.1].

For the second statement, let Γ be a differential graded algebra and $u : A\to\Gamma^0$ an algebra homomorphism. Define $u_* : \Omega_{nc}A\to\Gamma$ by

$$u_*(a_0\otimes a_1\otimes\cdots\otimes a_n) = u(a_0)du(a_1)\cdots du(a_n).$$

One checks that u_* is a homomorphism of differential graded algebras, and it is uniquely determined. □

A *square-zero extension* of A is an algebra R such that $A\cong R/I$ for some ideal I of R with $I^2=0$. The ideal I is an A-bimodule since $I^2=0$: Given an element $a=r+I\in A$ for some $r\in R$, and given $x\in I$, define $ax=rx$ and $xa=xr$. Every A-bimodule M determines a square-zero extension: Let $R=A\oplus M$ and define $(a_1,m_1)\cdot(a_2,m_2)=(a_1a_2,a_1m_2+m_1a_2)$ for all $a_1,a_2\in A$ and $m_1,m_2\in M$. This is called the *trivial extension*. More generally, if $f : A\otimes A\to M$ is a Hochschild 2-cocycle, that is,

$$af(b\otimes c)+f(a\otimes bc) = f(ab\otimes c)+f(a\otimes b)c,$$

for all $a,b,c\in A$ as in (1.2.1), then $R=A\oplus M$ is a ring with

$$(a_1,m_1)(a_2,m_2) = (a_1a_2, a_1m_2+m_1a_2+f(a_1\otimes a_2))$$

for all $a_1, a_2 \in A$ and $m_1, m_2 \in M$. (Associativity is equivalent to the above Hochschild 2-cocycle condition.) In fact, $\mathrm{HH}^2(A, M)$ is in one-to-one correspondence with equivalence classes of square-zero extensions of A by M [**Wei94**, Classification Theorem 9.3].

Square-zero extensions, noncommutative differential forms, and quasi-free algebras are all related as follows.

Theorem 3.2.3. *The following are equivalent:*

(i) *A is quasi-free.*

(ii) *$\Omega^1_{nc}A$ is a projective A^e-module.*

(iii) *$\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M.*

(iv) *For any square-zero extension R of A, there is a lifting $A \to R$.*

Proof. We have identified $\Omega^1_{nc}A$ with the first syzygy of A as an A^e-module, and so (i) and (ii) are equivalent.

By dimension shifting (Theorem A.2.3),

$$\mathrm{HH}^2(A, M) \cong \mathrm{Ext}^1_{A^e}(\Omega^1_{nc}A, M),$$

and so (ii) and (iii) are equivalent.

Finally, let R be a square-zero extension of A by an ideal I. Assume there is a lifting $A \to R$. The lifting splits the sequence $0 \to I \to R \to A \to 0$. So if any square-zero extension lifts, then $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M, that is, (iv) implies (iii). Conversely, if $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M, then every square-zero extension splits, and so there is a lifting $A \to R$, that is, (iii) implies (iv). □

Schelter [**Sch86**] proposed a condition similar to (iv) in the theorem as a definition of smoothness for noncommutative algebras.

In case A is commutative, consider the related condition that for every *commutative* square-zero extension R of A, there is a lifting $A \to R$. This is equivalent to smoothness for commutative algebras [**Wei94**, Section 9.3.1] but is a weaker condition than being quasi-free. In fact, a commutative algebra is smooth in the sense of Definition 3.1.4 if and only if it is smooth in this classical sense [**dB98**].

3.3. Van den Bergh duality and Calabi-Yau algebras

Some smooth noncommutative algebras have a notion of duality between Hochschild homology and cohomology, analogous to Poincaré duality in geometry. We state this duality in Theorem 3.3.1 below; see also Corollary 3.3.6. We note that in general if P is a left A^e-module then $P^* = \mathrm{Hom}_{A^e}(P, A^e)$ is a right A^e-module by setting $(f \cdot (a \otimes b))(p) = bf(p)a$ for all $a, b \in A$, $p \in P$, and $f \in \mathrm{Hom}_{A^e}(P, A^e)$. (There is a more general statement that may be easier to see: If B is an algebra and P is a left B-module, then $P^* = \mathrm{Hom}_B(P, B)$ is a right B-module, or equivalently a left B^{op}-module, with $(f \cdot b)(p) = bf(p)$ for all $b \in B$, $p \in P$, $f \in \mathrm{Hom}_B(P, B)$.)

An A-bimodule U is *invertible* if there is an A-bimodule V such that $U \otimes_A V \cong A$ and $V \otimes_A U \cong A$ as A-bimodules. (The invertible bimodules correspond one-to-one with autoequivalences of the category A-mod, that is, equivalences between the category of A-modules and itself, given by tensor product. The identity autoequivalence is given by the invertible bimodule A.)

Theorem 3.3.1. [**dB98**, Theorem 1] *Let A be a smooth algebra. Assume that there is a positive integer d for which $\mathrm{HH}^i(A, A^e) = 0$ unless $i = d$ and that $U = \mathrm{HH}^d(A, A^e)$ is an invertible A-bimodule. Then*

$$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, U \otimes_A M)$$

for all A-bimodules M and $n \in \{0, \ldots, d\}$.

Note that for values of n outside the stated range in the theorem, the cohomology $\mathrm{HH}^n(A, M)$ is automatically 0.

Definition 3.3.2. If the hypotheses of the theorem are satisfied, we call U the *dualizing bimodule* of A and we say that A has *Van den Bergh duality.*

Proof of Theorem 3.3.1. This proof is a special case of a proof in [**Krä07**]. Since A is smooth, there is an A^e-projective resolution $P_\bullet$ of A such that each P_i is finitely generated and $P_i = 0$ for $i > \dim(A)$. Let $Q_\bullet$ be an A^e-projective resolution of M. Since U is invertible, $U \otimes_A Q_\bullet$ is an A^e-projective resolution of $U \otimes_A M$. Let

$$C_{p,q} = \mathrm{Hom}_{A^e}(P_{-p}, Q_q)$$

for all $p \leq 0$, $q \geq 0$. We claim that since P_{-p} is finitely generated projective, for each p, q, there is an isomorphism of vector spaces,

$$\mathrm{Hom}_{A^e}(P_{-p}, A^e) \otimes_{A^e} Q_q \xrightarrow{\sim} C_{p,q} \tag{3.3.3}$$

given by $f \otimes y \mapsto (x \mapsto f(x)y)$ for all $f \in \mathrm{Hom}_{A^e}(P_{-p}, A^e)$ and $y \in Q_q$. Indeed, in case $P_{-p} = A^e$, this is clearly an isomorphism, as it is if P_{-p} is a free module of finite rank. It then follows for any finitely generated projective P_{-p}. Note that the tensor product here is taken over A^e instead of over A.

We now give two proofs of the isomorphism stated in the theorem, both using the bicomplex $C_{\bullet,\bullet}$. The proofs are essentially the same, however the first uses the Acyclic Assembly Lemma (Theorem A.4.4) in two ways, and the second uses a comparison of two spectral sequences for a bicomplex (Section A.6).

The first proof is as follows. Let $\varepsilon : Q_0 \to M$ be the augmentation of the projective resolution $Q_\bullet$ of M. There are maps $\mathrm{Tot}(C_{\bullet,\bullet}) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{A^e}(P_{-\bullet}, M)$ and, using the above isomorphism (3.3.3), there is a map

$$\mathrm{Tot}(C_{\bullet,\bullet}) \;\longrightarrow\; U \otimes_{A^e} Q_\bullet.$$

We claim that this map is a quasi-isomorphism, and with the corresponding degree shift, this results in the isomorphism stated in the theorem. To prove the claim, consider the columns in the following diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathrm{Hom}_{A^e}(P_d, Q_2) & \longleftarrow \cdots \longleftarrow & \mathrm{Hom}_{A^e}(P_0, Q_2) & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathrm{Hom}_{A^e}(P_d, Q_1) & \longleftarrow \cdots \longleftarrow & \mathrm{Hom}_{A^e}(P_0, Q_1) & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathrm{Hom}_{A^e}(P_d, Q_0) & \longleftarrow \cdots \longleftarrow & \mathrm{Hom}_{A^e}(P_0, Q_0) & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Since each P_i is projective and $Q_\bullet \xrightarrow{\varepsilon} M$ is exact, the ith column is exact when augmented with $\mathrm{Hom}_{A^e}(P_i, M)$. By the Acyclic Assembly Lemma (Theorem A.4.4), the resulting total complex is acyclic. Similarly, instead augmenting each jth row with $U \otimes_{A^e} Q_j$ on the left, by the hypothesis that $\mathrm{HH}^i(A, A^e) = 0$ for $i \neq d$ and the isomorphism (3.3.3), the rows will be exact. By the Acyclic Assembly Lemma (Theorem A.4.4), the resulting total complex is acyclic. We claim that $U \otimes_{A^e} Q_j \cong A \otimes_{A^e} (U \otimes_A Q_j)$ for all j. To see this, first note that it is true for a free module since $U \xrightarrow{\sim} A \otimes_{A^e} (U \otimes_A A^e)$ via the map $u \mapsto 1 \otimes (u \otimes (1 \otimes 1))$ which has inverse $a \otimes (u \otimes (b \otimes c)) \mapsto caub$. Each Q_j is projective, so is a direct summand of a free module, and this isomorphism preserves such a direct sum. This shows that the cohomology of the total complex of $C_{\bullet,\bullet}$ is $\mathrm{HH}^\bullet(A, M)$ on the one hand, and it is $\mathrm{HH}_{d-\bullet}(A, U \otimes_A M)$ on the other, proving the theorem.

For a second proof, we may apply the two spectral sequences of the double complex $C_{\bullet,\bullet}$ described in Section A.6. Let E'' denote the first described spectral sequence in which $E''_1 \cong \mathrm{H}''(C)$, $C''_2 \cong \mathrm{H}'(\mathrm{H}''(C))$. Using the same arguments as above, we find that E''_1 consists of $\mathrm{Hom}_{A^e}(P_\bullet, M)$ in the bottom row, with 0 in all other components, and thus E_2'' consists of $\mathrm{HH}^\bullet(A, M)$ in the bottom row, with zero differentials. So the spectral sequence collapses and this is the cohomology of $C_{\bullet,\bullet}$. Let E' denote the second described spectral sequence in which $E_1' \cong \mathrm{H}'(C)$, $E_2' \cong \mathrm{H}''(\mathrm{H}'(C))$. We find that E_1' consists of $U \otimes_{A^e} Q_\bullet$ in the left column and E_2' is thus $\mathrm{Tor}^{A^e}_{d-\bullet}(U, M)$, with zero differentials. Since $U \otimes_{A^e} Q_i \cong A \otimes_{A^e} (U \otimes_A Q_i)$ for all i (see above), this is $\mathrm{HH}_{d-\bullet}(A, U \otimes_A M)$, the cohomology of $C_{\bullet,\bullet}$, completing the second proof. □

We next find the dualizing bimodule in case A is a polynomial ring.

Example 3.3.4. Let $A = k[x]$. Consider the Koszul resolution (1.1.15) of A as an A^e-module. Apply $\mathrm{Hom}_{k[x]^e}(-, k[x]^e)$ to obtain

$$0 \longleftarrow \mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \longleftarrow \mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \longleftarrow 0.$$

Under the isomorphism $\mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \cong \mathrm{Hom}_k(k, k[x]^e) \cong k[x]^e$, this sequence is equivalent to

$$0 \longleftarrow k[x]^e \longleftarrow k[x]^e \longleftarrow 0,$$

the nonzero map being given by multiplication by $x \otimes 1 - 1 \otimes x$. So $\mathrm{HH}^0(k[x], k[x]^e) = 0$ and $\mathrm{HH}^1(k[x], k[x]^e) \cong k[x]$, an invertible $k[x]$-bimodule. The hypotheses of Theorem 3.3.1 are satisfied and so

$$\mathrm{HH}^n(k[x], M) \cong \mathrm{HH}_{1-n}(k[x], M)$$

for all $k[x]$-bimodules M and $n = 0, 1$. A similar argument applies to a polynomial ring in more indeterminates, and we find that

$$\begin{aligned} \mathrm{HH}^m(k[x_1, \ldots, x_m], k[x_1, \ldots, x_m]^e) &\cong k[x_1, \ldots, x_m] \\ \mathrm{HH}^n([k[x_1, \ldots, x_m], M) &\cong \mathrm{HH}_{m-n}(k[x_1, \ldots, x_m], M) \end{aligned}$$

for all $k[x_1, \ldots, x_m]$-bimodules M and $n = 0, \ldots, m$.

Definition 3.3.5. An algebra A is *Calabi-Yau* if it has Van den Bergh duality with dualizing bimodule $U \cong A$.

Calabi-Yau algebras were first defined by Ginzburg [**Gina**] as an analog in the noncommutative setting of rings of functions on Calabi-Yau varieties. There is also a notion of a twisted Calabi-Yau algebra in which $U \cong A_\sigma$, twisted by an automorphism σ on the right.

By replacing U by A in Theorem 3.3.1 and applying the isomorphism $A \otimes_A M \cong M$, we obtain the following corollary.

Corollary 3.3.6. *If A is a Calabi-Yau algebra of dimension d, then*

$$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, M)$$

for all A-bimodules M and integers n.

Some examples of Calabi-Yau algebras are the polynomial rings of Example 3.3.4, some Sklyanin algebras [**MS**], and some deformed preprojective algebras [**Ami**]. Skew group algebras can be Calabi-Yau, and we give details for some of these examples in the next section.

3.4. Skew group algebras

Let k be an algebraically closed field of characteristic 0, let G be a finite group, and let V be a kG-module of dimension d as a vector space. In this section we consider the skew group algebra $A = S(V)\rtimes G$ as an example of an algebra with Van den Bergh duality. In Section 2.5, we found expressions for the Hochschild homology and cohomology of A. Here we present Farinati's alternate computation of Hochschild cohomology that uses Van den Bergh duality [**Far05**]. By our work in Example 3.3.4,

$$\mathrm{HH}^d(S(V), S(V)^e) \cong S(V) \otimes \textstyle\bigwedge^d(V^*)$$

and $\mathrm{HH}^n(S(V), S(V)^e) = 0$ for $n \neq d$. The dual vector space $V^* = \mathrm{Hom}_k(V, k)$ is a kG-module via $({}^g f)(v) = f({}^{g^{-1}}v)$ for all $g \in G$, $g \in V^*$, and $v \in V$, and G acts factorwise on the tensor product $S(V) \otimes \bigwedge^d(V^*)$. We will see that $S(V) \rtimes G$ has Van den Bergh duality, and in case G acts on V via linear transformations of determinant 1, it is Calabi-Yau.

Applying Van den Bergh duality for $S(V)$ as in Example 3.3.4, the techniques in the proof of Theorem 2.5.1, and the $S(V)$-bimodule isomorphism $A^e \cong S(V) \otimes k(G \times G)$,

$$\begin{aligned} \mathrm{HH}^n(A, A^e) &\cong \mathrm{HH}^n(S(V), S(V)^e \otimes k(G \times G))^G \\ &\cong \mathrm{HH}_{d-n}(S(V), S(V)^e \otimes k(G \times G))^G. \end{aligned}$$

This is Hochschild homology, obtained from the tensor product of an $S(V)^e$-projective resolution of $S(V)$, over $S(V)^e$, with $S(V)^e \otimes k(G \times G)$. By definition, the differential is the identity on the factor $k(G \times G)$ and so tensoring with $k(G \times G)$ commutes with taking homology. That is,

$$\begin{aligned} \mathrm{HH}^n(A, A^e) &\cong \mathrm{HH}_{d-n}(S(V), S(V)^e) \otimes k(G \times G))^G \\ &\cong \mathrm{HH}^n(S(V), S(V)^e) \otimes k(G \times G))^G, \end{aligned}$$

the second isomorphism holding by another application of Van den Bergh duality. If $n \neq d$, this is 0, while if $n = d$, this is

$$(S(V) \otimes \textstyle\bigwedge^d(V^*) \otimes k(G \times G))^G$$

as vector spaces. We claim that the A-bimodule structure is as it is on $A \otimes \bigwedge^d(V^*)$: If G acts by linear transformations of determinant 1, the isomorphism is given by

$$\begin{array}{rcl} S(V) \rtimes G & \longrightarrow & (S(V) \otimes k(G \times G))^G \\ z \otimes g & \mapsto & \dfrac{1}{|G|} \sum_{h \in G} {}^h z \otimes (hg, h^{-1}) \end{array}$$

for all $z \in S(V)$ and $g \in G$. In the more general case where G does not act by linear transformations of determinant 1, a similar isomorphism yields $\mathrm{HH}^d(A, A^e) \cong A \otimes \bigwedge^d(V^*)$, and $\mathrm{HH}^n(A, A^e) = 0$ if $n \neq d$. Thus A has Van den Bergh duality with dualizing bimodule $A \otimes \bigwedge^d(V^*)$.

We now use Van den Bergh duality and the dualizing bimodule we just found to compute Hochschild cohomology from Hochschild homology. We begin with a computation of Hochschild homology. As in Theorem 2.5.1, the Hochschild homology of A is

$$\begin{array}{rcl} \mathrm{HH}_n(S(V) \rtimes G) & \cong & \mathrm{HH}_n(S(V), S(V) \rtimes G)^G \\ & \cong & \bigoplus_{g \in \overline{G}} \mathrm{HH}_n(S(V), S(V)g)^{C(g)} \end{array}$$

where $\overline{G}$ is a set of representatives of conjugacy classes of G and $C(g)$ is the centralizer in G of g. Letting V^g be the subspace of V invariant under g and $V_g = \mathrm{Im}(1-g)$, we have $V = V^g \oplus V_g$ and an $S(V)$-bimodule isomorphism $S(V)g \cong S(V^g) \otimes S(V_g)g$. Then

$$\mathrm{HH}_n(S(V), S(V)g)^{C(g)} \cong \left(\bigoplus_{p+q=n} \mathrm{HH}_p(S(V^g)) \otimes \mathrm{HH}_q(S(V_g), S(V_g)g) \right)^{C(g)}.$$

As before, $\mathrm{HH}_*(S(V^g)) \cong S(V^g) \otimes \bigwedge(V^g)$. We compute $\mathrm{HH}_*(S(V_g), S(V_g)g)$. Diagonalize the action of g on V_g, so that V_g has a basis of eigenvectors for g: namely $x_1, \ldots, x_r$ with eigenvalues $\lambda_1, \ldots, \lambda_r$. By the Künneth Theorem (Theorem A.4.1),

$$\mathrm{HH}_*(S(V_g), S(V_g)g) \cong \otimes_{i=1}^r \mathrm{HH}_*(k[x_i], k[x_i]g).$$

We claim that $\mathrm{HH}_0(k[x_i], k[x_i]g) = k$ and $\mathrm{HH}_1(k[x_i], k[x_i]g) \cong 0$ since $\lambda_i \neq 1$: If we consider the $k[x]^e$-module $k[x]g$ where ${}^g x = \lambda x$ for some scalar λ, then applying $- \otimes_{k[x]^e} k[x]g$ to (1.1.15), we obtain

$$0 \to k[x]g \to k[x]g \to 0$$

where the nonzero map is given by applying $x \otimes 1 - 1 \otimes x$, and this becomes multiplication by $(1-\lambda)x$. So $\mathrm{HH}_0(k[x], k[x]g) = k$ and $\mathrm{HH}_1(k[x], k[x]g) = 0$, as claimed. Note that also $h \in C(g)$ may be simultaneously diagonalized with g. So we now have

$$\mathrm{HH}_n(S(V), S(V)g)^{C(g)} \cong (S(V^g) \otimes \textstyle\bigwedge^n(V^g))^{C(g)}.$$

Now we may compute Hochschild cohomology via Van den Bergh duality:

$$\begin{aligned}\mathrm{HH}^n(S(V)\rtimes G) &\cong \mathrm{HH}_{d-n}(S(V)\rtimes G, (S(V)\rtimes G)\otimes \textstyle\bigwedge^d(V^*))\\ &\cong \bigoplus_{g\in\overline{G}}(S(V^g)\otimes \textstyle\bigwedge^{d-n}(V^g)\otimes\textstyle\bigwedge^d(V^*))^{C(g)}.\end{aligned}$$

Note that

$$\begin{aligned}\textstyle\bigwedge^{d-n}(V^g)\otimes\textstyle\bigwedge^d(V^*) &\cong \textstyle\bigwedge^{d-n}(V^g)\otimes\textstyle\bigwedge^{\dim V^g}((V^g)^*)\wedge\textstyle\bigwedge^{\operatorname{codim} V^g}((V_g)^*)\\ &\cong \textstyle\bigwedge^{n-\operatorname{codim} V^g}((V^g)^*)\otimes\textstyle\bigwedge^{\operatorname{codim} V^g}((V_g)^*)\end{aligned}$$

as $kC(g)$-modules via the evaluation map pairing V and V^*, and noting that the vector space dimension of $\bigwedge^{n-\operatorname{codim} V^g}((V^g)^*)$ is the same as that of $\bigwedge^{d-n}((V^g)^*)$ since $d=\dim_k V$. In comparison to our calculation of Example 2.5.3, the duality makes the degree shift due to the factor $\bigwedge^{\operatorname{codim} V^g}((V^g)^*)$ appear very naturally.

3.5. Connes differential and Batalin-Vilkovisky structure

We introduce the Connes differential on Hochschild homology. For Calabi-Yau algebras, we use the Connes differential in combination with Van den Bergh duality to define a Batalin-Vilkovisky structure on Hochschild cohomology. For finite dimensional symmetric algebras, we use the Connes differential in combination with a duality relation specific to this setting to define a Batalin-Vilkovisky structure. The former can be generalized to some twisted Calabi-Yau algebras, and the latter to some Frobenius algebras.

Consider the following maps on the bar resolution. Recall $s: A^{\otimes(n+1)}\to A^{\otimes(n+2)}$, the contracting homotopy of (1.1.3):

$$s(a_0\otimes\cdots\otimes a_n)=1\otimes a_0\otimes\cdots\otimes a_n$$

for all $a_0,\ldots,a_n\in A$. Define $t: A^{\otimes(n+1)}\to A^{\otimes(n+1)}$ to be the signed cyclic permutation of factors given by

$$t(a_0\otimes\cdots\otimes a_n)=(-1)^n a_n\otimes a_0\otimes\cdots\otimes a_{n-1},$$

and define $N: A^{\otimes(n+1)}\to A^{\otimes(n+1)}$ by

$$N=1+t+t^2+\cdots+t^n.$$

Definition 3.5.1. The *Connes differential* $\mathcal{B}: A^{\otimes(n+1)}\to A^{\otimes(n+2)}$ is defined by

$$\mathcal{B}=(1-t)sN.$$

One checks that B is a chain map of degree 1, on the bar complex, and so induces a map on Hochschild homology.

Assume A is a Calabi-Yau algebra of dimension d so that $\mathrm{HH}^n(A)\cong \mathrm{HH}_{d-n}(A)$ for all n. We define an operator $\Delta:\mathrm{HH}^n(A)\to\mathrm{HH}^{n-1}(A)$ to be that induced by the Connes differential $\mathcal{B}$ under the Van den Bergh duality isomorphism. That is, the following diagram commutes for all n:

$$\begin{array}{ccc} \mathrm{HH}^n(A) & \xrightarrow{\Delta} & \mathrm{HH}^{n-1}(A) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{HH}_{d-n}(A) & \xrightarrow{\mathcal{B}} & \mathrm{HH}_{d-n+1}(A) \end{array}$$

Definition 3.5.2. The map Δ defined above is the *Batalin-Vilkovisky operator* on the Hochschild cohomology ring of the Calabi-Yau algebra A.

There is a relationship with the Gerstenhaber bracket as follows.

Theorem 3.5.3. *Let A be a Calabi-Yau algebra and let α, β be homogeneous elements in* $\mathrm{HH}^*(A)$. *Then*

$$[\alpha, \beta] = \Delta(\alpha \smile \beta) - \Delta(\alpha) \smile \beta - (-1)^{|\alpha|}\alpha \smile \Delta(\beta).$$

For a proof, see [**Gina**, Section 9.3].

For a Calabi-Yau algebra A, we say that $(\mathrm{HH}^*(A), \smile, [\ ,\ ], \Delta)$ is a *Batalin-Vilkovisky algebra*. A Batalin-Vilkovisky structure on Hochschild cohomology has been defined for other types of algebras as well. For example, for some twisted Calabi-Yau algebras, see [**KK14**], for finite dimensional symmetric algebras, see [**Tra08**], and for some Frobenius algebras, see [**LZZ16**]. We give a description of this structure for symmetric algebras based on ideas in [**LZZ16**] where it is done more generally for Frobenius algebras with semisimple Nakayama automorphism.

Let A be a finite dimensional symmetric algebra, that is, there is a nondegenerate symmetric associative bilinear form $\langle\ ,\ \rangle : A \times A \to k$. Let $D(A) = \mathrm{Hom}_k(A, k)$ be an A^e-module via $(afb)(c) = f(bca)$ for all $a, b, c \in A$ and $f \in D(A)$. The form $\langle\ ,\ \rangle$ gives an isomorphism $A \cong D(A)$ as A^e-modules. Let B be the bar resolution of A as an A^e-module. Applying the Nakayama relations twice (the first time involving coinduction from k to A and the second time involving induction from A to A^e), we find

$$\begin{aligned} \mathrm{Hom}_k(A \otimes_{A^e} B, k) &\cong \mathrm{Hom}_A(A \otimes_{A^e} B, D(A)) \\ &\cong \mathrm{Hom}_{A^e}(B, D(A)) \\ &\cong \mathrm{Hom}_{A^e}(B, A), \end{aligned}$$

the last isomorphism due to the isomorphism $D(A) \cong A$ of A^e-modules. Taking cohomology, we obtain

$$D(\mathrm{HH}_*(A)) \cong \mathrm{HH}^*(A).$$

We use this duality to define a Batalin-Vilkovisky structure on $\mathrm{HH}^*(A)$ just as in the Calabi-Yau setting: Let $\mathcal{B}^t$ denote the transpose of the Connes differential $\mathcal{B}$, that is $\mathcal{B}^t(f) = f\mathcal{B}$ for all $f \in D(\mathrm{HH}_*(A))$. Then $\Delta : \mathrm{HH}^n(A) \to$

$\mathrm{HH}^{n-1}(A)$ is defined by the following commuting diagram:

$$\begin{array}{ccc} \mathrm{HH}^n(A) & \xrightarrow{\Delta} & \mathrm{HH}^{n-1}(A) \\ \downarrow \cong & & \downarrow \cong \\ D(\mathrm{HH}_n(A)) & \xrightarrow{\mathcal{B}^t} & D(\mathrm{HH}_{n-1}(A)) \end{array}$$

More details and examples, as well as a generalization to Frobenius algebras with semisimple Nakayama automorphism, may be found in [**LZZ16**] and the references given there.

3.6. Hochschild-Kostant-Rosenberg Theorem

The classical theorem of Hochschild, Kostant, and Rosenberg [**HKR62**] gives Hochschild (co)homology of a smooth finitely generated commutative algebra. The theorem has more general versions than what we present here; see, for example, [**Wei94**, Theorem 9.4.7].

Let $\Omega^1_{com}A$ be the A-module of Kähler differentials defined in Section 3.2.

Theorem 3.6.1. *(Hochschild-Kostant-Rosenberg) Let A be a smooth finitely generated commutative algebra. Then there are isomorphisms of graded algebras,*

$$\mathrm{HH}^*(A) \cong \textstyle\bigwedge_A(\mathrm{Der}(A)) \qquad \textit{and} \qquad \mathrm{HH}_*(A) \cong \textstyle\bigwedge_A(\Omega^1_{com}(A)).$$

Proof. We will prove the Hochschild cohomology isomorphism. The homology isomorphism is similar; see, e.g., [**Wei94**, Theorem 9.4.7]. Recall that $\mathrm{Der}(A) \cong \mathrm{Hom}_{A^e}(\Omega^1_{nc}A, A)$, and since A is commutative, this is isomorphic to $\mathrm{HH}^1(A)$. Thus there is a map

$$\psi : \textstyle\bigwedge_A \mathrm{Der}(A) \to \mathrm{HH}^*(A)$$

given by identifying $\bigwedge^0_A \mathrm{Der}(A)$ with $A \cong \mathrm{HH}^0(A)$ and $\bigwedge^1_A \mathrm{Der}(A)$ with $\mathrm{Der}(A) \cong \mathrm{HH}^1(A)$, and then extending to an algebra homomorphism. This is possible due to graded commutativity of $\mathrm{HH}^*(A)$. Note that ψ is an isomorphism if and only if $\psi \otimes_A A_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of A.

Since Ext commutes with localization [**Wei94**, Proposition 3.3.10], for any maximal ideal $\mathfrak{m}$ of A, $\mathrm{HH}^*(A_{\mathfrak{m}}) \cong \mathrm{HH}^*(A) \otimes_A A_{\mathfrak{m}}$. We will show that $\mathrm{HH}^*(A_{\mathfrak{m}}) \cong \bigwedge_{A_{\mathfrak{m}}}(\mathrm{Der}(A_{\mathfrak{m}}))$ for every maximal ideal $\mathfrak{m}$ of A, and the claimed isomorphism will follow. This will be the case once we have shown that ψ is an isomorphism whenever A is a local ring.

Now assume A is local with maximal ideal $\mathfrak{m}$. Let M be the preimage of $\mathfrak{m}$ in A^e under π, so that M is a maximal ideal of A^e. Since A is smooth, A^e is a regular ring [**Wei94**, Proposition 9.4.6]. Now $A \cong (A^e)_M/(\Omega^1_{nc}A)_M$

and since these are regular local rings, $(\Omega^1_{nc}A)_M$ is generated by a regular sequence $\mathbf{x}$ [**Wei94**, Exercise 4.4.2]. By Theorem 2.3.12, $K(\mathbf{x})$ is a free resolution of the $(A^e)_M$-module A. Restricting to A^e and applying $\mathrm{Hom}_{A^e}(-, A)$, we find the differentials are 0 and the cohomology is as claimed. $\square$

Chapter 4

Algebraic Deformation Theory

We will examine some types of deformations of associative algebras. We focus on the role played by Hochschild cohomology, and in particular by the Gerstenhaber bracket. Surveys of further aspects of the theory include [**Sch**] on deformations arising in noncommutative geometry and [**Gia11**] for deformation formulas arising from bialgebra actions and for deformations of bialgebras. Here we generally discuss formal deformations, infinitesimal deformations, deformation quantization, and graded deformations. We then summarize the theory of Braverman and Gaitsgory [**BG96**] for graded deformations of Koszul algebras, and derive the classical Poincaré-Birkhoff-Witt Theorem for Lie algebras as an application of this theory.

In this chapter A denotes an algebra over a field k.

4.1. Formal deformations

Let t be an indeterminate. Denote by $A[[t]]$ the algebra whose elements are formal power series $\sum_{i\geq 0} a_i t^i$ with coefficients $a_i \in A$. Multiplication is given by the Cauchy product:

$$(\sum_{i\geq 0} a_i t^i)(\sum_{j\geq 0} b_j t^j) = \sum_{l\geq 0} \sum_{i+j=l} a_i b_j t^l.$$

We are interested in new associative algebra structures on the $k[[t]]$-module $A[[t]]$ that agree with A upon taking the quotient by the ideal generated by t. Precisely, we have the following definition.

Definition 4.1.1. A *formal deformation* $(A_t, *)$ of A (or a *deformation of A over $k[[t]]$*) is an associative k-bilinear multiplication $*$ on the $k[[t]]$-module $A[[t]]$, such that modulo the ideal generated by t, the multiplication corresponds to that on A; the multiplication is required to be determined by a multiplication on elements of A and extended to $A[[t]]$ by the Cauchy product rule. Define similarly a *deformation of A over $k[t]$* or *over $k[t]/(t^n)$*.

To be clear, we intend in the definition a *new* multiplication on elements of A, taking values in $A[[t]]$, and extended to a multiplication on $A[[t]]$ by Cauchy products. We give an explicit description via a multiplication formula (4.1.3) below.

Remark 4.1.2. There are yet more general types of deformations: Let R be any commutative augmented k-algebra (such as $R = k[[t_1, \dots, t_m]]$), with augmentation map $\varepsilon : R \to k$, that is complete with respect to the $(\operatorname{Ker}\varepsilon)$-adic topology. A *deformation of A over R* is an associative R-algebra A_R that is isomorphic to the completed tensor product of A with R as an R-module and for which there is a k-algebra isomorphism $A_R/(\operatorname{Ker}\varepsilon) \xrightarrow{\sim} A$. One often assumes that A_R is free as an R-module, or more generally that A_R is flat as an R-module (for a *flat deformation*). In this book we will only consider deformations over $k[[t]]$, $k[t]$, or $k[t]/(t^n)$.

Any multiplication $*$ as in Definition 4.1.1 is determined by products of pairs of elements of A: For $a, b \in A$,

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \mu_3(a \otimes b)t^3 + \cdots , \tag{4.1.3}$$

where ab is the usual product in A and $\mu_1, \mu_2, \mu_3, \dots$ are functions from $A \otimes A$ to A giving the coefficients of $t, t^2, t^3, \dots$ as indicated. Sometimes we write $\mu_0(a \otimes b) = ab$ so that the formula becomes

$$a * b = \sum_{i \geq 0} \mu_i(a \otimes b)t^i.$$

The functions μ_i are necessarily k-linear. We call μ_i the *ith multiplication map* of the deformation. We sometimes denote the deformation $(A_t, *)$ by (A_t, μ_t), writing

$$\mu_t = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots$$

as a function on $A \otimes A$. When needed, we extend μ_t to be a function on the tensor product of the $k[[t]]$-module $A[[t]]$ with itself, completed so that expressions with formal power series as tensor factors make sense:

$$\Big(\sum_{i \geq 0} a_i t^i\Big) \otimes \Big(\sum_{j \geq 0} b_j t^j\Big) = \sum_{n \geq 0} \Big(\sum_{i+j=n} a_i \otimes b_j \Big) t^n.$$

The tensor product on the left side of the equation above is taken over $k[[t]]$; we have suppressed the subscript on the tensor symbol between elements for readability. Some authors write $\widehat{\otimes}$ in place of $\otimes$ to denote a completed tensor product.

We will next derive some properties of the functions μ_i. Let $a, b, c \in A$. Under the assumption of associativity of $*$, we have $(a * b) * c = a * (b * c)$. Calculating each side of this equation using formula (4.1.3), we find that

$$\begin{aligned} (a * b) * c &= abc + (\mu_1(ab \otimes c) + \mu_1(a \otimes b)c)t \\ &\quad + (\mu_2(ab \otimes c) + \mu_1(\mu_1(a \otimes b) \otimes c) + \mu_2(a \otimes b)c)t^2 + \cdots \end{aligned}$$

while

$$\begin{aligned} a * (b * c) &= abc + (\mu_1(a \otimes bc) + a\mu_1(b \otimes c))t \\ &\quad + (\mu_2(a \otimes bc) + \mu_1(a \otimes \mu_1(b \otimes c)) + a\mu_2(b \otimes c))t^2 + \cdots \end{aligned}$$

Equating coefficients of t, it follows that

$$\mu_1(ab \otimes c) + \mu_1(a \otimes b)c = \mu_1(a \otimes bc) + a\mu_1(b \otimes c) \tag{4.1.4}$$

for all $a, b, c \in A$. Comparing with equation (1.2.1), we see that μ_1 may be identified with a Hochschild 2-cocycle on the bar resolution. Equating coefficients of t^2, we have

$$\begin{aligned} &\mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c)) \\ &= a\mu_2(b \otimes c) - \mu_2(ab \otimes c) + \mu_2(a \otimes bc) - \mu_2(a \otimes b)c \end{aligned}$$

for all $a, b, c \in A$. Comparing with Definition 1.4.1, the left side of the above equation is the circle product $\mu_1 \circ \mu_1$, which in characteristic not 2 is half of the Gerstenhaber bracket $[\mu_1, \mu_1]$ applied to $a \otimes b \otimes c$. The right side is $d_3^*(\mu_2)$ applied to $a \otimes b \otimes c$. Thus associativity of $*$ implies that

$$[\mu_1, \mu_1] = 2d_3^*(\mu_2)$$

in characteristic not 2, while in characteristic 2 we must express the condition as $\mu_1 \circ \mu_1 = d_3^*(\mu_2)$. A similar analysis shows that

$$[\mu_1, \mu_2] = d_3^*(\mu_3)$$

and more generally that

$$\sum_{j=1}^{i-1} (\mu_j(\mu_{i-j}(a \otimes b) \otimes c) - \mu_j(a \otimes \mu_{i-j}(b \otimes c))) = d_3^*(\mu_i)(a \otimes b \otimes c) \tag{4.1.5}$$

for all $a, b, c \in A$ and $i \geq 2$. Alternatively we may apply Lemma 1.4.3(ii), with $\pi = \mu_0$, to rewrite the differential as $(-1)[-, \mu_0]$ and equation (4.1.5) becomes

$$\sum_{j=0}^{i} (\mu_j(\mu_{i-j}(a \otimes b) \otimes c) - \mu_j(a \otimes \mu_{i-j}(b \otimes c))) = 0. \tag{4.1.6}$$

We have thus found that there are infinitely many conditions that must be satisfied, one for each i, given by equation (4.1.5) or (4.1.6), in order that $*$ be an associative multiplication on $A[[t]]$. We call the left side of equation (4.1.5) the $(i-1)$*st obstruction*.

We give several examples next. Each example is a deformation over $k[[t]]$ or $k[t]$ which is then specialized to a particular value of the parameter t, a common source of new algebras arising from deformations of given algebras.

Example 4.1.7. Let $A = k[x, y]$. Define a multiplication $*$ on $A[t]$ or on $A[[t]]$ by

$$x * x = x^2, \quad y * y = y^2, \quad x * y = xy, \quad y * x = xy + t,$$

and extend using associativity. Substitute $t = 1$ in $A[t]$ to obtain the *Weyl algebra*

$$A_1 = k\langle x, y\rangle/(yx - xy - 1).$$

This may also be done with more indeterminates.

Example 4.1.8. Let $A = \mathbb{C}[x, y]$. Define a multiplication $*$ on $A[[t]]$ by

$$\begin{aligned} &x * x = x^2, \quad y * y = y^2, \quad x * y = xy, \\ &y * x = xy(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots) = xy \cdot \exp(t). \end{aligned}$$

Let $t_0 \in \mathbb{C}$ and substitute $t = t_0$ in the subalgebra of $A[[t]]$ generated by x and y. Let $q = \exp(t_0)$. The resulting algebra is the *quantum plane*

$$\mathbb{C}_q[x, y] = \mathbb{C}\langle x, y\rangle/(yx - qxy).$$

This may also be done with more indeterminates.

Example 4.1.9. Let $A = \mathbb{C}[e, f, h]$. Define a multiplication on $A[t]$ by

$$\begin{array}{lllllllll} e * e &=& e^2, & e * f &=& ef, & e * h &=& eh, \\ f * e &=& ef - ht, & f * f &=& f^2, & f * h &=& fh, \\ h * e &=& eh + 2et, & h * f &=& fh - 2ft, & h * h &=& h^2. \end{array}$$

Substitute $t = 1$ to obtain the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$:

$$U(\mathfrak{sl}_2) = \mathbb{C}\langle e, f, h\rangle/(fe - ef + h,\ he - eh - 2e,\ hf - fh + 2f).$$

The last example is essentially generalized below in a restatement of the classical Poincaré-Birkhoff-Witt (or PBW) Theorem: Recall that a *Lie algebra* $\mathfrak{g}$ is a vector space with a linear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that is antisymmetric, that is, $[x, y] = -[y, x]$, and satisfies the *Jacobi identity*,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$. The *universal enveloping algebra* of $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}).$$

In the proof of Theorem 4.5.5 we will see that $U(\mathfrak{g})$ is a particular type of deformation of a polynomial ring, termed a PBW deformation due to its appearance in this theorem.

4.2. Infinitesimal deformations and rigidity

An algebra is rigid if it cannot be deformed, and we make this notion precise in this section. We saw in the last section that every formal deformation has associated to it a Hochschild 2-cocycle. If Hochschild cohomology vanishes in degree 2, the algebra is necessarily rigid, as stated in Proposition 4.2.7 below. Otherwise we examine more closely the Hochschild 2-cocycles arising, starting with the next definition.

Definition 4.2.1. A k-linear function $\mu_1 : A \otimes A \to A$ is an *infinitesimal deformation* if (4.1.4) holds, that is $d^*\mu_1 = 0$ in $\mathrm{Hom}_k(A \otimes A \otimes A, A)$. Its *primary obstruction vanishes* if $[\mu_1, \mu_1]$ is a coboundary in $\mathrm{Hom}_k(A \otimes A \otimes A, A)$. It is *integrable* if there is a formal deformation (A_t, μ_t) for which μ_1 is the first multiplication map.

If μ_1 is an infinitesimal deformation, it defines an associative algebra structure on $A[t]/(t^2)$, that is, it defines a deformation of A over $k[t]/(t^2)$, the ring of dual numbers: Let

$$a * b = ab + \mu_1(a \otimes b)t$$

for all $a, b \in A$ and extend $k[t]/(t^2)$-bilinearly to $A[t]/(t^2)$. Conversely, a deformation of A over $k[t]/(t^2)$ is determined by the coefficient of t in the above equation, which necessarily satisfies equation (4.1.4). Sometimes when we refer to an infinitesimal deformation, we mean this corresponding algebra structure on $A[t]/(t^2)$.

In Section 6.1, we more generally define an infinitesimal n-deformation for any $n \geq 2$; our infinitesimal deformations here will be called infinitesimal 2-deformations there. We will show in Theorem 6.1.8 that a Hochschild n-cocycle corresponds to an infinitesimal n-deformation.

We will next define a notion of equivalence of formal deformations. We will see that equivalent deformations have corresponding Hochschild 2-cocycles that are cohomologous.

Definition 4.2.2. Two formal deformations (A_t, μ_t), (A'_t, μ'_t) are *equivalent* if there is a $k[[t]]$-linear function $\phi_t : A_t \to A'_t$ of the form

$$\phi_t(a) = a + \phi_1(a)t + \phi_2(a)t^2 + \cdots$$

for functions $\phi_i : A \to A$ such that

$$\phi_t \mu_t(a \otimes b) = \mu'_t(\phi_t(a) \otimes \phi_t(b)) \tag{4.2.3}$$

for all $a, b \in A$. A formal deformation (A_t, μ_t) is *trivial* if it is equivalent to $A[[t]]$.

Note that the function ϕ_t is an isomorphism of algebras by its definition above. (Any function of the given form is necessarily invertible as a formal power series.)

Lemma 4.2.4. *If (A_t, μ_t), (A'_t, μ'_t) are equivalent via a function ϕ_t as in Definition 4.2.2, then $\mu'_1 = \mu_1 - d^*\phi_1$. In particular, if (A_t, μ_t) is trivial, then μ_1 is a coboundary.*

Proof. Expanding equation (4.2.3), we have

$$ab + (\phi_1(ab) + \mu_1(a \otimes b))t + \cdots = ab + (\mu'_1(a \otimes b) + \phi_1(a)b + a\phi_1(b))t + \cdots$$

for all $a, b \in A$. Equating coefficients of t, we see that $\mu'_1 = \mu_1 - d^*\phi_1$ as claimed. If (A_t, μ_t) is trivial, then it is equivalent to (A'_t, μ'_t) with $\mu'_0 = \mu_0$ and $\mu'_i = 0$ for all $i > 0$. Thus $\mu_1 = d^*\phi_1$. □

Lemma 4.2.5. *A nontrivial formal deformation (A_t, μ_t) of A is equivalent to a formal deformation (A'_t, μ'_t) with the property that the first nonvanishing cochain μ'_n is a Hochschild 2-cocycle that is not a coboundary.*

Proof. Suppose (A_t, μ_t) is a formal deformation of A whose first nonvanishing cochain is a coboundary. Write

$$\mu_t(a \otimes b) = ab + \mu_n(a \otimes b)t^n + \mu_{n+1}(a \otimes b)t^{n+1} + \cdots$$

for all $a, b \in A$, where $\mu_n = d^*(\beta)$ for some $\beta \in \mathrm{Hom}_k(A, A)$. Let

$$\phi_t(a) = a + \beta(a)t^n$$

for all $a \in A$, and note that

$$\phi_t^{-1}(a) = a - \beta(a)t^n + \beta^2(a)t^{2n} - \beta^3(a)t^{3n} + \cdots ,$$

where $\beta^2(a) = \beta(\beta(a))$, $\beta^3(a) = \beta(\beta(\beta(a)))$, etc. Set $\mu'_t = \phi_t\mu_t(\phi_t^{-1} \otimes \phi_t^{-1})$. Then, since $\mu_n = d^*(\beta)$, there is some function μ'_{n+1} such that

$$\begin{aligned}
&\mu'_t(a \otimes b) \\
&= \phi_t\mu_t((a - \beta(a)t^n + \cdots) \otimes (b - \beta(b)t^n + \cdots)) \\
&= \phi_t(ab + (\mu_n(a \otimes b) - a\beta(b) - \beta(a)b)t^n + \cdots) \\
&= ab + (\beta(ab) + \mu_n(a \otimes b) - a\beta(b) - \beta(a)b)t^n + \mu'_{n+1}(a \otimes b)t^{n+1} + \cdots \\
&= ab + \mu'_{n+1}(a \otimes b)t^{n+1} + \cdots .
\end{aligned}$$

If μ'_{n+1} is a coboundary, then similarly (A'_t, μ'_t) is equivalent to (A''_t, μ''_t) where

$$\mu''_t(a \otimes b) = ab + \mu''_{n+2}(a \otimes b)t^{n+2} + \cdots$$

via a function ϕ'_t with $\phi'_t(a) = a + \beta'(a)t^{n+1}$. Continuing in this fashion, we may let Φ_t be the function

$$\Phi_t(a) = a + \Phi_n(a)t^n + \Phi_{n+1}(a)t^{n+1} + \cdots$$

defined as the composition of all $\ldots, \phi''_t, \phi'_t, \phi_t$. This composition is well-defined: The coefficient function of each power of t in Φ_t is a finite polynomial in $\beta, \beta', \beta'', \ldots$. For example, if $n = 1$, then

$$\Phi_t(a) = a + \beta(a)t + \beta'(a)t^2 + (\beta''(a) + \beta'(\beta(a)))t^3 + \cdots .$$

Applying Φ_t, we see that (A_t, μ_t) is trivial as claimed: Note that for any given power of t, its coefficient function in the resulting equivalent deformation only involves a composition of finitely many such equivalences, and we find that the coefficient function is indeed 0. □

We now focus on algebras having no such deformations.

Definition 4.2.6. An algebra is *rigid* if it has no nontrivial formal deformations.

As an immediate consequence of Lemma 4.2.5, we have the following proposition.

Proposition 4.2.7. *If* $\mathrm{HH}^2(A) = 0$*, then* A *is rigid.*

Examples of algebras A for which $\mathrm{HH}^2(A) = 0$ include separable algebras, universal enveloping algebras of semisimple Lie algebras, Weyl algebras, and tensor algebras over a field. Thus all of these algebras are rigid. We point out however that universal enveloping algebras do have bialgebra deformations [**EK96**], and bialgebra cohomology governs these deformations in an analogous theory.

4.3. Maurer-Cartan equation and Poisson structures

For this section, we assume the characteristic of k is not 2. We will give another interpretation of the conditions (4.1.5) for a deformation, deriving the Maurer-Cartan equation (4.3.1) below. We will in particular examine the first obstruction condition in the case of commutative algebras.

Write $\mu_t = \sum_{i\geq 0} \mu_i t^i$ as before. Then all conditions (4.1.6) may combined and reinterpreted as

$$[\mu_t, \mu_t] = 0.$$

Set $\mu_t = \mu_0 + \mu'$. Since A is associative, $[\mu_0, \mu_0] = 0$, and since μ_0, μ' may be viewed as 2-cochains, $[\mu', \mu_0] = [\mu_0, \mu']$. So the above equation is equivalent to

$$2[\mu_0, \mu'] + [\mu', \mu'] = 0.$$

By Lemma 1.4.3(ii), the differential on the Hochschild complex is $(-1)[-, \mu_0]$, so this is equivalent to

$$-d^*\mu' + \frac{1}{2}[\mu', \mu'] = 0. \tag{4.3.1}$$

With appropriate sign conventions, this is the *Maurer-Cartan equation* (also called the *Berikkashvili equation*), and we have shown that the deformed multiplication μ_t is associative if and only if μ' satisfies the Maurer-Cartan equation. Focusing on μ_1 and μ_2 (giving rise to coefficients of t and t^2), this equation implies μ_1 is a Hochschild 2-cocycle and $d^*\mu_2 = \frac{1}{2}[\mu_1, \mu_1]$, as we saw before.

We next look more closely at commutative algebras and their potentially noncommutative deformations.

Definition 4.3.2. A *Poisson algebra* is a commutative associative algebra A that is also a Lie algebra under a binary operation $\{\ ,\ \}$ for which

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

for all $a, b, c \in A$. We call $\{\ ,\ \}$ a *Poisson bracket*.

One source of Poisson brackets on commutative algebras is Hochschild cohomology: Let $\mu_1 : A \otimes A \to A$ be a Hochschild 2-cocycle on A whose primary obstruction vanishes at the chain level, that is, $[\mu_1, \mu_1] = 0$. Let

$$\{a, b\} = \frac{1}{2}(\mu_1(a \otimes b) - \mu_1(b \otimes a))$$

for all $a, b \in A$. Calculations using commutativity of A and the Hochschild 2-cocycle condition show that $\{\ ,\}$ is a Poisson bracket on A.

Definition 4.3.3. Let A be a Poisson algebra. A *deformation quantization* of A is a formal deformation $(A_t, *)$ for which

$$a * b - b * a = \{a, b\}t \pmod{t^2}$$

for all $a, b \in A$.

Example 4.3.4. The Weyl algebra A_1 of Example 4.1.7 is the specialization of a deformation quantization of the polynomial ring $k[x, y]$ with Poisson bracket $\{x, y\} = -1$. Similar statements hold for Weyl algebras in more indeterminates.

We will discuss in Section 6.6 some general conditions under which it is known that a Poisson algebra has a deformation quantization. In the noncommutative setting, an infinitesimal deformation μ_1 of a not necessarily commutative algebra A is sometimes regarded as a noncommutative Poisson structure on A when its primary obstruction vanishes.

4.4. Graded deformations

Let A be an $\mathbb{N}$-graded algebra. Let t be an indeterminate, and assign t the degree 1. Consider the resulting $\mathbb{N}$-graded algebra $A[t]$, in which $|at^i| = |a|+i$ for all homogeneous $a \in A$ and $i \in \mathbb{N}$. A *graded deformation* of A over $k[t]$ is a deformation A_t of A over $k[t]$ that is also a graded algebra. Necessarily then, since $|t| = 1$, each function μ_j of equation (4.1.3) is homogeneous of degree $-j$. An *ith level graded deformation of* A is a graded associative algebra A_i over $k[t]/(t^{i+1})$ with underlying $k[t]/(t^{i+1})$-module $A[t]/(t^{i+1})$ and multiplication given on elements $a, b \in A$ by

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \cdots + \mu_i(a \otimes b)t^i$$

for k-linear functions $\mu_j : A \otimes A \to A$. A first level graded deformation of A then corresponds to an infinitesimal deformation of A that is graded. We say that an $(i-1)$st level graded deformation A_{i-1} of A *lifts* to an ith level graded deformation A_i of A if the jth multiplication maps of A_{i-1} and A_i agree for all $j \leq i-1$. The obstruction to existence of such a lifting is equation (4.1.5).

Braverman and Gaitsgory [**BG96**] made the following proposition regarding lifting graded deformations. We will need to use an induced grading on Hochschild cohomology, defined next.

The bar resolution of A is graded, with

$$|a_0 \otimes \cdots \otimes a_{n+1}| = |a_0| + \cdots + |a_{n+1}|$$

for all homogeneous $a_0, \ldots, a_{n+1} \in A$. The A^e-module $A^{\otimes(n+2)}$ is a graded A^e-module in this way, and the differentials are graded maps. More generally we may consider a projective resolution of A as an A^e-module consisting of graded modules and graded differentials. The cohomology inherits this grading. Hochschild cohomology $\mathrm{HH}^*(A)$ is graded in this way by the grading on A as well as by homological degree. We call the grading on $\mathrm{HH}^*(A)$ coming from A its *internal grading*. Denote by $\mathrm{HH}^{i,j}(A)$ the subspace of $\mathrm{HH}^i(A)$ consisting of homogeneous elements of internal degree j.

Proposition 4.4.1. *A first level graded deformation of A corresponds to an element of* $\mathrm{HH}^{2,-1}(A)$. *An obstruction to lifting an $(i-1)$st level graded deformation of A to an ith level graded deformation of A is in* $\mathrm{HH}^{3,-i}(A)$, *and an $(i-1)$st level graded deformation lifts to an ith level deformation if and only if the $(i-1)$st obstruction given by the left side of equation (4.1.5) is 0 in cohomology.*

Proof. We have noted before that a first level deformation corresponds to an infinitesimal deformation, that is a Hochschild 2-cocycle. If, in addition, it is graded, then necessarily the internal degree of the Hochschild 2-cocycle is -1, as noted before. Two first level graded deformations are isomorphic if and only if their corresponding cocycles are cohomologous. Therefore the first statement of the proposition holds.

A graded deformation of A necessarily involves functions μ_i that are homogeneous of degree $-i$, and the same is true of an ith level graded deformation. We have already noted the obstructions (4.1.5), and in this graded setting, we see that the equation (4.1.5) indeed involves functions of internal degree $-i$. Thus the second statement holds. □

4.5. Braverman-Gaitsgory theory and the PBW Theorem

In this section, we present the theory of Braverman and Gaitsgory [**BG96**] and its application to the classical Poincaré-Birkhoff-Witt Theorem on the structure of universal enveloping algebras of Lie algebras. We only consider here the original setting of [**BG96**] where A is a connected graded Koszul algebra, although this theory has been generalized in a number of directions (see, for example, the survey [**SW15**]).

Write $A = T(V)/(R)$ where V is a finite dimensional vector space whose elements are given degree 1, and R is a subspace of $V \otimes V$. Assume A is a Koszul algebra, as defined in Section 2.3. Let $\alpha : R \to V$ and $\beta : R \to k$ be k-linear maps, and consider the subspace

$$P = \{x - \alpha(x) - \beta(x) \mid x \in R\} \tag{4.5.1}$$

of $T_0(V) \oplus T_1(V) \oplus T_2(V)$. Set $A' = T(V)/(P)$, where (P) denotes the ideal of $T(V)$ generated by P. Then A' is a filtered algebra: Letting F_nA' be the image in A' of the subspace $\oplus_{0\le i\le n} T_i(V)$ of $T(V)$, we have

$$F_0A' \subset F_1A' \subset F_2A' \subset \cdots$$

and $(F_iA')(F_jA') \subset F_{i+j}A'$. Denote the associated graded algebra to A' by $\mathrm{gr}\, A'$, that is, as a vector space,

$$\mathrm{gr}\, A' = F_0A' \oplus (F_1A'/F_0A') \oplus (F_2A'/F_1A') \oplus \cdots ,$$

and the product of an element in $F_iA'/F_{i-1}A'$ with one in $F_jA'/F_{j-1}A'$ is defined by multiplying inverse images in F_iA' and F_jA', and taking the resulting image in $F_{i+j}A'/F_{i+j-1}A'$. By definition of P then, the relations R hold in $\mathrm{gr}\, A'$, that is, there is an algebra homomorphism

$$A \to \mathrm{gr}\, A' \tag{4.5.2}$$

induced by mapping the generating space V to its image in $\mathrm{gr}\, A'$.

Definition 4.5.3. The algebra $A' = T(V)/(P)$ is a *PBW deformation* (also called a *filtered deformation*) of A if the algebra homomorphism (4.5.2) is an isomorphism.

A PBW deformation may be seen to be a specialization of a graded deformation to a value of the parameter t: Given P as in (4.5.1), let

$$P[t] = \{x - \alpha(x)t - \beta(x)t^2 \mid x \in R\},$$

a homogeneous subspace of $T(V)[t]$ of degree 2. The specializations of $T(V)[t]/P[t]$ to $t = 0$ and to $t = 1$ are isomorphic to A and to A', respectively. The condition that A' be a PBW deformation of A implies that $T(V)[t]/P[t]$ is a graded deformation of A.

Braverman and Gaitsgory gave necessary and sufficient conditions for $A' = T(V)/(P)$ to be a PBW deformation of A. The following theorem is essentially a combination of [**BG96**, Lemma 0.4, Theorem 0.5, and Lemma 3.3].

Theorem 4.5.4. *The algebra $A' = T(V)/(P)$ is a PBW deformation of A if and only if the following conditions hold:*

(a) $P \cap F_1(T(V)) = \{0\}$,
(b) $\mathrm{Im}(\alpha \otimes 1 - 1 \otimes \alpha) \subset R$,
(c) $\alpha(\alpha \otimes 1 - 1 \otimes \alpha) = -(\beta \otimes 1 - 1 \otimes \beta)$,
(d) $\beta(\alpha \otimes 1 - 1 \otimes \alpha) = 0$,

where the maps $\alpha \otimes 1 - 1 \otimes \alpha$ and $\beta \otimes 1 - 1 \otimes \beta$ are defined on the subspace $(R \otimes V) \cap (V \otimes R)$ of $T(V)$.

Proof. Assume that A' is a PBW deformation of A. Then necessarily condition (a) holds, in order that $k \oplus V$ map isomorphically onto a copy of itself in A'. We will show that the other three conditions also hold. These can be interpreted in terms of the Hochschild cohomology $\mathrm{HH}^*(A)$ as follows.

Let $K_{\bullet}$ be the Koszul resolution (2.3.4) of A and let $\iota : K_{\bullet} \hookrightarrow B_{\bullet}$ be its inclusion into the bar resolution of A, as described in Section 2.3. Let $\psi : B_{\bullet} \to K_{\bullet}$ be a chain map lifting the identity map on A, and note that ψ may be chosen so that $\psi\iota = 1_K$. Identify α and β with functions on the degree 2 term K_2 of the Koszul resolution by extending to A-bimodule homomorphisms. Set

$$\mu_1 = \alpha\psi, \quad \mu_2 = \beta\psi$$

to obtain maps in $\mathrm{Hom}_{A^e}(A^{\otimes 4}, A) \cong \mathrm{Hom}_k(A^{\otimes 2}, A)$.

If A' is a PBW deformation of A then

$$T(V)[t]/(P[t],\ t^2) = T(V)[t]/(x - \alpha(x)t,\ t^2 \mid x \in R)$$

corresponds to an infinitesimal deformation of A. Considering the elements of $(R \otimes V) \cap (V \otimes R)$, associativity implies condition (b). Further, the quotient $T(V)[t]/(P[t],\ t^3)$ is a second level graded deformation of A, which implies that

$$\mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1) = d_3^*(\mu_2),$$

considered as functions on $A \otimes A \otimes A$. Applying both sides of this equation to $(R \otimes V) \cap (V \otimes R)$, since $\psi\iota = 1_K$, this is equivalent to condition (c). Similarly, there is a μ_3 such that

$$\mu_1(\mu_2 \otimes 1 - 1 \otimes \mu_2) + \mu_2(\mu_1 \otimes 1 - 1 \otimes \mu_1) = d_3^*(\mu_3).$$

Applying both sides of this equation to elements of $(R \otimes V) \cap (V \otimes R)$, for degree reasons (since $|\mu_3| = -3$, $|\mu_2| = -2$, $|\mu_1| = -1$), the terms $\mu_1(\mu_2 \otimes 1 - 1 \otimes \mu_2)$ have image zero, and we obtain $\beta(\alpha \otimes 1 - 1 \otimes \alpha) = 0$, which is condition (d).

Conversely, suppose conditions (a)–(d) hold. Then $\mu_1\iota = \alpha$ since $\psi\iota = 1_K$. It follows that μ_1 is a Hochschild 2-cocycle on the bar resolution, and so defines an infinitesimal deformation of A. Similarly $\mu_2\iota = \beta$. We modify μ_2 so that it satisfies condition (4.1.5) with $i = 2$ as a function on the bar resolution: Let

$$\gamma = -\mu_2 d_3 + \mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1).$$

Then $\gamma\iota = -\beta d_3 + \alpha(\alpha \otimes 1 - 1 \otimes \alpha) = 0$ on K_3, which implies γ is a coboundary on the bar resolution, that is, $\gamma = \mu d_3$ for some μ. Now

$$\mu\iota d = \mu d\iota = \gamma\iota = 0,$$

so $\mu\iota$ is a cocycle on $K_{\bullet}$. Consequently there is a cocycle μ', of internal degree -2, on the bar resolution with $\mu'\iota = \mu\iota$. Then $(\mu_2 - \mu + \mu')\iota = \beta$ and

$$(\mu_2 - \mu + \mu')d + \mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1)$$

is zero on the bar resolution since $\mu' d = 0$ and

$$-(\mu_2 - \mu)d = \mu_1(\mu_1 \otimes 1 - 1 \otimes \mu_1).$$

We replace μ_2 by $\mu_2 - \mu + \mu'$.

Now, the map μ_1 and the new map μ_2 satisfy (4.1.5) with $i = 1$, $i = 2$. Thus there is a second level graded deformation of A defined by μ_1, μ_2. By condition (d), considering internal degree, the obstruction

$$\mu_2(\mu_1 \otimes 1 - 1 \otimes \mu_1) + \mu_1(\mu_2 \otimes 1 - 1 \otimes \mu_2)$$

to lifting to a third level deformation of A becomes 0 as a cochain on $K_{\bullet}$, on applying ι. So this is a coboundary on the bar resolution, and there is a μ_3 of internal degree -3 satisfying (4.1.5) with $i = 3$.

Now by Proposition 4.4.1, the obstruction to lifting to a fourth level graded deformation lies in $\mathrm{HH}^{3,-4}(A)$. However, this is 0 since K_3 consists of elements of internal degree 3. It follows that there is a μ_4 defining a fourth level deformation. The same argument shows that this may be lifted to a fifth level deformation and so on. Letting A_t denote the graded deformation obtained in this manner, we see that A' is isomorphic to $A_t|_{t=1}$: A map of vector spaces from V to $A_t|_{t=1}$ induces a map $A' \to A_t|_{t=1}$, which may be seen to be an isomorphism. $\square$

As an application, we obtain the classical Poincaré-Birkhoff-Witt Theorem for Lie algebras next. The proof shows that the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is a PBW deformation of a polynomial ring. This is $S(\mathfrak{g})$, the symmetric algebra on the underlying vector space of a Lie algebra $\mathfrak{g}$, that is,

$$S(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x \mid x, y \in \mathfrak{g}).$$

Theorem 4.5.5 (Poincaré-Birkhoff-Witt Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. The associated graded algebra of $U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$.*

Proof. Let $V = \mathfrak{g}$ and $A = S(\mathfrak{g}) = T(V)/(R)$ where R is the vector subspace of $V \otimes V$ spanned by all $x \otimes y - y \otimes x$ for $x, y \in \mathfrak{g}$. Let

$$P = \{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\},$$

so that $U(\mathfrak{g}) = T(\mathfrak{g})/(P)$ by definition. Set $\alpha(x \otimes y - y \otimes x) = [x, y]$ and $\beta(x \otimes y - y \otimes x) = 0$. By antisymmetry, condition (a) of Theorem 4.5.4 holds. One checks that $(R \otimes V) \cap (V \otimes R)$ consists of linear combinations of elements of the form

$$x \otimes y \otimes z - y \otimes x \otimes z + y \otimes z \otimes x - z \otimes y \otimes x + z \otimes x \otimes y - x \otimes z \otimes y,$$

for $x, y, z \in \mathfrak{g}$, and that condition (b) of Theorem 4.5.4 holds by a calculation. Condition (c) is equivalent to the Jacobi identity, and condition (d) automatically holds since β is 0. By Theorem 4.5.4, $U(\mathfrak{g})$ is a PBW deformation of $S(\mathfrak{g})$, and in particular, $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g})$. $\square$

For a survey of some of the many generalizations of this classical Poincaré-Birkhoff-Witt Theorem, as well as other methods of proof, see [**SW15**].

Chapter 5

Gerstenhaber Bracket

We will give several interpretations of the Lie structure on Hochschild cohomology. Some lead to computational techniques. We begin with a realization of Hochschild cohomology as the homology of a complex of coderivations on the tensor coalgebra of A; the Gerstenhaber bracket is a graded commutator of coderivations ([**Qui89, Sta93**]). This tensor coalgebra is similar to the bar resolution. We then explain a formula for brackets of elements in degree 1 with those in arbitrary degree n [**SA**]. The degree 1 elements are identified with derivations and thus with functions on the bar resolution, while degree n elements and the bracket formula are given on an arbitrary resolution. Thus we begin to depart from the historical setting of the bar resolution, and the remainder of this chapter involves techniques for other resolutions and exact sequences. We present homotopy liftings that allow the bracket to be expressed on an arbitrary resolution as essentially a graded commutator of function compositions [**Vol**]. We discuss related computational techniques as applied in particular to Koszul algebras [**NW16**]. For a topological approach, we outline a construction of brackets as loops in an extension category [**Sch98**]. Hochschild cohomology may also be realized as the Lie algebra of the derived Picard group of an algebra, the Gerstenhaber bracket being its graded Lie bracket [**Kel04**]; we will not present this work here.

We use the Koszul sign convention in this chapter, that is for graded vector spaces V, V', W, W' and k-linear graded functions $f : V \to V'$, $g : W \to W'$ of degrees $|f|$, $|g|$, the function $f \otimes g$ on $V \otimes W$ is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) \tag{5.0.1}$$

for all homogeneous $v \in V$, $w \in W$.

5.1. Coderivations

Let $T = T(A) = \oplus_{n\geq 0} A^{\otimes n}$, considered as a complex with differential d_T given by

$$d_T(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

for all $a_1, \ldots, a_n \in A$. Denote by $\mathrm{Hom}_k(T(A), T(A))$ the complex with differential ∂ given by

$$\partial(f) = d_T f - (-1)^{|f|} f d_T$$

for all homogeneous functions f.

The complex $\mathrm{Hom}_k(T(A), T(A))$ has a binary operation given by the graded commutator

$$[f, g] = fg - (-1)^{(|f|-1)(|g|-1)} gf \tag{5.1.1}$$

for all homogeneous $f, g \in \mathrm{Hom}_k(T(A), T(A))$. By virtue of being a graded commutator, it enjoys a graded Jacobi identity just as in Lemma 1.4.2(ii).

Define a k-linear map $\Delta_T : T(A) \to T(A) \otimes T(A)$ by

$$\begin{aligned} \Delta_T(a_1 \otimes \cdots \otimes a_n) &= 1 \otimes (a_1 \otimes \cdots \otimes a_n) + (a_1 \otimes \cdots \otimes a_n) \otimes 1 \\ &\quad + \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n) \end{aligned}$$

for all $a_1, \ldots, a_n \in A$. Under this map, $T(A)$ is a differential graded coalgebra, sometimes written $T^c(A)$ to distinguish it from the algebra on the same underlying vector space, that is: Δ_T is a chain map, $(\Delta_T \otimes 1)\Delta_T = (1 \otimes \Delta_T)\Delta_T$, and $(\varepsilon \otimes 1)\Delta_T = 1 = (1 \otimes \varepsilon)\Delta_T$ where $\varepsilon : T(A) \to k$ projects onto $T(A)_0 = k$. Let

$$\Delta_T^{(2)} = (\Delta_T \otimes 1_T)\Delta_T = (1_T \otimes \Delta_T)\Delta_T.$$

Definition 5.1.2. A *graded coderivation* on $T(A)$ is a graded k-linear map $f : T(A) \to T(A)$, of some degree j, for which

$$\Delta_T f = (f \otimes 1_T + 1_T \otimes f)\Delta_T.$$

Denote by $\mathrm{Coder}(T(A))$ the vector space spanned by the graded coderivations on $T(A)$.

Note the space $\mathrm{Coder}(T(A))$ is closed under the graded commutator bracket (5.1.1) by its definition. Also note that the differential ∂ is itself a coderivation since Δ_T is a chain map. By the graded Jacobi identity, ∂ is also a graded derivation with respect to this bracket.

The following connection with Hochschild cohomology goes back to work of Quillen [**Qui89**] and Stasheff [**Sta93**]. Let $B = B(A)$ be the bar resolution (1.1.4) of A as an A^e-module, so that $B_i = A^{\otimes(i+2)}$. We take the differential $\delta = d^*$ on $\mathrm{Hom}_k(T(A), A) \cong \mathrm{Hom}_{A^e}(B, A)$ to be that induced by the differential on the bar resolution of A, which in turn is related to the differential d_T on $T(A)$:

$$\begin{aligned}&\delta(f)(a_0 \otimes \cdots \otimes a_m)\\ &= a_0 f(a_1 \otimes \cdots \otimes a_m) + f d_T(a_0 \otimes \cdots \otimes a_m) + (-1)^m f(a_0 \otimes \cdots \otimes a_{m-1})a_m\end{aligned}$$

for all $a_0, \ldots, a_m \in A$ and $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$. Such a function f may be extended uniquely to a coderivation $D_f : T(A) \to T(A)[1-m]$ as follows:

$$D_f = (1_T \otimes f \otimes 1_T)\Delta_T^{(2)},$$

where if $l < m$, we interpret D_f to be 0 on $A^{\otimes l}$. On elements then, using the Koszul sign convention (5.0.1), we have

(5.1.3)

$$D_f(a_1 \otimes \cdots \otimes a_l) = \sum_{i=1}^{l-m+1} (-1)^{(m-1)(i-1)} a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_l$$

for all $a_1, \ldots, a_l \in A$. (Existence and uniqueness of D_f is due to the corresponding truncated complex being cofree in a certain category of coalgebras; see, for example, [**MSS02**, Section II.3.7].)

Theorem 5.1.4. *The complex* $(\mathrm{Coder}(T(A)), \partial)$ *is a subcomplex of the complex* $(\mathrm{Hom}_k(T(A), T(A)), \partial)$ *that is isomorphic, as a differential graded vector space, to* $(\mathrm{Hom}_k(T(A), A), \delta)$.

Proof. We have already seen that the space $\mathrm{Coder}(T(A))$ is a subcomplex of $\mathrm{Hom}_k(T(A), T(A))$, since the differential ∂ is a coderivation and $\mathrm{Coder}(T(A))$ is closed under bracket. Given an element of $\mathrm{Hom}_k(T(A), A)$, one extends uniquely to a coderivation from $T(A)$ to $T(A)$ as specified above. On the other hand, given a coderivation from $T(A)$ to $T(A)$, its composition with projection onto A is an element of $\mathrm{Hom}_k(T(A), A)$. One checks that the differential ∂ on $\mathrm{Coder}(T(A))$ corresponds to δ on $\mathrm{Hom}_k(T(A), A)$. □

As a consequence of the theorem, Hochschild cohomology $\mathrm{HH}^*(A)$ is the homology of the complex $(\mathrm{Coder}(T(A)), \partial)$. We may realize the Gerstenhaber bracket in a natural way on $\mathrm{Coder}(T(A))$ as follows.

Theorem 5.1.5. *The bracket (5.1.1) induces the Gerstenhaber bracket under the isomorphism of Theorem 5.1.4.*

Proof. In view of formula (5.1.3), identifying cochains f, g on B with their corresponding coderivations D_f, D_g on $T(A)$, the formula (5.1.1) coincides with Definition 1.4.1 of Gerstenhaber bracket. (To make this comparison, we may rewrite formula (5.1.1) as $[f, g] = fD_g - (-1)^{(|f|-1)(|g|-1)} gD_f$.) □

5.2. Derivation operators

In this section, we present Suárez-Álvarez' methods from [**SA**] for computing Gerstenhaber brackets with elements of homological degree 1 via an arbitrary resolution. These methods may be used for example to find the Lie structure of $\mathrm{HH}^1(A)$ and the structure of Hochschild cohomology $\mathrm{HH}^*(A)$ as a Lie module for $\mathrm{HH}^1(A)$. Suárez-Álvarez worked in a broader context of derivation operators and actions on Ext. Here we consider only that part of his theory that is directly relevant to the Gerstenhaber bracket on $\mathrm{HH}^*(A)$, and refer to [**SA**] for more general results.

Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A. Let $f : A \to A$ be a derivation, so that it represents an element of $\mathrm{HH}^1(A)$, as explained in Section 1.2. Let $f^e : A^e \to A^e$ be defined by $f^e = f \otimes 1 + 1 \otimes f$, and note that f^e is a derivation on A^e. Functions satisfying equation (5.2.2) below are termed f^e*-operators* in [**SA**]. More generally, the notion of a δ-operator, for any derivation δ on an algebra, is defined there.

The following lemma is related to work of Gopalakrishnan and Sridharan [**GS66**].

Lemma 5.2.1. *Let $f : A \to A$ be a derivation. There is a k-linear chain map $\tilde{f}_\bullet : P_\bullet \to P_\bullet$ lifting f with the property that for each n,*

$$\tilde{f}_n((a \otimes b) \cdot x) = f(a)xb + a\tilde{f}_n(x)b + axf(b) \tag{5.2.2}$$

for all $a, b \in A$ and $x \in P_n$. Moreover, $\tilde{f}_\bullet$ is unique up to A^e-module chain homotopy.

Proof. We wish to define each $\tilde{f}_i$ so that it satisfies equation (5.2.2), and so that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu_P} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu_P} & A & \longrightarrow & 0 \end{array}$$

If P_0 is free as an A^e-module, choose a free basis $\{x_j \mid j \in J\}$, where J is some indexing set. Since μ_P is surjective, there exist $y_j \in P_0$ such that $\mu_P(y_j) = f(\mu_P(x_j))$. Set $\tilde{f}_0(x_j) = y_j$. Extend to P_0 by requiring

$$\tilde{f}_0((a \otimes b) \cdot x_j) = f(a)x_jb + ay_jb + ax_jf(b)$$

for all $a, b \in A$ and $j \in J$. Note the rightmost square in the diagram indeed commutes since f is a derivation and the action of A^e on A is by left and right multiplication. If P_0 is not free, we may realize it as a direct summand of a free module and argue similarly.

Now $\tilde{f}_0 d_1$ has image contained in the image of d_1, since $\mu_P \tilde{f}_0 d_1 = f\mu_P d_1 = 0$. We may apply the same argument as above to define $\tilde{f}_1$, and so on.

If $\tilde{f}_\bullet$ and $\tilde{f}'_\bullet$ are two such k-linear chain maps, then $\tilde{f}_\bullet - \tilde{f}'_\bullet$ is a chain map lifting the zero map from A to A. Since each of $\tilde{f}_\bullet$, $\tilde{f}'_\bullet$ satisfies (5.2.2), their difference $\tilde{f}_\bullet - \tilde{f}'_\bullet$ is A^e-linear, and so it is A^e-chain homotopic to 0. □

Example 5.2.3. Let B be the bar resolution on A, and let $f : A \to A$ be a derivation. For each i, let

$$\tilde{f}_i(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i+1} a_0 \otimes \cdots \otimes a_{j-1} \otimes f(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_{i+1}$$

for all $a_0, \ldots, a_{i+1} \in A$. Then $\tilde{f}_\bullet : B \to B$ is a k-linear chain map satisfying (5.2.2).

The following theorem is due to Suárez-Álvarez [**SA**].

Theorem 5.2.4. *Let $f : A \to A$ be a derivation. Let P be a projective resolution of A as an A^e-module. Let $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be a cocycle, and let $\tilde{f}_n : P_n \to P_n$ be a map satisfying (5.2.2). The Gerstenhaber bracket of f and g is represented by*

$$[f, g] = fg - g\tilde{f}_n \qquad (5.2.5)$$

as a cocycle on P_n.

Proof. First note that if P is the bar resolution B, and $\tilde{f}_n$ is chosen as in Example 5.2.3, then formula (5.2.5) is the historical one for the Gerstenhaber bracket with a 1-cocycle. Since $gd = 0$ and $\tilde{f}_\bullet$ is unique up to chain homotopy as stated in Lemma 5.2.1, the element of Hochschild cohomology given by formula (5.2.5) does not depend on choice of $\tilde{f}_\bullet$.

More generally let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps, that is, chain maps lifting the identity map on A. Identify the derivation f with a cocycle on B. The Gerstenhaber bracket of f and g is by definition $[f, g\theta]\iota$, where $[f, g\theta]$ denotes the Gerstenhaber bracket defined as usual on B. Let $\tilde{f}'_\bullet : B \to B$ be a k-linear chain map satisfying (5.2.2) for B. A calculation shows that for each i, the function $\theta\tilde{f}'_i - \tilde{f}_i\theta$ is in fact an A^e-module homomorphism. Since $\theta\tilde{f}'_\bullet - \tilde{f}_\bullet\theta$ lifts the zero map from A to A, it must be A^e-chain homotopic to 0. By our arguments in the first paragraph above, $[f, g\theta] = fg\theta - g\theta\tilde{f}'_n$ represents the Gerstenhaber bracket of f and $g\theta$

at the chain level on B. Using the notation $\sim$ to indicate that cocycles are cohomologous, on P we have

$$\begin{aligned}[f, g\theta]\iota &\sim fg\theta\iota - g\theta\tilde{f}'_n\iota \\ &\sim fg\theta\iota - g\tilde{f}_n\theta\iota \\ &\sim fg - g\tilde{f}_n,\end{aligned}$$

since $\theta\iota$ is chain homotopic to the identity map and $gd = 0$. □

Example 5.2.6. Let $A = k[x, y]$. We will illustrate the derivation operator method by finding a general formula for the Gerstenhaber bracket of a 1-cocycle with a 2-cocycle on the Koszul resolution P. Other brackets may be found similarly. Let $f = x^i y^j \frac{\partial}{\partial x}$, a derivation on A. Let $g = qx^* \wedge y^*$ for some $q \in A$. We first find functions $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2$ as in Lemma 5.2.1 (whose proof is constructive):

$$\begin{aligned}\tilde{f}_0(a \otimes b) &= f(a) \otimes b + a \otimes f(b), \\ \tilde{f}_1(a \otimes x \otimes b) &= f(a) \otimes x \otimes b + \sum_{l=1}^{j} ax^i y^{j-l} \otimes y \otimes y^{l-1} b \\ &\quad + \sum_{l=1}^{i} ax^{i-l} \otimes x \otimes x^{l-1} y^j b + a \otimes x \otimes f(b), \\ \tilde{f}_1(a \otimes y \otimes b) &= f(a) \otimes y \otimes b + a \otimes y \otimes f(b), \\ \tilde{f}_2(a \otimes x \wedge y \otimes b) &= f(a) \otimes x \wedge y \otimes b + \sum_{l=1}^{i} ax^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j b \\ &\quad + a \otimes x \wedge y \otimes f(b),\end{aligned}$$

for all $a, b \in A$. By Theorem 5.2.4, setting $p = x^i y^j$,

$$\begin{aligned}[f, g](x \wedge y) &= (fg - g\tilde{f}_2)(x \wedge y) \\ &= f(q) - g\left(\sum_{l=1}^{i} x^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j\right) \\ &= p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p).\end{aligned}$$

So $[f, g] = (p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p))x^* \wedge y^*$.

5.3. Homotopy liftings

In this section we present Volkov's approach to brackets on Hochschild cohomology expressed directly on an arbitrary resolution. More details and applications may be found in [**Vol**].

Let $P \xrightarrow{\mu_P} A$ be a projective resolution of A as an A^e-module. We work with the Hom complex $\mathrm{Hom}_{A^e}(P, P)$ in which the differential $\mathbf{d}$ is given by

$$\mathbf{d}(f) = df - (-1)^m fd$$

for all A^e-maps $f : P \to P[-m]$. (See Section A.5.) The Hom complex is quasi-isomorphic to $\mathrm{Hom}_{A^e}(P, A)$ via the augmentation μ_P. We use the notation $\sim$ in this section to indicate that two functions are cohomologous in the Hom complex, or equivalently in $\mathrm{Hom}_{A^e}(P, A)$ on application of μ_P.

Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles, that is, $fd = 0$ and $gd = 0$. We wish to express the Gerstenhaber bracket $[f, g]$ as something analogous to a graded commutator of function compositions,

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f, \tag{5.3.1}$$

for functions $\psi_f : P \to P[1 - m]$ and $\psi_g : P \to P[1 - n]$ determined by f and g in some way. (We will be able to view these functions as analogs of Stasheff's coderivations, but defined on an arbitrary resolution. We take Volkov's different route of development, as Stasheff's theory does not appear to generalize directly.) We caution that we have chosen slightly different notation from that of Volkov [**Vol**]. Our functions will differ from his by signs: Our ψ_f will be $\pm\phi_f$ in [**Vol**].

We will impose a condition on the functions ψ_f, ψ_g inspired by a property of the circle product stated in Lemma 1.4.3(i). For an an m-cocycle f and an n-cocycle g, this is:

$$(-1)^m \delta(g \circ f) = (-1)^{mn} f \smile g - g \smile f. \tag{5.3.2}$$

Let $\Delta_P : P \to P \otimes_A P$ be a diagonal map, that is an A^e-module chain map lifting the identity map on A. The cup product of f and g may be represented by $(f \otimes g)\Delta_P$. We would like ψ_f to satisfy the following equation analogous to (5.3.2) for all cocycles g in $\mathrm{Hom}_{A^e}(P_n, A)$:

$$(-1)^m g\psi_f d = ((-1)^{mn} f \otimes g - g \otimes f)\Delta_P. \tag{5.3.3}$$

We will derive from our imposed conditions (5.3.1) and (5.3.3) some further conditions on the functions ψ_f, leading to Definition 5.3.6 below of homotopy lifting. We will show in Theorem 5.3.10 that the conditions are sufficient to define the Gerstenhaber bracket as (5.3.1).

We consider the second imposed condition (5.3.3) first. Fixing f, since $gd = 0$ and $|\psi_f| = m - 1$, the condition is

$$g\mathbf{d}(\psi_f) = (-1)^m g\psi_f d = ((-1)^{mn} f \otimes g - g \otimes f)\Delta_P = g(f \otimes 1_P - 1_P \otimes f)\Delta_P$$

for all $n \geq 0$ and all n-cocycles g. This will hold if

$$\mathbf{d}(\psi_f) = (f \otimes 1_P - 1_P \otimes f)\Delta_P. \tag{5.3.4}$$

We consider the first imposed condition (5.3.1) in the case that g is the 0-cocycle μ_P. We may rewrite condition (5.3.1) in case $g = \mu_P$ as follows. Let B denote the bar resolution on A, and let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps. Then $f\theta$ is a cocycle on B, and so is cohomologous to a cocycle taking the value 0 whenever one of the tensor factors in an argument is in the field k. Thus the Gerstenhaber bracket $[f\theta, \mu_B]$ becomes 0 in cohomology. Using the historical definition of Gerstenhaber bracket and the comparison maps ι, θ to translate to cocycles on P, the Gerstenhaber bracket of f and μ_P is

$$[f, \mu_P] = [f\theta, \mu_P\theta]\iota = [f\theta, \mu_B]\iota \sim 0.$$

So, if $[f, \mu_P]$ may be expressed according with equation (5.3.1), then setting $\psi = \psi_{\mu_P}$, we have

$$f\psi + (-1)^m \mu_P \psi_f \sim 0. \tag{5.3.5}$$

Note that by its definition, $f\psi + (-1)^m\mu_P\psi_f$ is a function on P_{m-1}, so condition (5.3.5) is simply requiring ψ_f to take values in P_0 consistent with those of $f\psi$.

In fact these two conditions (5.3.4) and (5.3.5) are sufficient to define the bracket via formula (5.3.1), as we will see in Theorem 5.3.10. Next we will give a name to functions ψ_f having these properties, as in [**Vol**].

Definition 5.3.6. Let P be a projective resolution of A as an A^e-module, and let $f \in \operatorname{Hom}_{A^e}(P_m, A)$ be a cocycle. An A^e-module homomorphism $\psi_f : P \to P[1-m]$ is a *homotopy lifting of f with respect to* Δ_P if conditions (5.3.4) and (5.3.5) hold for some $\psi : P \to P[1]$ for which $\mathbf{d}(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$.

We will often use the term *homotopy lifting of f* without explicit reference to Δ_P if it is clear from context which map Δ_P is intended, or in situations where the choice of Δ_P does not matter. We caution again that our homotopy lifting differs from that of Volkov [**Vol**] by signs.

It may be checked directly that if ψ_f, ψ_g are homotopy liftings for cocycles f, g with respect to Δ_P, then $[f, g]$ as defined in (5.3.1) is a cocycle. It may also be checked that if either f or g is a coboundary, then so is $[f, g]$ as defined in (5.3.1): If $f = hd$ for some cochain h, set $\psi_f = (-1)^m(h \otimes 1_P - 1_P \otimes h)\Delta_P$. A calculation shows that ψ_f is a homotopy lifting for f. With this choice, $f\psi_g \sim (-1)^{(m-1)(n-1)} g\psi_f$.

Example 5.3.7. Let $P = B$, the bar resolution of A, and let Δ_B be the standard diagonal map on B, defined by extending the map Δ_T of Section 5.1 to an A^e-module homomorphism, noting that $B = A \otimes T(A) \otimes A$. Then $(\mu_B \otimes 1_B - 1_B \otimes \mu_B)\Delta_B = 0$, and we may take $\psi = 0$ in Definition 5.3.6. Let $f \in \operatorname{Hom}_{A^e}(B_m, A)$ be a cocycle. We may assume without loss of generality that $f(a_0 \otimes \cdots \otimes a_{m+1})$ is 0 whenever at least one of $a_1, \ldots, a_m$ is in the field k, since f is cohomologous to such a function. Let

$$\begin{aligned}
&\psi_f(a_0 \otimes \cdots \otimes a_{l+1}) \\
&= \sum_{i=1}^{l-m+1} (-1)^u a_0 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{l+1},
\end{aligned}$$

where $u = (m-1)(i-1)$, for all $l \geq m$ and $a_0, \ldots, a_{l+1} \in A$, and take ψ_f to be the zero map on B_l for $l \leq m-1$. Then ψ_f is a homotopy lifting of f with respect to Δ_B. A calculation shows that with this choice of ψ_f and a similar choice of ψ_g, the bracket $[f, g]$ as given by formula (5.3.1) is precisely the Gerstenhaber bracket as defined on the bar resolution.

We may view ψ_f defined by the above formula as a coderivation on B, or restrict to $T(A) \cong k \otimes T(A) \otimes k \hookrightarrow A \otimes T(A) \otimes A = B$ to obtain a coderivation $\psi_f|_{T(A)}$ on $T(A)$ as in Definition 5.1.2; see also formula (5.1.3). If f is a 1-cocycle, then $\psi_f|_{T(A)}$, viewed another way, may be extended to an f^e-operator in the sense of Lemma 5.2.1 (see Example 5.2.3). Thus homotopy liftings encompass these two views—coderivations on the tensor coalgebra and derivation operators on the bar resolution—that were introduced in the previous two sections.

Lemma 5.3.8. *Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle. Homotopy liftings $\psi_f : P \to P[1-m]$ exist and are unique up to chain homotopy.*

Proof. First we show existence of ψ, a homotopy lifting of μ_P with respect to Δ_P. Consider the function $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$ in the Hom complex $\mathrm{Hom}_{A^e}(P, P)$. Apply the quasi-isomorphism μ_P to $\mathrm{Hom}_{A^e}(P, A)$. Note that $\mu_P(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = \mu_P \otimes \mu_P - \mu_P \otimes \mu_P = 0$, and so under the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$ to $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, the map $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ becomes 0. Since $\mathbf{d}(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0$, it is therefore a boundary in $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$. Precomposing with the chain map Δ_P, we see that $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P = \mathbf{d}(\psi)$ for some $\psi : P \to P[1]$, as claimed, where $P[1]$ denotes a shift in degree (see Section A.1).

Next we show existence of functions ψ_f satisfying the conditions (5.3.4) and (5.3.5). Now $(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a chain map from P to $P[-m]$ since $fd = 0$. Applying μ_P, since $|\mu_P| = 0$, we have

$$\begin{aligned}
\mu_P(f \otimes 1_P - 1_P \otimes f)\Delta_P &= f(1_P \otimes \mu_P - \mu_P \otimes 1_P)\Delta_P \\
&= -f\mathbf{d}(\psi) \quad = \quad -f\psi d,
\end{aligned}$$

that is, applying the quasi-isomorphism μ_P from $\mathrm{Hom}_{A^e}(P, P)$ to $\mathrm{Hom}_{A^e}(P, A)$, we find that $\mu_P(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a coboundary. Consequently, in $\mathrm{Hom}_{A^e}(P, P)$, the function $(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a coboundary, so that

$$(f \otimes 1_P - 1_P \otimes f)\Delta_P = \mathbf{d}(\psi_f)$$

for some ψ_f, that is, condition (5.3.4) holds. We will show that some of the functions ψ_f satisfying (5.3.4) also satisfy condition (5.3.5). As above we now have

$$\mu_P \mathbf{d}(\psi_f) = -f\psi d,$$

and since $\mathbf{d}(\psi_f) = d\psi_f + (-1)^m \psi_f d$ and $\mu_P d = 0$, this is equivalent to

$$((-1)^m \mu_P \psi_f + f\psi)d = 0.$$

However, we want $(-1)^m \mu_P \psi_f + f\psi \sim 0$. Set $g = (-1)^m \mu_P \psi_f + f\psi$, viewed as a map from P_{m-1} to A. We have seen that g is a cocycle, and thus it corresponds to a chain map $g_\bullet$ from P to $P[1-m]$. Define $\psi'_f = \psi_f - (-1)^m g_\bullet$. Since $g_\bullet$ is a chain map, $\mathbf{d}(\psi'_f) = \mathbf{d}(\psi_f)$, and so ψ'_f also satisfies (5.3.4). Additionally we now have

$$(-1)^m \mu_p \psi'_f + f\psi = (-1)^m \mu_P \psi_f + f\psi - g = 0,$$

by definition of g, and so ψ'_f also satisfies (5.3.5). Without loss of generality then, ψ_f takes the correct values in P_0 and so (5.3.5) holds, as well as (5.3.4).

Finally, we show uniqueness up to chain homotopy. Let ψ_f and ψ'_f be two homotopy liftings of f with respect to Δ_P. Then $\mathbf{d}(\psi_f - \psi'_f) = 0$ and $\mu_P(\psi_f - \psi'_f) \sim 0$. Again, μ_P gives rise to the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P, P)$ to $\mathrm{Hom}_{A^e}(P, A)$ and this implies $\psi_f - \psi'_f \sim 0$, as claimed. Note that this argument does not depend on choice of homotopy ψ for $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$, again as any two will be homotopic. □

The following theorem and proof are [**Vol**, Theorem 4].

Theorem 5.3.9. *Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. The element of Hochschild cohomology $\mathrm{HH}^*(A)$ represented by $[f, g]$ as defined by formula (5.3.1) is independent of choice of resolution P, of diagonal map Δ_P, and of homotopy liftings ψ_f and ψ_g.*

Proof. We will prove independence of choices in the reverse order from what is stated. Independence of choice of ψ_f and ψ_g is immediate from the uniqueness of ψ_f and ψ_g up to chain homotopy stated in Lemma 5.3.8, since $fd = 0$ and $gd = 0$.

Let Δ_P and Δ'_P be two diagonal maps. Then $\Delta'_P - \Delta_P = \mathbf{d}(u)$ for some $u : P \to (P \otimes_A P)[1]$. Let ψ_f and ψ_g be homotopy liftings of f and g with respect to Δ_P. Let

$$\psi'_f = \psi_f + (-1)^m (f \otimes 1_P - 1_P \otimes f)u,$$

and similarly ψ'_g. A calculation shows that ψ'_f and ψ'_g are homotopy liftings of f and g with respect to Δ'_P, respectively. We find that

$$\begin{aligned}
&f\psi'_g - (-1)^{(m-1)(n-1)} g\psi'_f \\
&= f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f + (-1)^n f(g \otimes 1_P - 1_P \otimes g)u \\
&\qquad - (-1)^{(m-1)(n-1)}(-1)^m g(f \otimes 1_P - 1_P \otimes f)u \\
&= f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f,
\end{aligned}$$

so these two expressions give the same bracket $[f, g]$ via formula (5.3.1). Thus the formula is independent of choice of diagonal map.

Let $Q \xrightarrow{\mu_Q} A$ be another projective resolution of A as an A^e-module, and let $\Delta_Q : Q \to Q \otimes_A Q$ be a diagonal map. Let $\iota : P \to Q$ and $\theta : Q \to P$ be chain maps lifting the identity map on A. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles on P. Then $f\theta$ and $g\theta$ are cocycles on Q. Let $\psi_{f\theta}$ be a homotopy lifting for $f\theta$ with respect to Δ_Q. Set $\psi_f = \theta\psi_{f\theta}\iota$. We first check that ψ_f is a homotopy lifting for f with respect to $\Delta_P = (\theta \otimes \theta)\Delta_Q\iota$:

$$\begin{aligned}
\mathbf{d}(\psi_f) &= \theta\mathbf{d}(\psi_{f\theta})\iota \\
&= \theta(f\theta \otimes 1_Q - 1_Q \otimes f\theta)\Delta_Q\iota \\
&= (f \otimes 1_P - 1_P \otimes f)(\theta \otimes \theta)\Delta_Q\iota \\
&= (f \otimes 1_P - 1_P \otimes f)\Delta_P,
\end{aligned}$$

so ψ_f satisfies (5.3.4).

Set $\psi_P = \theta\psi_Q\iota$ where ψ_Q satisfies $\mathbf{d}(\psi_Q) = (\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q$ as well as $(-1)^m\mu_Q\psi_{f\theta} + f\theta\psi_Q \sim 0$. We may check then that

$$\begin{aligned}
\mathbf{d}(\psi_P) = \theta\mathbf{d}(\psi_Q)\iota &= \theta(\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q\iota \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)(\theta \otimes \theta)\Delta_Q\iota \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P,
\end{aligned}$$

since $\theta\mu_Q = \mu_P\theta$. Then, as $(-1)^m\mu_Q\psi_{f\theta} + f\theta\psi_Q \sim 0$, we have by the definitions of ψ_f and of ψ_P above,

$$\begin{aligned}
(-1)^m\mu_P\psi_f + f\psi_P &= (-1)^m\mu_Q\psi_{f\theta}\iota + f\theta\psi_Q\iota \\
&= ((-1)^m\mu_Q\psi_{f\theta} + f\theta\psi_Q)\iota \quad \sim \quad 0,
\end{aligned}$$

that is, ψ_f satisfies (5.3.5). Therefore ψ_f is a homotopy lifting of f with respect to Δ_P, and we may similarly define a homotopy lifting of g.

Finally, formula (5.3.1) applied to f, g on P yields

$$\begin{aligned}
[f, g] &= f\theta\psi_{g\theta}\iota - (-1)^{(m-1)(n-1)} g\theta\psi_{f\theta}\iota \\
&= [f\theta, g\theta]\iota,
\end{aligned}$$

so the chain map ι takes $[f\theta, g\theta]$ to $[f, g]$. Thus the bracket does not depend on choice of resolution. □

Theorem 5.3.10. *Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles on P, and let ψ_f and ψ_g be homotopy liftings of f and g, as in Definition 5.3.6. The bracket given by*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f$$

at the chain level induces the Gerstenhaber bracket on Hochschild cohomology $\mathrm{HH}^\bullet(A)$.

Proof. In Example 5.3.7, we saw that standard choices produce the Gerstenhaber bracket from formula (5.3.1). By Theorem 5.3.9, it is independent of choices. □

Remark 5.3.11. In practice, often $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$, and one can take $\psi = 0$ in Definition 5.3.6 of homotopy lifting. See [**Vol**, Remark 1] and the next section.

5.4. Computational techniques

Some of the results of the previous sections lead to effective computational techniques for the Lie structure on Hochschild cohomology. In particular, we explain here some settings in which the theory of homotopy liftings can be simplified for computational purposes.

Let $\phi_P : P \otimes_A P \to P[1]$ be a homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. That is,

$$\mathbf{d}(\phi_P) = \mu_P \otimes 1_P - 1_P \otimes \mu_P. \tag{5.4.1}$$

(To see that such a homotopy exists, consider the quasi-isomorphism μ_P from $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$ to $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, which takes $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ to 0; since μ_P is a chain map, $\mathbf{d}(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0$.) Let

$$\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P.$$

Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ with $fd = 0$, and let $\psi_f : P \to P[1-m]$ be defined by

$$\psi_f = \phi_P(1_P \otimes f \otimes 1_P)\Delta_P^{(2)}. \tag{5.4.2}$$

Under some conditions, ψ_f is a homotopy lifting of f with respect to Δ_P, as we see next.

Example 5.4.3. Consider the bar resolution B of A. We may identify $B_i \otimes_A B_j$ with $A \otimes A^{\otimes i} \otimes A \otimes A^{\otimes j} \otimes A$ and define ϕ_B by

$$\begin{aligned}&\phi_B(a_0 \otimes (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1}) \otimes (a_{i+2} \otimes \cdots \otimes a_{i+j+1}) \otimes a_{i+j+2})\\ &= (-1)^i a_0 \otimes \cdots \otimes a_{i+j+2}\end{aligned}$$

for all $a_0, \dots, a_{i+j+2} \in A$. Then ψ_f defined as in (5.4.2) agrees with that given in Example 5.3.7.

Our main examples in the rest of this section will be Koszul algebras A. For these algebras, a result of Buchweitz, Green, Snashall and Solberg [**BGSS08**] guarantees that the standard embedding $\iota : P \to B$ of the Koszul resolution P into the bar resolution B of A preserves the diagonal map in the sense that Δ_B takes $\iota(P)$ to $\iota(P) \otimes_A \iota(P)$, and so we may define the diagonal map Δ_P on P via this embedding. It follows that Δ_P is coassociative and that $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$. In Definition 5.3.6 of homotopy lifting of an m-cocycle f on P, we may thus take $\psi = 0$, and so condition (5.3.5) becomes $\mu_P \psi_f \sim 0$. We may in fact assume that $\psi_f|_{P_{m-1}} = 0$. This simplifies the work of finding homotopy liftings (and it simplifies many of the proofs of the previous section under these hypotheses). In fact formula (5.4.2) always gives a homotopy lifting in the case that P has a coalgebra structure, as we see next.

Lemma 5.4.4. *Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A for which there is a diagonal map $\Delta_P : P \to P \otimes_A P$ such that*

$$(\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$$

and $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$. (That is, Δ_P satisfies the coassociative and counit properties.) Let $\phi_P : P \otimes_A P \to P[1]$ be a homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$, that is, equation (5.4.1) holds. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ with $fd = 0$, and let $\psi_f : P \to P[1-m]$ be defined by formula (5.4.2). Then ψ_f is a homotopy lifting of f.

Proof. Set $\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$. Since $\Delta_P^{(2)}$ and $1_P \otimes f \otimes 1_P$ are chain maps,

$$\begin{aligned}
\mathbf{d}(\psi_f) &= \mathbf{d}(\phi_P)(1_P \otimes f \otimes 1_P)\Delta_P^{(2)} \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)(1_P \otimes f \otimes 1_P)\Delta_P^{(2)} \\
&= (\mu_P \otimes f \otimes 1_P - 1_P \otimes f \otimes \mu_P)\Delta_P^{(2)} \\
&= ((f \otimes 1_P)(\mu_P \otimes 1_P \otimes 1_P) - (1_P \otimes f)(1_P \otimes 1_P \otimes \mu_P))\Delta_P^{(2)} \\
&= (f \otimes 1_P - 1_P \otimes f)\Delta_P.
\end{aligned}$$

Note that $\psi_f|_{P_{m-1}} = 0$ by definition, and as explained above, we may take $\psi = 0$ in Definition 5.3.6. Therefore ψ_f is a homotopy lifting of f. □

Compare the following theorem to [**NW16**, Theorem 3.2.5], which has stronger hypotheses, and to [**NW16**, Lemma 3.4.1], which has somewhat different hypotheses.

Theorem 5.4.5. *Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A for which there is a diagonal map $\Delta_P : P \to P \otimes_A P$ satisfying the coassociative and counit properties. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. Define ψ_f by formula (5.4.2), and similarly ψ_g. Then*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f$$

induces the Gerstenhaber bracket on Hochschild cohomology.

Proof. This follows immediately from Lemma 5.4.4 and Theorem 5.3.10. □

Remark 5.4.6. More generally, a homotopy ϕ_P for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ may be used to define the Gerstenhaber bracket in a similar way, with the addition of some error terms. See [**Vol**, Corollary 5 and Remark 1] in which the Gerstenhaber bracket is given generally as

(5.4.7)

$$\begin{aligned}[f, g] = &-f\phi_P(g \otimes 1_P \otimes 1_P - 1_P \otimes g \otimes 1_P + 1_P \otimes g \otimes 1_P)\Delta_P^{(2)} \\ &+ (-1)^{(m-1)(n-1)} g\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}.\end{aligned}$$

We caution that the function

$$-\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}$$

is not necessarily a homotopy lifting of f; the formula (5.4.7) instead results from a more complicated homotopy lifting as explained in the proof of [**Vol**, Corollary 5].

In the remainder of this section, we apply Theorem 5.4.5 to an example, a polynomial ring in two indeterminates. The case of n indeterminates is similar, if more notationally unwieldy, and is handled in [**NW16**, Section 4], showing that formula (5.3.1) indeed yields the familiar Gerstenhaber bracket on the Hochschild cohomology of a polynomial ring. In other settings the first computation of Gerstenhaber brackets, or of a related Batalin-Vilkovisky structure, used these techniques (see, for example, [**Gri, GNW, NW, Vol**]).

Example 5.4.8. Let $A = k[x, y]$ and let P be its Koszul resolution. Setting $V = \mathrm{Span}_k\{x, y\}$, we may write $P = A \otimes \textstyle\bigwedge^{\bullet} V \otimes A$. Identify $P_i \otimes_A P_j$ with $A \otimes \textstyle\bigwedge^i V \otimes A \otimes \textstyle\bigwedge^j V \otimes A$ for each i, j, and identify $\textstyle\bigwedge^0 V$ with k and $\textstyle\bigwedge^1 V$ with V. We first find a homotopy $\phi_P : P \otimes_A P \to P[1]$ for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. In degree 2, the map ϕ_P is necessarily 0 since $P_3 = 0$. We define ϕ_P in degrees 0 and 1 on free basis elements:

$$\begin{aligned}
\phi_P(1\otimes x^iy^j\otimes 1) &= \sum_{l=1}^{j} x^iy^{j-l}\otimes y\otimes y^{l-1} + \sum_{l=1}^{i} x^{i-l}\otimes x\otimes x^{l-1}y^j,\\
\phi_P(1\otimes x^iy^j\otimes x\otimes 1) &= -\sum_{l=1}^{j} x^iy^{j-l}\otimes x\wedge y\otimes y^{l-1},\\
\phi_P(1\otimes x^iy^j\otimes y\otimes 1) &= 0,\\
\phi_P(1\otimes x\otimes x^iy^j\otimes 1) &= 0,\\
\phi_P(1\otimes y\otimes x^iy^j\otimes 1) &= \sum_{l=1}^{i} x^{i-l}\otimes x\wedge y\otimes x^{l-1}y^j.
\end{aligned}$$

We use this function ϕ_P, formula (5.4.2) for ψ_f, and the formula of Theorem 5.4.5 to compute some brackets in degree 1. The diagonal map Δ_P is defined by the standard embedding of P into the bar resolution, followed by the standard diagonal map there. First we find some values of $\psi_{x^iy^jx^*}$ and $\psi_{x^iy^jy^*}$:

$$\begin{aligned}
\psi_{x^iy^jx^*}(1\otimes x\otimes 1) &= \phi_P(x^iy^j), &\qquad \psi_{x^iy^jx^*}(1\otimes y\otimes 1) &= 0,\\
\psi_{x^iy^jy^*}(1\otimes x\otimes 1) &= 0, &\qquad \psi_{x^iy^jy^*}(1\otimes y\otimes 1) &= \phi_P(x^iy^j).
\end{aligned}$$

It follows that, for example,

$$\begin{aligned}
&[x^iy^jx^*,\ x^my^nx^*](1\otimes x\otimes 1)\\
&= x^iy^jx^*\psi_{x^my^nx^*}(1\otimes x\otimes 1) - x^my^nx^*\psi_{x^iy^jx^*}(1\otimes x\otimes 1)\\
&= x^iy^jx^*\phi_P(x^my^n) - x^my^nx^*\phi_P(x^iy^j)\\
&= \sum_{l=1}^{m} x^iy^jx^{m-l}x^{l-1}y^n - \sum_{l=1}^{i} x^my^nx^{i-l}x^{l-1}y^j\\
&= mx^iy^jx^{m-1}y^n - ix^my^nx^{i-1}y^j\\
&= x^iy^j\frac{\partial}{\partial x}(x^my^n) - x^my^n\frac{\partial}{\partial x}(x^iy^j).
\end{aligned}$$

Another calculation shows that the value of this bracket function on $1\otimes y\otimes 1$ is zero. Therefore, for all $p, q\in A$, we have

$$[px^*,\ qx^*] = (p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p))x^*.$$

Similarly we find that

$$\begin{aligned}
[px^*,\ qy^*] &= p\frac{\partial}{\partial x}(q)y^* - q\frac{\partial}{\partial y}(p)x^*,\\
[py^*,\ qy^*] &= (p\frac{\partial}{\partial y}(q) - q\frac{\partial}{\partial y}(p))y^*.
\end{aligned}$$

We may calculate other brackets using the same techniques.

5.5. Extensions

In this section, we consider Schwede's exact sequence interpretation of the Lie structure on Hochschild cohomology [**Sch98**]. Hermann [**Her16c**] generalized Schwede's construction to some exact monoidal categories, and gave a description of the bracket with degree 0 elements in [**Her16a**], completing Schwede's interpretation. We refer to these papers for most of the technical details and proofs, instead giving just a skimming here.

Let $n \geq 1$ and let $\mathcal{E}xt^n_{A^e}(A, A)$ denote the category whose objects are n-extensions of A by A as an A^e-module, and morphisms are maps of n-extensions. View $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ as equivalence classes of objects in $\mathcal{E}xt^n_{A^e}(A, A)$. We adapt notation from [**Sch98**]: Consider an m-extension and an n-extension,

$$M: \qquad 0 \longrightarrow A \xrightarrow{i_M} M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{\mu_M} A \longrightarrow 0,$$

$$N: \qquad 0 \longrightarrow A \xrightarrow{i_N} N_{n-1} \longrightarrow \cdots \longrightarrow N_0 \xrightarrow{\mu_N} A \longrightarrow 0.$$

We will assume that all M_i, N_i are projective as left A-modules, and as right A-modules, where needed. See [**Sch98**] for a discussion about such an assumption.

Let P be a projective resolution of A as an A^e-module. Let f and g be an m-cocycle and an n-cocycle on P, corresponding to M and N, respectively. So $f \in \mathrm{Hom}_{A^e}(P_m, A)$ may be defined via the following commuting diagram. We denote by $\hat{f}_{\bullet} : P_{\bullet} \to M_{\bullet}$ the chain map indicated below, so that $f = \hat{f}_m$.

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P_m & \xrightarrow{d_m} & P_{m-1} & \xrightarrow{d_{m-1}} & \cdots & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu_P} & A & \longrightarrow 0 \\
& \downarrow f & & \downarrow \hat{f}_{m-1} & & & & \downarrow \hat{f}_0 & & \downarrow = & \\
0 \longrightarrow & A & \xrightarrow{i_M} & M_{m-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \xrightarrow{\mu_M} & A & \longrightarrow 0
\end{array}$$

We will later replace M by $K(f)$, an m-extension defined via P and a pushout diagram.

We write $N\#M$ for the following Yoneda splice:

$$N\#M: \quad 0 \longrightarrow A \xrightarrow{i_M} M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{i_N\mu_M} N_{n-1} \longrightarrow \cdots \longrightarrow N_0 \xrightarrow{\mu_N} A \longrightarrow 0$$

Note that an element of $\mathrm{Hom}_{A^e}(P_{m+n}, A)$ corresponding to this $(m+n)$-extension is $f \smile g$ defined via a diagonal map $\Delta_P : P \to P \otimes_A P$. Accordingly, by the notation $f \smile g$ in this section we mean $\pi(f \otimes g)\Delta_P$, that is, (1.3.9) for a fixed diagonal map Δ_P.

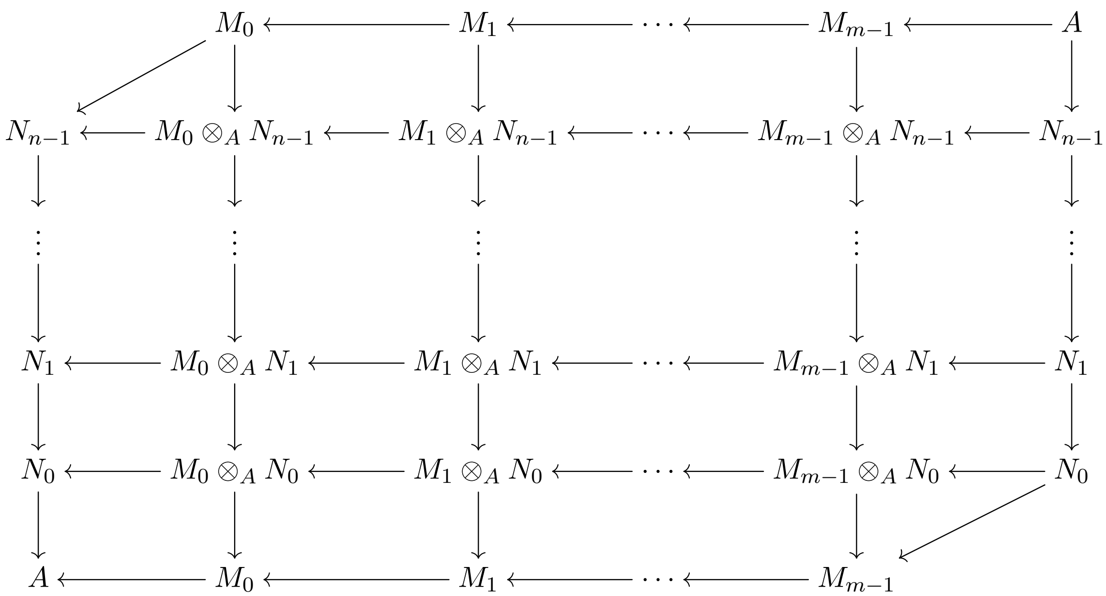
We write $M \otimes_A N$ for the $(m+n)$-extension corresponding to the total complex of the tensor product of the two truncated sequences. In degree $m+n-1$, for example, we have the module $(M_m \otimes_A N_{n-1}) \oplus (M_{m-1} \otimes_A N_n) \cong N_{n-1} \oplus M_{m-1}$, and the extension is

$$M \otimes_A N: \qquad 0 \longrightarrow A \longrightarrow M_{m-1} \oplus N_{n-1} \longrightarrow \cdots \longrightarrow M_0 \otimes_A N_0 \longrightarrow A \longrightarrow 0.$$

The cup product of f and g corresponds to any of the $(m+n)$-extensions $N\#M$, $M \otimes_A N$, $(-1)^{mn}M\#N$, $(-1)^{mn}N \otimes_A M$. (Here we take as additive inverse $-M\#N$ to the extension $M\#N$ the extension whose modules agree with those of $M\#N$, where map μ_M is replaced by $-\mu_M$, and all other maps agree with those of $M\#N$.) These extensions are all equivalent and there are maps as indicated in the following diagram:

$$\begin{array}{ccccc} & \swarrow^{\lambda_{M,N}} & M \otimes_A N & \searrow^{\rho_{M,N}} & \\ N\#M & & & & (-1)^{mn}M\#N \\ & \nwarrow_{\rho_{N,M}} & (-1)^{mn}N \otimes_A M & \nearrow_{\lambda_{N,M}} & \end{array} \tag{5.5.1}$$

Such maps may be described as follows. Consider the augmented double complex:



All but the leftmost column and bottom row constitute the double complex corresponding to $M \otimes_A N$, and the outermost rows and columns are $N \# M$ (left column and top row) and $(-1)^{mn} M \# N$ (right column and bottom row). The maps $\lambda_{M,N} : M \otimes_A N \to N \# M$ and $\rho_{M,N} : M \otimes_A N \to (-1)^{mn} M \# N$, are given as follows: For $n \le i \le m+n$, $\lambda_{M,N}$ projects $(M \otimes_A N)_i$ onto $M_{i-n} \otimes_A N_n \cong M_{i-n}$. For $0 \le i \le n-1$, $\lambda_{M,N}$ first projects $(M \otimes_A N)_i$ onto $M_0 \otimes_A N_i$, then maps to $A \otimes_A N_i \cong N_i$ via $\mu_N \otimes 1$. For $m \le i \le m+n$, $\rho_{M,N}$ projects $(M \otimes_A N)_i$ onto $(-1)^{m(m+n-i)} M_m \otimes_A N_{i-m} \cong N_{i-m}$. For $0 \le i \le m-1$, $\rho_{M,N}$ projects $(M \otimes_A N)_i$ onto $(-1)^{mn} M_i \otimes_A N_0$, then maps to $M_i \otimes_A A \cong M_i$ via $1 \otimes \mu_N$. Similarly there are maps $\rho_{N,M}$: $(-1)^{mn} N \otimes_A M \to N \# M$ and $\lambda_{N,M} : (-1)^{mn} N \otimes_A M \to (-1)^{mn} M \# N$.

Diagram (5.5.1) represents a loop in the extension category $\mathcal{E}xt^{m+n}_{A^e}(A, A)$. By a result of Retakh [**Ret86**], such loops are in one-to-one correspondence with $\mathrm{Ext}^{m+n-1}_{A^e}(A, A)$. By a result of Schwede [**Sch98**], under Retakh's correspondence, the loop (5.5.1) corresponds to the Gerstenhaber bracket $[f, g]$. We refer to Retakh [**Ret86**] and Schwede [**Sch98**] for most details and proofs. Here we give some of the algebraic ideas underlying Schwede's result. Specifically, we look closely at some maps $P \to N \# M$ arising from the maps comprising the loop (5.5.1).

Schwede replaced loop (5.5.1) with another equivalent loop as follows. Let $K(f \smile g)$ denote the $(m+n)$-extension of A by A given by a pushout diagram:

$K(f \smile g)$:

$$0 \longrightarrow A \longrightarrow K(f \smile g)_{m+n-1} \longrightarrow P_{m+n-2} \longrightarrow \cdots \longrightarrow$$

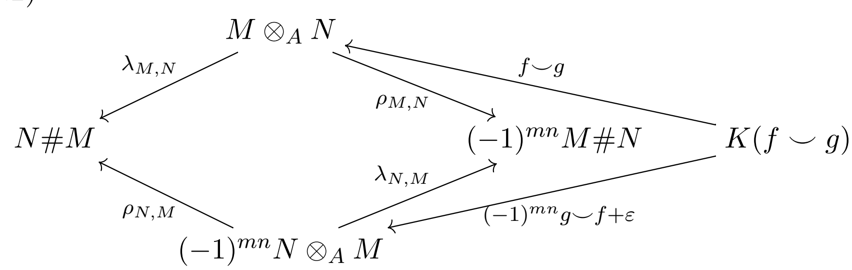
$$P_0 \longrightarrow A \longrightarrow 0 \qquad ,$$

with

$$K(f \smile g)_{m+n-1} = (P_{m+n-1} \oplus A)/\{(-d(x),\ (f \smile g)(x)) \mid x \in P_{m+n}\}.$$

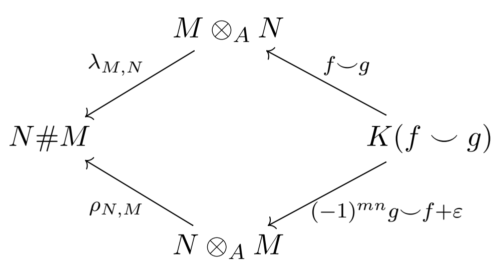
(In his proof, Schwede took P to be the bar resolution, since his goal was to show that the loop (5.5.1) corresponds to the historical definition of Gerstenhaber bracket on the bar resolution.) Schwede defined explicitly a map ε in such a way that the rightmost quadrilateral in the following diagram commutes:

(5.5.2)



Schwede deleted the component $(-1)^{mn}M\#N$ from diagram (5.5.2), thus replacing loop (5.5.1) with loop (5.5.3) below. For this purpose, commutativity of the rightmost quadrilateral in diagram (5.5.2) was needed.

(5.5.3)



Schwede proved that there is a chain homotopy $s : P \to N\#M$, factoring through $K(f \smile g)$, between $\lambda_{M,N}(f \smile g)$ and $\rho_{N,M}((-1)^{mn}g \smile f + \varepsilon)$, for which

$$s_{m+n-1} = (-1)^{n-1}[f, g].$$

Chapter 6

Infinity Algebras

There are several appearances in Hochschild cohomology of higher order operations, the original idea of which is due to Stasheff [**Sta63**]. Some of these operations extend those on underlying chain complexes, such as the cup product operation. In this chapter, we look at a few settings where such infinity algebras arise in relation to Hochschild cohomology. There are many more applications in the literature than those we present here.

Indexing and sign conventions vary somewhat in the literature; we make some of the more standard choices.

6.1. A_∞-algebras

In this section we define A_∞-algebras (also called strongly homotopy associative algebras) and their morphisms, and give some examples relevant to Hochschild cohomology.

Definition 6.1.1. An A_∞-*algebra* is a graded vector space $A = \oplus_{i\in\mathbb{Z}} A_i$ together with graded linear maps

$$m_n : A^{\otimes n} \to A$$

of degree $|m_n| = 2 - n$ for all $n \geq 1$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0. \tag{6.1.2}$$

The equations (6.1.2) are called *Stasheff identities*. Sometimes we denote the A_∞-algebra by $(A, m_\bullet)$ to emphasize the notation chosen for these higher multiplication maps. If $m_1 = 0$, then A is called *minimal*.

We consider the implications of equation (6.1.2) for small values of n: If $n = 1$, we must take $s = 1$ and $r = t = 0$, and so the equation is

$$m_1^2 = 0,$$

that is, m_1 is a differential on A. We will thus sometimes write $d = m_1$. If $n = 2$, we may take $s = 2$ and $r = t = 0$, or $s = 1$ and $r + t = 1$, to obtain

$$m_1 m_2 - m_2(m_1 \otimes 1) - m_2(1 \otimes m_1) = 0.$$

To express this equation on elements of A, we may write $m_1(a) = d(a)$ and $m_2(a \otimes b) = a \cdot b$, and the equation becomes

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$

for all homogeneous $a, b \in A$, due to the Koszul sign convention. That is, $m_1 = d$ is a (graded) derivation with respect to m_2. If $n = 3$, equation (6.1.2) becomes

$$\begin{aligned} &m_1 m_3 + m_2(m_2 \otimes 1) - m_2(1 \otimes m_2) \\ &+ m_3(m_1 \otimes 1^{\otimes 2}) + m_3(1 \otimes m_1 \otimes 1) + m_3(1^{\otimes 2} \otimes m_1) = 0, \end{aligned}$$

which may be rewritten

$$\delta(m_3) = m_2(1 \otimes m_2) - m_2(m_2 \otimes 1),$$

where δ is the differential induced by $m_1 = d$ on the complex $\mathrm{Hom}_k(A^{\otimes 3}, A)$. That is, m_2 is associative up to a coboundary in this Hom complex. It follows that the cohomology of (A, m_1) is a graded associative algebra with multiplication induced by m_2.

Example 6.1.3. If A is a differential graded algebra, we may take m_1 to be its differential, m_2 its multiplication, and $m_n = 0$ for $n \geq 3$, to define an A_∞-algebra structure on A. In particular, an associative algebra may be viewed as a differential graded algebra with zero differential, and thus as an A_∞-algebra in this way. If an A_∞-algebra A is concentrated in degree 0, that is $A_i = 0$ for all $i \neq 0$, the maps m_n are necessarily zero maps for all $n \neq 2$ since $|m_n| = 2 - n$, so A is simply an associative algebra.

Example 6.1.4. (This example is due to Penkava and Schwarz [**PS95**].) Let B be an associative algebra, let $A = B[x]/(x^2)$, and let n be a positive integer. We take A to be graded with $|b| = 0$ for all $b \in B$ and $|x| = 2 - n$. Let $g : B^{\otimes n} \to B$ be a Hochschild n-cocycle. Define a linear function $xg : A^{\otimes n} \to A$ by

$$(xg)(a_1 \otimes \cdots \otimes a_n) = \begin{cases} xg(a_1 \otimes \cdots \otimes a_n), & \text{if } a_1, \ldots, a_n \in B \\ 0, & \text{if } x \text{ is a factor of } a_1 \cdots a_n. \end{cases}$$

Let $\pi : A^{\otimes 2} \to A$ denote multiplication on A. If $n = 2$, let $m_2 = \pi + xg$ and $m_i = 0$ for all $i \neq 2$. If $n \neq 2$, let $m_2 = \pi$, $m_n = xg$, and $m_i = 0$ for all $i \notin \{2, n\}$. Calculations show that A is an A_∞-algebra. We will see a connection to algebraic deformation theory via Definition 6.1.7 below.

Example 6.1.5. Let n be a positive integer, $n > 2$, let $B = k[x]/(x^n)$, and let $A = \mathrm{Ext}^*_B(k, k)$. As shown in Example 1.6.7, $A \cong k[y, z]/(y^2)$ with $|y| = 1$ and $|z| = 2$. We take m_2 to be multiplication on A, m_i to be the zero map if $i \notin \{2, n\}$, and

$$m_n(y^{i_1} z^{j_1} \otimes \cdots \otimes y^{i_n} z^{j_n}) = \begin{cases} z^{j_1 + \cdots + j_n + 1}, & \text{if } i_1 = \cdots = i_n = 1 \\ 0, & \text{otherwise} \end{cases}$$

for all nonnegative integers $j_1, \ldots, j_n$ and all $i_1, \ldots, i_n \in \{0, 1\}$. Calculations show that A is an A_∞-algebra. This example may be constructed via the general method outlined in the proof of Theorem 6.2.2 below. In Section 6.3 we will discuss a distinction between this A_∞-structure on $\mathrm{Ext}^*_B(k, k)$ and the structure of $\mathrm{Ext}^*_B(k, k)$ when $B = k[x]/(x^2)$, which is a Koszul algebra.

We say that an A_∞-algebra A is *generated* by a subset S if A coincides with its smallest subspace containing S that is closed under all m_n.

Example 6.1.6. In Example 6.1.5, $m_n(y \otimes \cdots \otimes y) = z$. Thus A is generated by y as an A_∞-algebra.

Definition 6.1.7. Let n be a positive integer. Let B be an associative algebra, and let $A = B[x]/(x^2)$ where $|b| = 0$ for all $b \in B$ and $|x| = 2 - n$. An *infinitesimal n-deformation* of B is a $k[x]/(x^2)$-multilinear A_∞-algebra structure on A that lifts the multiplication of B. That is, under composition with the quotient map from A to $A/(x) \cong B$, $m_2|_B$ becomes multiplication on B and $m_i|_B$ becomes 0 for all $i > 2$.

One checks that an infinitesimal 2-deformation may be identified with an infinitesimal deformation as in Definition 4.2.1: Writing

$$m_2(b_1 \otimes b_2) = m_2'(b_1 \otimes b_2) + m_2''(b_1 \otimes b_2)x$$

for all $b_1, b_2 \in B$, it follows from the definitions that m_2' is the original multiplication on B and m_2'' is a Hochschild 2-cocycle. Similarly, an infinitesimal n-deformation corresponds to a Hochschild n-cocycle: Since $|x| = 2 - n$ and $|m_i| = 2 - i$, the only possible nonzero operations m_i are m_2 and m_n. Additionally, m_n takes elements of $B^{\otimes n}$ to Bx if $n > 2$. Calculations show that the resulting coefficient function of x must be an n-cocycle. Thus this is essentially a converse to Example 6.1.4, and is a proof of the following theorem.

Theorem 6.1.8. *Let B be an algebra. For each $n \geq 2$, Hochschild n-cocycles on B are in one-to-one correspondence with infinitesimal n-deformations of B.*

Generalizing the case $n = 2$, one checks that cohomologous Hochschild n-cocycles correspond to isomorphic infinitesimal n-deformations; the appropriate notion of isomorphism is given by Definition 6.1.9 below. Specifically, let $(A_g, m_\bullet)$ be the infinitesimal n-deformation of B given in Example 6.1.4. Suppose $g' = g + hd$ for some $(n-1)$-cochain h and $(A_{g'}, m'_\bullet)$ is the infinitesimal n-deformation of B corresponding to g'. Set $f_1 : A_g \to A_{g'}$ to be the identity map on the underlying vector space, set $f_i = 0$ if $i \notin \{1, n-1\}$, and $f_{n-1} = -xh$. Recalling that $m_1 = 0$ and $m'_1 = 0$, we see that the only conditions (6.1.10) with nonzero terms are the second and $(n-1)$st such equations. The second such equation automatically holds since f_1 is the identity map and $m_2 = m'_2$. The $(n-1)$st equation olds since $g' = g + hd$.

By way of Theorem 6.1.8, we relate Hochschild cohomology $\mathrm{HH}^*(B)$ of an associative algebra B with infinitesimal deformations of B as an A_∞-algebra, in the same way that degree 2 Hochschild cohomology $\mathrm{HH}^2(B)$ corresponds to infinitesimal deformations of B as an associative algebra.

We now return to the general setting of A_∞-algebras.

Definition 6.1.9. Let $(A, m^A_\bullet)$, $(B, m^B_\bullet)$ be A_∞-algebras. A *morphism* of A_∞-algebras $f_\bullet : (A, m^A_\bullet) \to (B, m^B_\bullet)$ consists of graded linear maps

$$f_n : A^{\otimes n} \to B$$

of degree $|f_n| = 1 - n$ for all $n \geq 1$ such that

(6.1.10)

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m^A_s \otimes 1^{\otimes t}) = \sum_{i_1+\cdots+i_r=n} (-1)^u m^B_r(f_{i_1} \otimes \cdots \otimes f_{i_r})$$

where $u = (r-1)(i_1-1) + (r-2)(i_2-1) + \cdots + 2(i_{r-2}-1) + (i_{r-1}-1)$. The *identity morphism* $f_\bullet : A \to A$ is defined by $f_1 = 1_A$ and $f_n = 0$ for $n \neq 1$. The composition of two morphisms $g : A \to B$ and $f : B \to C$ is given by

$$(fg)_n = \sum_{i_1+\cdots+i_r=n} (-1)^u f_r(g_{i_1} \otimes \cdots \otimes g_{i_r})$$

for all n, with $u = u(i_1, \ldots, i_r)$ as above. An A_∞-morphism $f_\bullet$ is a *quasi-isomorphism* if f_1 is a quasi-isomorphism, that is, f_1 induces an isomorphism on cohomology, $\mathrm{H}^*(A) \xrightarrow{\sim} \mathrm{H}^*(B)$.

We interpret the definition of A_∞-morphism for small values of n. If $n = 1$, equation (6.1.10) is

$$f_1 m^A_1 = m^B_1 f_1,$$

in other words, f_1 is a cochain map. If $n = 2$, it is

$$f_1 m^A_2 = m^B_2(f_1 \otimes f_1) + m^B_1 f_2 + f_2(m^A_1 \otimes 1 + 1 \otimes m^A_1),$$

which may be rewritten

$$f_1 m_2^A = m_2^B(f_1 \otimes f_1) + \delta(f_2).$$

That is, f_1 is an algebra homomorphism, with respect to multiplication m_2, up to a coboundary in the Hom complex $\mathrm{Hom}_k(A^{\otimes \bullet}, B)$ given by f_2.

6.2. Minimal models

For some applications, an A_∞-algebra may be replaced by a minimal A_∞-algebra to which it is quasi-isomorphic. We outline this technique here.

Definition 6.2.1. Let A be an A_∞-algebra. A *minimal model* for A is a minimal A_∞-algebra B together with a quasi-isomorphism of A_∞-algebras $f_\bullet : B \to A$.

Let A be an A_∞-algebra, and $\mathrm{H}^*(A)$ its cohomology. The following theorem of Kadeishvili [**Kad82**] states that the cohomology $\mathrm{H}^*(A)$ has the structure of a minimal A_∞-algebra corresponding to the A_∞-algebra structure of A. In particular, the theorem implies existence of a minimal model.

Theorem 6.2.2. *The cohomology* $\mathrm{H}^*(A)$ *of an* A_∞*-algebra* A *may be given the structure of an* A_∞*-algebra under which it is a minimal model for* A*. This structure is unique up to isomorphism of* A_∞*-algebras.*

Proof. We give a proof only in the special case that $(A, m_\bullet)$ is a differential graded algebra. Thus we assume that $m_n = 0$ for $n > 2$. For the general case, see [**Kad82**]. We will define maps $m'_n : \mathrm{H}^*(A)^{\otimes n} \to \mathrm{H}^*(A)$, for each n, under which $(\mathrm{H}^*(A), m'_\bullet)$ becomes an A_∞-algebra. At the same time we will define maps $f_n : \mathrm{H}^*(A)^{\otimes n} \to A$ that will constitute a quasi-isomorphism $f_\bullet : (\mathrm{H}^*(A), m'_\bullet) \to (A, m_\bullet)$. Let $m'_1 = 0$ and let $f_1 : \mathrm{H}^*(A) \to A$ be any k-linear section of the surjection $p : \mathrm{Z}^*(A) \to \mathrm{H}^*(A)$ from the space of cocycles $\mathrm{Z}^*(A)$ to the cohomology $\mathrm{H}^*(A)$ of A. That is, f_1 takes values in $Z^*(A)$ and $pf_1 = 1_{\mathrm{H}^*(A)}$. Let m'_2 be multiplication on $\mathrm{H}^*(A)$ as induced by m_2 on A. Then by definition, for each $\alpha, \beta \in \mathrm{H}^*(A)$, the elements $m_2(f_1(\alpha) \otimes f_1(\beta))$ and $f_1(m'_2(\alpha \otimes \beta))$ are cohomologous in A. Put another way, letting $\Phi_2 = m_2(f_1 \otimes f_1)$, we see that $f_1 m'_2 - \Phi_2$ is a coboundary, that is there is some k-linear map $f_2 : \mathrm{H}^*(A)^{\otimes 2} \to A$ for which

$$f_1 m'_2 - \Phi_2 = m_1 f_2.$$

Since $m'_1 = 0$, we may rewrite this as

$$\Phi_2 = f_1 m'_2 - \delta(f_2),$$

the required condition (6.1.10), with $n = 2$, for an A_∞-morphism. Since $m'_1 = 0$ and m'_2 is associative, condition (6.1.2) holds with $n = 3$.

The remainder of the proof proceeds by induction on n. We explain the case $n = 3$ first for clarity. Let

$$\Phi_3 = m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(1 \otimes m_2' - m_2' \otimes 1).$$

A calculation shows that Φ_3 takes values in the space $Z^*(A)$ of cocycles of A. Let $m_3' : \mathrm{H}^*(A)^{\otimes 3} \to \mathrm{H}^*(A)$ be a k-linear function such that $m_3'(\alpha \otimes \beta \otimes \gamma)$ represents $\Phi_3(\alpha \otimes \beta \otimes \gamma)$ for all $\alpha, \beta, \gamma \in \mathrm{H}^*(A)$. Then by definition of m_3', the elements $f_1 m_3'(\alpha \otimes \beta \otimes \gamma)$ and $\Phi_3(\alpha \otimes \beta \otimes \gamma)$ are cohomologous. It follows that

$$f_1 m_3' - \Phi_3 = m_1 f_3$$

for some $f_3 : \mathrm{H}^*(A)^{\otimes 3} \to A$. Thus equation (6.1.10) holds when $n = 3$. Now consider the left side of equation (6.1.2) with $n = 4$ for $m_1' = 0$, m_2', m_3':

$$(6.2.3)\quad m_2'(-m_3' \otimes 1 - 1 \otimes m_3') + m_3'(m_2' \otimes 1 \otimes 1 - 1 \otimes m_2' \otimes 1 + 1 \otimes 1 \otimes m_2').$$

Compose with f_1 and apply (6.1.10) repeatedly to obtain a coboundary in A. Since f_1 is a section of the quotient map from $Z^*(A)$ to $\mathrm{H}^*(A)$, this implies that the expression (6.2.3) is equal to 0.

More generally, let $n > 3$ and suppose we have defined m_i', f_i for all $i < n$. Let

$$\Phi_n = \sum_{i_1+i_2=n} (-1)^{i_1-1} m_2(f_{i_1} \otimes f_{i_2}) - \sum_{\substack{r+s+t=n \\ s>1,\ r+t>0}} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m_s' \otimes 1^{\otimes t}),$$

that is the difference of the right and left sides of equation (6.1.10), excluding the terms $f_1 m_n'$ and $m_1 f_n$ (since $m_i = 0$ for $i > 2$). One checks that Φ_n takes values in the space $Z^*(A)$ of cocycles of A. Let $m_n' : \mathrm{H}^*(A)^{\otimes n} \to \mathrm{H}^*(A)$ be a k-linear function such that $m_n'(\alpha_1 \otimes \cdots \alpha_n)$ represents $\Phi_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$ for all $\alpha_1, \ldots, \alpha_n \in \mathrm{H}^*(A)$. Then $f_1 m_n'(\alpha_1 \otimes \cdots \otimes \alpha_n)$ and $\Phi_n(\alpha_1 \otimes \cdots \alpha_n)$ are cohomologous, so

$$f_1 m_n' - \Phi_n = m_1 f_n$$

for some $f_n : \mathrm{H}^*(A)^{\otimes n} \to A$. Thus equation (6.1.10) holds for n.

By construction, $f_\bullet$ is an A_∞-morphism. Condition (6.1.2) for $(\mathrm{H}^*(A), m_\bullet')$ automatically holds when $n = 1$ or $n = 2$ since $m_1' = 0$, and as we saw before, it holds when $n = 3$ since m_2' is an associative multiplication on $\mathrm{H}^*(A)$. More generally, one applies f_1 to the left side of equation (6.1.2) for $m_\bullet'$ and shows that the result is a coboundary in A by repeated use of equation (6.1.10) to eliminate terms involving $m_\bullet'$ from the expression, as explained for $n = 4$ above. Since f_1 is a quasi-isomorphism and $m_1' = 0$, the left side of equation (6.1.2) for $m_\bullet'$ is indeed 0 for each n.

The uniqueness statement can be proven by invoking a k-linear projection from A to $\mathrm{H}^*(A)$ whose composition with f_1 is the identity on $\mathrm{H}^*(A)$. This may be extended to an A_∞-morphism from A to $\mathrm{H}^*(A)$; in fact any A_∞-quasi-isomorphism has a homotopy inverse [**Hue10**]. Given two copies of

$\mathrm{H}^*(A)$, with possibly different higher multiplications $m'_\bullet$ and A_∞-morphism $f_\bullet$, by mapping each to A and then to the other, we obtain maps between the two copies whose compositions in both directions must be identity maps. □

Remark 6.2.4. For a more conceptual proof that $(\mathrm{H}^*(A), m'_\bullet)$ is an A_∞-algebra, equivalent to the above proof but avoiding explicit calculations, one observes that $f_\bullet$ constitutes an injective coalgebra morphism from the reduced bar construction of $\mathrm{H}^*(A)$ to that of A. Since $m_\bullet$ satisfies the conditions (6.1.2), it follows that $m'_\bullet$ does as well.

Example 6.2.5. Let $B = k[x]/(x^n)$, let $P_\bullet$ be the standard periodic free resolution of k as a B-module as in Example 1.6.7. Let $A = \mathrm{Hom}_B(P_\bullet, P_\bullet)$, so that $\mathrm{H}^*(A) \cong \mathrm{Ext}^*_B(k,k) \cong k[y,z]/(y^2)$. Applying the algorithm suggested by the proof of Theorem 6.2.2 leads to the A_∞-algebra structure on $\mathrm{H}^*(A)$ as given in Example 6.1.5.

Remark 6.2.6. In Theorem 6.2.2, letting B be an associative algebra, we may take A to be the differential graded algebra $\oplus_{i\geq 0} \mathrm{Hom}_{B^e}(B^{\otimes(i+2)}, B)$, that is, $C^*(B,B)$. Then $\mathrm{H}^*(A)$ is the Hochschild cohomology $\mathrm{HH}^*(B)$, and Theorem 6.2.2 implies that we may realize this Hochschild cohomology as a minimal model. The proof of the theorem indicates how to define the needed higher operations. See also [**Hue10**] for a proof using homological perturbation and further results.

Definition 6.2.7. An A_∞-algebra A is *formal* if its minimal model has the property that $m_n = 0$ for all $n \geq 3$.

We will see in the next section a large class of formal A_∞-algebras given by cohomology of Koszul algebras.

6.3. Formality and Koszul algebras

In this section, we present Keller's results on formality and Koszul algebras [**Kel02**] in the special case that A is connected and graded. Let $B = T(V)/(R)$ for a finite dimensional vector space V with homogeneous relations $R \subset \oplus_{n\geq 2} T_n(V)$. Let k be a B-module where each element of V acts as 0, and let $P = P_\bullet$ be a projective resolution of k as a B-module. We view $\mathrm{Ext}^*_B(k,k)$ as an A_∞-algebra via Theorem 6.2.2, that is as the minimal model for the differential graded algebra $\mathrm{Hom}_B(P,P)$, so that in general it will have higher multiplication maps. When we mention the A_∞-algebra $\mathrm{Ext}^*_B(k,k)$, it is this A_∞-structure that is intended. A use of comparison maps between resolutions shows that, up to isomorphism, this A_∞-structure will not depend on choice of resolution $P_\bullet$.

We first have a lemma about a generating set for this A_∞-algebra.

Lemma 6.3.1. *The A_∞-algebra $\mathrm{Ext}^*_B(k,k)$ is generated in degree 1.*

A proof uses homological properties of such algebras; see, e.g., [**Kel02**, §2.2].

Example 6.3.2. Let $B = k[x]/(x^n)$ as in Example 6.1.6. As we saw in Example 6.1.6, $\mathrm{Ext}^{\bullet}_B(k,k)$ is generated by a single element y as an A_∞-algebra. The degree of y is 1.

Recall from Theorem 2.3.6 that the Ext algebra of a Koszul algebra is generated in degree 1 as an associative algebra, and this is essentially a defining characteristic of Koszul algebras. From this theorem, we obtain the following formality result.

Theorem 6.3.3. *Let B be a finitely generated graded connected algebra. The A_∞-algebra $\mathrm{Ext}^*_B(k,k)$ is formal if and only if B is a Koszul algebra.*

Proof. First assume that B is Koszul. The Koszul resolution $\widetilde{K}_{\bullet} = \widetilde{K}_{\bullet}(B)$ defined by (2.3.5) gives rise to the complex $\mathrm{Hom}_B(\widetilde{K}_{\bullet}, \widetilde{K}_{\bullet})$, which has grading both by homological degree and that induced by the grading on the algebra B. There is a subcomplex of $\mathrm{Hom}_B(\widetilde{K}_{\bullet}, \widetilde{K}_{\bullet})$ consisting of all elements whose homological degree agrees with grading degree. By Theorem 2.3.6, since B is Koszul, $\mathrm{Ext}^i_B(k,k) = \mathrm{Ext}^{i,i}_B(k,k)$, and we see that it embeds into $\mathrm{Hom}_B(\widetilde{K}_{\bullet}, \widetilde{K}_{\bullet})$ as an associative algebra. Therefore $\mathrm{Ext}^*_B(k,k)$ is formal.

Now assume that $\mathrm{Ext}^*_B(k,k)$ is formal. By Lemma 6.3.1, it is generated in degree 1 as an A_∞-algebra. As $\mathrm{Ext}^*_B(k,k)$ is formal, we conclude that it is generated in degree 1 as an associative algebra. By Theorem 2.3.6, B is Koszul. □

Example 6.3.4. See Example 6.1.5 for an illustration of the distinction made by the theorem: Let $B = k[x]/(x^n)$. If $n = 2$, then B is Koszul, and $\mathrm{Ext}^*_B(k,k)$ is formal. If $n > 2$, then B is not Koszul and $\mathrm{Ext}^*_B(k,k)$ is not formal. Accordingly we had found a nonzero higher multiplication m_n in this case.

6.4. A_∞-center

There is a notion of center of an A_∞-algebra that arises in an important way in Hochschild cohomology, as we will see in Theorem 6.4.3. There is more than one reasonable way to define the center of an A_∞-algebra. We follow Briggs and Gélinas [**BG**] for a definition for minimal A_∞-algebras that is invariant under quasi-isomorphism.

We will need to introduce the graded symmetric and exterior algebras that will be used for the rest of this chapter. The notation $S(V)$ and $\bigwedge(V)$ below agree with earlier uses of the notation in this book in the case that V is concentrated in degree 0.

Let V be a graded vector space over a field k of characteristic not 2. The *graded symmetric algebra* is

$$S(V) = T(V)/(u \otimes v - (-1)^{|u||v|} v \otimes u \mid u, v \text{ homogeneous elements of } V).$$

The *graded exterior algebra* is

$$\bigwedge(V) = T(V)/(u \otimes v + (-1)^{|u||v|} v \otimes u \mid u, v \text{ homogeneous elements of } V).$$

One checks that $S(V)$ is universal with respect to graded symmetric maps and $\bigwedge(V)$ is universal with respect to graded anti-symmetric maps. If the characteristic of k is 2, one modifies the above definitions of $S(V)$ and $\bigwedge(V)$ so that they satisfy these universal properties. In the former case we must additionally mod out by all $v \otimes v$ for which $|v|$ is odd, and in the latter by all $v \otimes v$ for which $|v|$ is even. There is some resulting redundancy in these larger sets of relations.

In the rest of this chapter, all our symmetric and exterior algebras will be graded in this sense.

Let S_n denote the symmetric group on n symbols. For each $\sigma \in S_n$ and homogeneous $v_1, \dots, v_n \in V$, define $\chi(\sigma; v_1, \dots, v_n)$ by

$$v_{\sigma(1)} \cdots v_{\sigma(n)} = \chi(\sigma; v_1, \dots, v_n) v_1 \cdots v_n \tag{6.4.1}$$

where the product occurs in the graded exterior algebra $\bigwedge(V)$. We sometimes write $\chi(\sigma)$ when it is clear which vectors $v_1, \dots, v_n$ are involved. If V is concentrated in degree 0, then $\chi(\sigma)$ is simply $\operatorname{sgn}(\sigma)$.

For an A_∞-algebra A, define *higher commutators* $[-;-]_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A$ by

$$\begin{aligned}&[a_1, \dots, a_p; a_{p+1}, \dots, a_{p+q}]_{p,q}\\ &= \sum_{\sigma \in S_{p,q}} \chi(\sigma; a_1, \dots, a_{p+q}) m_{p+q}(a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)})\end{aligned}$$

for all homogeneous $a_1, \dots, a_n \in A$, where $S_{p,q}$ is the set of all (p,q)-shuffles in the symmetric group S_{p+q} as in Definition 1.1.18. Note that $[-;-]_{1,1}$ is the usual commutator for m_2, that is,

$$[a_1; a_2]_{1,1} = m_2(a_1 \otimes a_2) - (-1)^{|a_1||a_2|} m_2(a_2 \otimes a_1).$$

Definition 6.4.2. Let A be a minimal A_∞-algebra and let a be a homogeneous element of A. Then a is *central* in A if for all $n \geq 1$,

$$\begin{aligned}[a; -]_{1,n} = \sum_{r+s+t=n} &(-1)^{r(|a|+s)+t(|a|+1)} m_{r+1+t}(1^{\otimes r} \otimes p_s \otimes 1^{\otimes t})\\ &- (-1)^{|a|}(-1)^{rs+t} p_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t})\end{aligned}$$

for some k-linear maps $p_i : A^{\otimes i} \to A$ of degree $|p_i| = |a| - i$ for $i \geq 1$. Put more concisely, $\operatorname{ad}(a) = \partial(p)$, for a suitable notions of adjoint map ad and differential ∂ (see [**BG**, Definition 3.7] for this and more details). The A_∞-*center* of A, denoted $Z_\infty(A)$, is the vector space spanned by all homogeneous central elements of A.

The maps $p_\bullet$ are called *homotopy derivations*. The related notion of *strong homotopy derivation* [**KS06**] is such a map $p_\bullet$ for which the right side of the equation in the definition above is 0, that is, $\partial(p) = 0$. Note that the higher commutator on the left side of the equation has the effect of inserting the element a between factors in all possible ways, for example

$$\begin{aligned}[a; a_1, a_2]_{1,2} &= m_3(a \otimes a_1 \otimes a_2) \\ &\quad - (-1)^{|a||a_1|} m_3(a_1 \otimes a \otimes a_2) + (-1)^{|a|(|a_1|+|a_2|)} m_3(a_1 \otimes a_2 \otimes a).\end{aligned}$$

It follows from the definition of a central element a that $[a; -]_{1,1} = 0$, and thus a is in the graded center of A. In other words, $Z_\infty(A) \subseteq Z_{\mathrm{gr}}(A)$.

The following theorem is due to Briggs and Gélinas [**BG**], as a consequence of more general results. For a proof, see [**BG**]. Recall that an augmented algebra is one having an algebra homomorphism to k, and k is considered to be a module via this map.

Theorem 6.4.3. *Let B be an augmented algebra. The image of Hochschild cohomology* $\mathrm{HH}^*(B)$ *in the Ext algebra* $\mathrm{Ext}^*_B(k,k)$ *is precisely the A_∞-center,* $Z_\infty(\mathrm{Ext}^*_B(k,k))$.

The theorem generalizes a result of Buchweitz, Green, Snashall, and Solberg [**BGSS08**] from Koszul algebras to augmented algebras: They proved that if B is a Koszul algebra, then the image of Hochschild cohomology $\mathrm{HH}^*(B)$ in the Ext algebra $\mathrm{Ext}^*_B(k,k)$ is the graded center $Z_{\mathrm{gr}}(\mathrm{Ext}^*_B(k,k))$. As we saw in the last section, the Ext algebra of a Koszul algebra B is formal, and thus the graded center of $\mathrm{Ext}^*_B(k,k)$ coincides with its A_∞-center.

Example 6.4.4. Let $B = k[x]/(x^n)$, $n > 2$. Then $\mathrm{Ext}^*_B(k,k) \cong k[y,z]/(y^2)$ and

$$Z_\infty(\mathrm{Ext}^*_B(k,k)) = \begin{cases} k[z], & \text{if } \mathrm{char}(k) \nmid n \\ k[y,z]/(y^2), & \text{if } \mathrm{char}(k) \mid n. \end{cases}$$

(See [**BG**, Example 4.8] for details.)

There are many more applications of A_∞-algebras, as well as the dual notion of A_∞-coalgebras, in Hochschild cohomology. See, for example, [**Her**].

6.5. L_∞-algebras

Analogous to A_∞-algebras are the L_∞-algebras in which the operations are higher order Lie brackets. This notion first appeared in the paper [**SS85**] by Schlessinger and Stasheff.

Definition 6.5.1. An L_∞-*algebra* is a graded vector space L together with graded linear maps

$$\ell_n : \bigwedge^n L \to L$$

of degree $|\ell_n| = 2 - n$ for all n such that

$$\sum_{i=1}^{n} \sum_{\sigma \in S_{i,n-i}} (-1)^{i(n-i)} \chi(\sigma) \ell_{n-i+1}(\ell_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

for all homogeneous $v_1, \ldots, v_n \in L$, where $\chi(\sigma) = \chi(\sigma; v_1, \ldots, v_n)$ is defined by (6.4.1) and $S_{i,n-i}$ is the set of all $(i, n-i)$-shuffles in the symmetric group S_n as in Definition 1.1.18. For simplicity of notation, we have separated elements of L by commas when their product in $\bigwedge(L)$ is the argument of a function. Sometimes we denote the L_∞-algebra by $(L, \ell_\bullet)$ to emphasize the notation chosen for the higher operations. An L_∞-algebra is *minimal* if $\ell_1 = 0$.

Remark 6.5.2. In the literature, the elements of $S_{i,n-i}$ are sometimes called i-*unshuffles* in this context. Also, a graded symmetric algebra sometimes appears in definitions of L_∞-structures, in place of a graded exterior algebra as we have here, due to different choices of grading and indexing.

We interpret these equations in low degrees. We write $d(v) = \ell_1(v)$ and $[u, v] = \ell_2(u, v)$, $[u, v, w] = \ell_3(u, v, w)$, etc., for elements u, v, w of L. If $n = 1$, the equation in Definition 6.5.1 is $d^2(v) = 0$ for all $v \in L$, so d is a differential on L. If $n = 2$, the condition is

$$d([u, v]) = [d(u), v] + (-1)^{|u|}[u, d(v)],$$

for all homogeneous $u, v \in L$, so that d is a graded derivation with respect to $\ell_2 = [\ ,\ ]$. If $n = 3$, the condition may be written

$$\begin{aligned}&(-1)^{|u||w|}[[u, v], w] + (-1)^{|u||v|}[[v, w], u] + (-1)^{|v||w|}[[w, u], v] \\ &= (-1)^{|u||w|}(d\ell_3 + \ell_3 d)(u, v, w)\end{aligned}$$

for all homogeneous $u, v, w \in L$, that is, up to homotopy, the graded Jacobi identity holds.

Example 6.5.3. If L is a differential graded Lie algebra, we may take ℓ_1 to be its differential, ℓ_2 to be its Lie bracket, and $\ell_n = 0$ for all $n \geq 3$, for an L_∞-structure on L. In particular, a graded Lie algebra may be viewed as a differential graded Lie algebra with zero differential, and thus as an L_∞-algebra in this way. If an L_∞-algebra is concentrated in degree 0, that is $L_i = 0$ for all $i \neq 0$, the maps ℓ_n are necessarily zero maps for all $n \neq 2$ since $|\ell_n| = 2 - n$, so L is simply a Lie algebra.

A large class of examples is provided by A_∞-algebras together with graded commutators, as the following theorem of Lada and Markl [**LM95**] shows. This generalizes one relationship between associative algebras and Lie algebras. The theorem may be proven by direct computation, or by invoking a connection between L_∞-structures and coderivations as in [**LM95**].

Theorem 6.5.4. *Let $(A, m_\bullet)$ be an A_∞-algebra. Let*

$$\ell_n(a_1, \dots, a_n) = \sum_{\sigma \in S_n} \chi(\sigma) m_n(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

for all homogeneous $a_1, \dots, a_n \in A$. Then $(A, \ell_\bullet)$ is an L_∞-algebra.

Morphisms of L_∞-algebras involve the folllowing generalization of shuffles. There are more conceptual alternative descriptions, as well as equivalent formulas in characteristic 0 that involve sums over all permutations and division by factorials. See, for example, [**All10**] or [**KS06**].

Let $i_1, \dots, i_t$ be positive integers. A permutation σ of $S_{i_1+\cdots+i_j}$ is an $(i_1, \dots, i_t)$-*shuffle* if

$$\begin{aligned} &\sigma(1) < \cdots < \sigma(i_1), \\ &\sigma(i_1+1) < \cdots < \sigma(i_1+i_2), \dots, \\ \text{and } &\sigma(i_1 + \cdots + i_{t-1} + 1) < \cdots < \sigma(i_1 + \cdots + i_t). \end{aligned}$$

Definition 6.5.5. Let $(L, \ell_\bullet)$, $(L', \ell'_\bullet)$ be L_∞-algebras. A *morphism* of L_∞-algebras $f_\bullet : L \to L'$ consists of graded linear maps

$$f_n : \bigwedge^n L \to L'$$

of degree $|f_n| = 1-n$ for all $n \geq 1$ such that for all homogeneous $v_1, \dots, v_n \in L$,

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{\sigma \in S_{i,n-i}} (-1)^{i(n-i)} \chi(\sigma) f_{n-i+1}(\ell_i \otimes 1^{\otimes(n-i)})(v_{\sigma(1)}, \dots, v_{\sigma(n)}) \\ &= \sum_{\substack{1 \leq r \leq n \\ i_1+\cdots+i_r=n}} \sum_{\tau} (-1)^u \chi(\tau) \ell'_t(f_{i_1} \otimes \cdots \otimes f_{i_r})(v_{\tau(1)}, \dots, v_{\tau(n)}) \end{aligned}$$

where τ runs over all $(i_1, \dots, i_t)$-shuffles for which

$$\tau(i_1 + \cdots + i_{l-1} + 1) < \tau(i_1 + \cdots + i_l + 1)$$

if $i_l = i_{l+1}$, and $u = (r-1)(i_1-1)+(r-2)(i_2-1)+\cdots+2(i_{r-2}-1)+(i_{r-1}-1)$. An L_∞-morphism $f_\bullet$ is a *quasi-isomorphism* if f_1 is a quasi-isomorphism.

We interpret the definition for some small values of n: If $n = 1$, the equation is

$$f_1 \ell_1 = \ell'_1 f_1,$$

that is, f_1 is a cochain map. If $n = 2$, we obtain

$$\begin{aligned} &- f_2(d(v_1), v_2) - (-1)^{|v_1||v_2|} f_2(d(v_2), v_1) + f_1([v_1, v_2]) \\ &= d'(f_2(v_1, v_2)) + [f_1(v_1), f_1(v_2)] \end{aligned}$$

for all homogeneous $v_1, v_2 \in L$. Rewriting, the equation is

$$\begin{aligned} &f_1([v_1, v_2]) \\ &= [f_1(v_1), f_1(v_2)] + d'(f_2(v_1, v_2)) + f_2(d(v_1), v_2) + (-1)^{|v_1||v_2|} f_2(d(v_2), v_1), \end{aligned}$$

that is, f_1 preserves the bracket up to the coboundary $\partial(f_2)$.

Just as for A_∞-algebras, there is a notion of minimal model and formality, as are discussed specifically for the case of Hochschild cohomology in the next section. For more general results, see for example [**Hin03**, Lemma 4.2.1].

6.6. Formality and algebraic deformations

In this section, we assume the characteristic of the field k is 0 so that exponential maps are defined. The main ideas in this section are due to Kontsevich [**Kon03**]. Let A be an associative algebra and let $C(A, A) = \oplus_{i\geq 0} \operatorname{Hom}_k(A^{\otimes i}, A)$, a differential graded Lie algebra as described in Section 1.4. In this section, we suppress notation indicating homological degree in order to avoid confusion with the tensor powers and exterior powers arising in the infinity structures.

Definition 6.6.1. An *HKR map* is a graded k-linear injective map

$$\phi : \mathrm{HH}(A) \to C(A, A),$$

of degree $|\phi| = 0$, with image contained in the space of cocycles, that is a section of the quotient map from the space of cocycles to $\mathrm{HH}(A)$.

By Lemma 1.4.2, we may view both $\mathrm{HH}(A)$ and $C(A, A)$ as differential graded Lie algebras, the former having differential 0. An HKR map is in general not a morphism of differential graded Lie algebras. However, viewing $\mathrm{HH}(A)$ and $C(A, A)$ as L_∞-algebras (with higher brackets 0), an HKR map can sometimes be extended to a quasi-isomorphism of L_∞-algebras, as we will see below.

Definition 6.6.2. The associative algebra A is *formal* if there is a quasi-isomorphism of L_∞-algebras $\Phi_\bullet : (\mathrm{HH}(A), \ell_\bullet) \to (C(A, A), \ell'_\bullet)$ for which Φ_1 is an HKR map. Such a map Φ is called a *formality map*.

Note that this definition is analogous to Definition 6.2.7 when we replace the A_∞-structure with the L_∞-structure, taking $\mathrm{HH}(A)$ to be the minimal model of $C(A, A)$.

To be clear, the grading on HH(A) and on $C(A, A)$ in the definition is shifted by 1 since we are dealing with the Lie structure. So for example, the degree of a Hochschild 2-cocycle is now 1, an important distinction to make in the proof of the next theorem.

Theorem 6.6.3. *Let A be a formal associative algebra, and let α be an infinitesimal deformation of A for which the first obstruction vanishes. Then α lifts to a formal deformation of A.*

Proof. Let $\alpha \in \mathrm{HH}^2(A)$ for which the first obtruction vanishes, that is, $[\alpha, \alpha] = 0$. Consider the following element in $S(\mathrm{HH}(A))[[t]]$:

$$\exp(t\alpha) = 1 + t\alpha + \frac{1}{2!}t^2\alpha^2 + \frac{1}{3!}t^3\alpha^3 + \cdots$$

where we consider the ith term to be in $S^i(\mathrm{HH}^*(A))[[t]]$. Let Φ be a formality map for A. Consider the image of $\exp(t\alpha)$ under Φ extended to formal power series in t. Explicitly, we write this as

$$\Phi(\exp(t\alpha)) = 1 + t\Phi_1(\alpha) + \frac{1}{2!}t^2\Phi_2(\alpha^2) + \frac{1}{3!}t^3\Phi_3(\alpha^3) + \cdots,$$

where $\Phi_2(\alpha^2)$ may also be written $\Phi_2(\alpha, \alpha)$, $\Phi_3(\alpha^3)$ as $\Phi_3(\alpha, \alpha, \alpha)$, and so on. Due to the degree requirement on L_∞-morphisms, the elements $\Phi_i(\alpha^i)$ each have degree 1, that is, they are Hochschild 2-cochains. Since Φ is an L_∞-morphism and $[\alpha, \alpha] = 0$, it follows from the definitions that $\Phi(\exp(t\alpha))$ satisfies the Maurer-Cartan equation (4.3.1). Write $\mu_i = \frac{1}{i!}\Phi_i(\alpha^i)$ and $\mu = \mu_0 + \mu_*$ where $\mu_* = t\mu_1 + t^2\mu_2 + \cdots$. Then $(A[[t]], \mu)$ is a formal deformation of (A, μ_0). □

The following is a special case of general results of Kontsevich [**Kon03**] about Poisson manifolds. See also [**DTT07**]. We take k to be $\mathbb{C}$ or $\mathbb{R}$ here.

Theorem 6.6.4. *Let V be a finite dimensional vector space. Its (ungraded) symmetric algebra $S(V)$ is formal.*

For an outline of the proof, see [**BM08**, §5.2], where the theorem is then generalized to universal enveloping algebras of some Lie algebras. For more details and geometric context, see the excellent survey [**Sch**].

There are related further infinity structures arising on Hochschild cohomology HH(A) and the complex $C(A, A)$. In particular, combining the A_∞- and L_∞-structures we obtain what is called a G_∞-algebra, related to a conjecture of Deligne and various proofs [**KS00, MS02, Tam, Vor00**].

Chapter 7

Support Varieties for Finite Dimensional Algebras

In this chapter we present an application of Hochschild cohomology to the representation theory of finite dimensional algebras, that is algebras that are finite dimensional as vector spaces (over an algebraically closed field k). For many such algebras A, though not all, Hochschild cohomology $\mathrm{HH}^*(A)$ is finitely generated as an algebra. Consequently its maximal ideal spectrum is an affine algebraic variety. It has subvarieties associated to A-modules via actions of Hochschild cohomology $\mathrm{HH}^*(A)$, and the geometry of these varieties has implications in representation theory. These "varieties for modules" based on Hochschild cohomology were introduced by Snashall and Solberg [**SS04**] to mimic support varieties for finite groups (based on group cohomology). The theory is particularly well-behaved for self-injective algebras, as further developed by Erdmann, Holloway, Snashall, Solberg, and Taillefer [**EHS$^+$04**]. See also Solberg's excellent survey [**Sol06**].

We begin this chapter by briefly introducing the needed geometric notions and finite generation conditions. We take A to be a finite dimensional algebra and k an algebraically closed field, although support variety theory has been developed for other algebras and fields as well. In some of the references, A is assumed to be indecomposable. We do not make this assumption since we aim at more general applications, and there are some minor differences. We refer the reader to [**GJ89, MR88**] for the general theory of noncommutative noetherian rings; we will only need this theory in the graded commutative setting where it is essentially the same as for commutative rings.

7.1. Affine varieties

In this section we give a very brief introduction to the geometry that we will use in this chapter. For details, see any text on algebraic geometry or commutative algebra, or [**Ben91b**, Section 5.4]. Let k be an algebraically closed field.

Let H be a finitely generated commutative algebra over k. Equivalently, $H \cong k[x_1, \ldots, x_n]/I$ for some ideal I of a polynomial ring $k[x_1, \ldots, x_n]$. Let $\mathrm{Max}(H)$ denote the set of maximal ideals of H, so $\mathrm{Max}(H)$ is in one-to-one correspondence with the set of maximal ideals of $k[x_1, \ldots, x_n]$ containing I. In particular,

$$\mathrm{Max}(k[x_1, \ldots, x_n]) = \{(x_1 - a_1, \ldots, x_n - a_n) \mid a_1, \ldots, a_n \in k\},$$

so that the set of maximal ideals of the polynomial ring $k[x_1, \ldots, x_n]$ is in one-to-one correspondence with k^n.

The set $\mathrm{Max}(H)$ becomes a topological space under the *Zariski topology*: Closed sets are the sets

$$V(J) = \{J' \in \mathrm{Max}(H) \mid J' \supset J\},$$

determined by ideals J of H. Sometimes we write $V_H(J)$ in place of $V(J)$ to emphasize dependence on H. These sets satisfy the relations

$$V(J_1 J_2) = V(J_1) \cup V(J_2) \quad \text{and} \quad V\left(\sum_\alpha J_\alpha\right) = \bigcap_\alpha V(J_\alpha),$$

where α ranges over an indexing set and J_1, J_2, J_α are ideals of H. We call $\mathrm{Max}(H)$ with this topology the *maximal ideal spectrum* of H, also called an *affine variety*. In particular, $\mathrm{Max}(k[x_1, \ldots, x_n])$ is the *affine space* k^n. One is also interested in projective varieties and prime ideals in representation theory, but in this book, affine varieties and maximal ideals will suffice.

The following lemma will be used to define dimensions of affine varieties.

Lemma 7.1.1 (Noether Normalization Lemma)**.** *Let H be a finitely generated commutative algebra over k. There are elements $y_1, \ldots, y_n \in H$ generating a subalgebra of H that is isomorphic to the polynomial ring $k[y_1, \ldots, y_n]$ and over which H is finitely generated as a module.*

For a proof, see [**Mat86**]. Note that by its definition, the integer n in the lemma is unique.

Definition 7.1.2. Let H be a finitely generated commutative algebra over k, and $\mathrm{Max}(H)$ its maximal ideal spectrum. The *dimension* of $\mathrm{Max}(H)$, also called the dimension of H, is the integer n of the Noether Normalization Lemma (Lemma 7.1.1).

It can be shown that if H is finitely generated, then the dimension of H defined above is the same as the Krull dimension of H, defined next. (See [**Mat86**].)

Definition 7.1.3. For any commutative ring H, its *Krull dimension* is the largest nonnegative integer n for which there exist prime ideals

$$I_0 \supset I_1 \supset \cdots \supset I_n$$

of H such that $I_j \neq I_{j+1}$ for $0 \leq j \leq n-1$.

There is a further equivalent definition of dimension for finitely generated graded commutative algebras that we will need:

Definition 7.1.4. Let $V = \oplus_{i\geq 0} V_i$ be a graded vector space. The *rate of growth* $\gamma(V)$ is the smallest nonnegative integer c such that there is a real number b and a positive integer m for which $\dim_k V_n \leq bn^{c-1}$ for all $n \geq m$.

The dimension of a finitely generated graded commutative algebra is precisely its rate of growth.

7.2. Finiteness properties

Let $\mathfrak{r} = \mathrm{rad}(A)$, the Jacobson radical of the finite dimensional algebra A. We will use the action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M,M)$ for an A-module M, as described in Section 1.6, beginning with the special case $M = A/\mathfrak{r}$. We will assume in Sections 7.3 through 7.5 that A satisfies the following finite generation condition:

(fg) $\mathrm{HH}^*(A)$ is a noetherian ring and $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is a finitely generated $\mathrm{HH}^*(A)$-module.

Since $\mathrm{HH}^*(A)$ is graded and A is a finite dimensional algebra, by [**Eve61**, Proposition 2.4], if $\mathrm{HH}^*(A)$ is noetherian, then it is finitely generated as an algebra by homogeneous elements. Conversely, if $\mathrm{HH}^*(A)$ is finitely generated as an algebra, then it is noetherian, since free graded commutative rings are noetherian and quotients of noetherian rings are noetherian. We will refine these statements in Theorem 7.2.3 below.

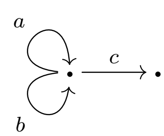
There are many finite dimensional algebras A satisfying the condition (fg), as well as some that do not, as we will see.

Example 7.2.1. Let $A = k[x_1, \ldots, x_m]/(x_1^{n_1}, \ldots, x_m^{n_m})$. We saw in Example 2.1.5 that $\mathrm{HH}^*(A)$ is finitely generated. (See also Example 2.2.5 for some related noncommutative examples.) The Jacobson radical $\mathfrak{r}$ of A is generated by $x_1, \ldots, x_m$, and so $A/\mathfrak{r} \cong k$, the trivial module. If $m = 1$, as a consequence of our work in Example 1.6.7, the Ext algebra $\mathrm{Ext}_A^*(k, k)$ is finitely generated as an $\mathrm{HH}^*(A)$-module. The same is seen to be true if $m > 1$ by combining the techniques of Examples 1.6.7 and 2.1.5.

Many other algebras satisfy condition (fg), such as finite group algebras [**Eve61, Gol59, Ven59**] and Hecke algebras [**Lin11**]. Others satisfy related finiteness conditions, such as monomial algebras [**GS06**], self-injective algebras of finite representation type [**GSS03**], and algebras of finite global dimension [**Hap89**]. Some types of Hopf algebras satisfy condition (fg); Hopf algebras generally are discussed in Chapter 8.

We next give examples of algebras that do not satisfy condition (fg). In characteristic 2, the example below is due to Xu [**Xu08**], who presented it as the first counterexample to a related conjecture of Snashall and Solberg. Here we describe a generalization of Xu's example to arbitrary characteristic, due to Snashall [**Sna09**]. It was generalized further by Gawell and Xantcha [**GX16**] to many more algebras defined by quivers and relations.

Example 7.2.2. Let $A = kQ/I$ where Q is the quiver



with two vertices as indicated, arrows a, b, c, and $I = (a^2,\ b^2,\ ab - ba,\ ac)$. Then A is a Koszul algebra [**Sna09**], so $\mathrm{Ext}_A^*(A/\mathfrak{r}, A/\mathfrak{r}) \cong A^!$, and by definition

$$A^! = kQ^{\mathrm{op}}/I^{\perp}$$

where Q^{op} is the quiver Q with arrows reversed, labeled α, β, γ and $I^{\perp} = (\alpha\beta + \beta\alpha,\ \beta\gamma)$. A calculation shows that the graded center of $A^!$ is

$$Z_{\mathrm{gr}}(A^!) \cong \begin{cases} k \oplus k[\alpha, \beta]\beta, & \text{if } \mathrm{char}(k) = 2, \\ k \oplus k[\alpha^2, \beta^2]\beta^2, & \text{if } \mathrm{char}(k) \neq 2, \end{cases}$$

where β has degree 1, $\alpha\beta$ has degree 2, β^2 has degree 2, and $\alpha^2\beta^2$ has degree 4. By [**BGSS08**], the image of $\mathrm{HH}^*(A)$ under the map $\phi_{A/\mathfrak{r}}$ defined by (1.6.2) is precisely $Z_{\mathrm{gr}}(A^!)$. This is not finitely generated as an algebra, since for each i, the element $\alpha^i\beta$ in characteristic 2 (respectively, $\alpha^{2i}\beta^2$ in characteristic not 2) is not in any subalgebra generated by elements of lower homological degree. Therefore $\mathrm{HH}^*(A)$ is not finitely generated as an algebra, and consequently does not satisfy (fg).

For the purpose of defining affine varieties, one can essentially ignore nilpotent elements, and indeed Snashall and Solberg [**SS04**] had originally conjectured that the quotient of $\mathrm{HH}^*(A)$ by its ideal generated by all homogeneous nilpotent elements is noetherian. Example 7.2.2 is a counterexample. Hermann [**Her16b**] asked if a weaker condition might be satisfied by more finite dimensional algebras: Replace the condition that $\mathrm{HH}^*(A)$ be noetherian with the condition that the quotient by its Gerstenhaber ideal generated by all homogeneous nilpotent elements be noetherian. (By this ideal, we mean the ideal generated via the binary operations of cup product and Gerstenhaber bracket.) We will not consider these weaker conditions here.

The following theorem is from [**EHS$^+$04**, Proposition 1.4] and [**Sol06**, Proposition 5.7]. We will use it to define support varieties in the next section. The flexibility in choosing an algebra H satisfying the conditions of the theorem below will be helpful.

Theorem 7.2.3. *The finite dimensional algebra A satisfies condition (fg) if and only if there exists a graded subalgebra H of $\mathrm{HH}^*(A)$ such that*

(fg1) *H is finitely generated commutative and $H^0 = \mathrm{HH}^0(A, A)$, and*

(fg2) *$\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is a finitely generated H-module.*

There is a trade-off between these two conditions (fg1) and (fg2): Condition (fg1) says that H is small enough for some geometric applications, and condition (fg2) says that H is large enough for some more specific applications. The two taken together say that H is just right for the theory of support varieties that we will define in Section 7.3.

In order to prove the theorem, we need the following lemmas. If M, N are A-modules, then $\mathrm{Hom}_k(M, N)$ is an A-bimodule with action given by

$$(afb)(m) = a(f(bm))$$

for all $f \in \mathrm{Hom}_k(M, N)$, $a, b \in A$, and $m \in M$.

Lemma 7.2.4. *For all finite dimensional A-modules M, N, there is a graded vector space isomorphism*

$$\mathrm{Ext}^*_A(M, N) \cong \mathrm{HH}^*(A, \mathrm{Hom}_k(M, N)).$$

Proof. Let $P_{\bullet} \to A$ be a free resolution of A as an A^e-module, and assume each P_i is finite dimensional. (For example, we could take the bar resolution.) Without loss of generality, for each i, $P_i \cong A \otimes P_i' \otimes A$ for a vector space P_i'. Then applying $- \otimes_A M$ yields a projective resolution of M as an A-module, and applying $\mathrm{Hom}_A(-, N)$, we see that for each i, there are vector space isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(P_i \otimes_A M, N) &\cong \mathrm{Hom}_k(P_i' \otimes M, N)\\ &\cong \mathrm{Hom}_k(P_i', \mathrm{Hom}_k(M,N))\\ &\cong \mathrm{Hom}_{A^e}(P_i, \mathrm{Hom}_k(M,N)).\end{aligned}$$

The differentials correspond under these isomorphisms, and so the isomorphism stated in the lemma holds. □

The next lemma is [**EHS$^+$04**, Proposition 1.4].

Lemma 7.2.5. *Let H be a finitely generated commutative subalgebra of $\mathrm{HH}^*(A)$. The following are equivalent:*

(i) $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ *is a finitely generated H-module.*

(ii) $\mathrm{Ext}^*_A(M,N)$ *is a finitely generated H-module for all finite dimensional A-modules M, N.*

(iii) $\mathrm{HH}^*(A,B)$ *is a finitely generated H-module for all finite dimensional A-bimodules B.*

Proof. By Lemma 7.2.4, $\mathrm{Ext}^*_A(M,N) \cong \mathrm{HH}^*(A, \mathrm{Hom}_k(M,N))$. The actions of $\mathrm{HH}^*(A)$ on these two spaces correspond under this isomorphism, and so (iii) implies (ii). By setting $M = N = A/\mathfrak{r}$, we see that (ii) implies (i). It remains to show that (i) implies (iii). By Lemma 7.2.4 with $M = N = A/\mathfrak{r}$, there is an isomorphism

$$\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r}) \cong \mathrm{HH}^*(A, \mathrm{Hom}_k(A/\mathfrak{r}, A/\mathfrak{r})).$$

Each simple A-module S is a direct summand of $A/\mathfrak{r}$, and simple A^e-modules are all of the form $\mathrm{Hom}_k(S,T)$ for simple A-modules S, T. Letting B be any finite dimensional A^e-module, it has a composition series with simple factors B_i. By (i) and the above observations, $\mathrm{HH}^*(A, B_i)$ is a finitely generated H-module for each i. By induction on the length of the composition series of B and a long exact sequence for Ext, the H-module $\mathrm{HH}^*(A,B)$ is finitely generated. Specifically, for the induction step, suppose $0 \to U \to V \to W \to 0$ is a short exact sequence of A^e-modules and $\mathrm{HH}^*(A,U)$, $\mathrm{HH}^*(A,W)$ are both finitely generated H-modules. Choose a finite set of generators for $\mathrm{HH}^*(A,U)$, and consider their images in $\mathrm{HH}^*(A,V)$ under the map induced by $U \to V$. Since H is noetherian and the image of $\mathrm{HH}^*(A,V)$ in $\mathrm{HH}^*(A,W)$, under the map induced by $V \to W$, is an H-submodule, the image of $\mathrm{HH}^*(A,V)$ in $\mathrm{HH}^*(A,W)$ is finitely generated. Choose a finite set of generators of this image, and choose an inverse image of each one in $\mathrm{HH}^*(A,V)$. Then the finite set of all these generators taken together generates $\mathrm{HH}^*(A,V)$. □

Proof of Theorem 7.2.3. Assume A satisfies (fg). If $\mathrm{char}(k) = 2$, let $H = \mathrm{HH}^*(A)$, and if $\mathrm{char}(k) \neq 2$, let $H = \mathrm{HH}^{\mathrm{ev}}(A)$, the subalgebra of $\mathrm{HH}^*(A)$ generated by all homogeneous elements of even degree. Then $H^0 = \mathrm{HH}^0(A)$ by definition and H is commutative since $\mathrm{HH}^*(A)$ is graded commutative. In addition, H is finitely generated: Take a finite set of homogeneous generators of $\mathrm{HH}^*(A)$, and replace those of odd degree by all products of pairs of odd degree generators. The resulting finite set generates H. So (fg1) holds. Condition (fg2) also holds since H is a subalgebra of $\mathrm{HH}^*(A)$ and we have assumed that $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is finitely generated over the noetherian ring $\mathrm{HH}^*(A)$.

Conversely, assume there is a graded subalgebra H of $\mathrm{HH}^*(A)$ that satisfies (fg1) and (fg2). Note that (fg2) is precisely condition (i) of Lemma 7.2.5. By Lemma 7.2.5(iii) with $B = A$, $\mathrm{HH}^*(A)$ is a finitely generated H-module, and so it is finitely generated as an algebra (take the algebra generators of H together with the H-module generators of $\mathrm{HH}^*(A)$). Since $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is finitely generated as an H-module, it will be finitely generated as an $\mathrm{HH}^*(A)$-module, and so (fg) holds. □

We will be interested in the maximal ideal spectrum of H and of $\mathrm{HH}^*(A)$. Due to (graded) commutativity of these algebras, in either case the nilpotent elements constitute an ideal that is contained in all maximal ideals. Thus for the purpose of considering the maximal ideal spectrum, we may as well work modulo this ideal. The following theorem, due to Snashall and Solberg [**SS04**, Proposition 4.6], gives some information about nilpotent elements of $\mathrm{HH}^*(A)$. Recall that the *radical* of an ideal I of a (graded) commutative ring R is

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some positive integer } n\}.$$

Corollary 1.6.6, with $M = A/\mathfrak{r}$, states that $\phi_{A/\mathfrak{r}}$, as defined in (1.6.2), is a ring homomorphism from $\mathrm{HH}^*(A)$ to $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ with image in the graded center. We now examine this ring homomorphism further.

Theorem 7.2.6. *Let $\mathcal{N}$ be the ideal of* $\mathrm{HH}^*(A)$ *generated by all homogeneous nilpotent elements. Then*

$$\mathcal{N} = \sqrt{\mathrm{Ker}(\phi_{A/\mathfrak{r}})}.$$

Proof. A nilpotent element is in the radical of every ideal by definition, so the containment $\mathcal{N} \subset \sqrt{\mathrm{Ker}(\phi_{A/\mathfrak{r}})}$ is automatic. It remains to prove that $\mathrm{Ker}(\phi_{A/\mathfrak{r}}) \subset \mathcal{N}$, as the containment $\sqrt{\mathrm{Ker}(\phi_{A/\mathfrak{r}})} \subset \mathcal{N}$ will follow since $\mathcal{N}$ contains all nilpotent elements. Let η be a homogeneous element in $\mathrm{Ker}(\phi_{A/\mathfrak{r}})$, and suppose η is represented by a function $f : P_m \to A$ where $P_{\bullet}$ is a minimal projective resolution of A as an A^e-module. Since $\phi_{A/\mathfrak{r}}(\eta) = 0$, at the chain level, $\phi_{A/\mathfrak{r}}(f)$ has image in $\mathfrak{r}$. For each m, the power η^m is represented by f^m, whose image is thus in $\mathfrak{r}^m$. Since $\mathfrak{r}$ is a nilpotent ideal, it follows that η is nilpotent. □

7.3. Support varieties

We are now ready to define support varieties. Assume A satisfies condition (fg) of the previous section. Applying Theorem 7.2.3, we fix a subalgebra H of $\mathrm{HH}^*(A)$ for which (fg1) and (fg2) hold. We do not assume that A is indecomposable as is done in some of the references, the main difference being that we need to keep track of extra points in a support variety of a non-indecomposable module, corresponding to primitive central idempotents in A that are not in the annihilator of the module. Specifically, as in Section 1.2, the Hochschild cohomology ring of A decomposes as a direct sum of Hochschild cohomology rings of the indecomposable algebras e_jA where $\{e_1, \ldots, e_i\}$ is a complete set of primitive central idempotents of A.

For finite dimensional A-modules M, N, let $I_H(M, N)$ be the annihilator in H of $\mathrm{Ext}^*_A(M, N)$, that is

$$I_H(M, N) = \{\alpha \in H \mid \alpha \cdot \beta = 0 \text{ for all } \beta \in \mathrm{Ext}^*_A(M, N)\},$$

where the (left) action of the subalgebra H of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M, N)$ is defined in Section 1.6. By its definition, $I_H(M, N)$ is an ideal of H. Let $I_H(M) = I_H(M, M)$. There is an analogous definition of support variety of a right module, and we will sometimes use it as well.

Definition 7.3.1. Let M, N be finite dimensional A-modules. The *support variety* of the pair M, N is

$$V_H(M, N) = \mathrm{Max}(H/I_H(M, N)),$$

the maximal ideal spectrum of $H/I_H(M, N)$. The *support variety* of M is $V_H(M) = V_H(M, M)$.

Note that by its definition, a support variety may be identified with a subvariety of $\mathrm{Max}(H)$, the maximal ideal spectrum of H.

Example 7.3.2. Let $A = k[x]/(x^n)$. Let $M = k$, the trivial module (on which x acts as 0). As we saw in Example 1.6.7, $\mathrm{Ext}^*_A(k, k)$ is isomorphic to $k[y]$ in case $n = 2$ and is isomorphic to $k[y, z]/(y^2)$ otherwise. We also found the image of $\mathrm{HH}^*(A)$ in $\mathrm{Ext}^*_A(k, k)$ under ϕ_k. Let $H = \mathrm{HH}^*(A)$ in characteristic 2 and otherwise $H = \mathrm{HH}^{\mathrm{ev}}(A)$. Then in all cases, $H/I_H(k)$ has Krull dimension 1 and so $V_H(k)$ is a line.

The following lemma gives a relationship between the variety of a pair of modules and the varieties of the modules.

Lemma 7.3.3. *For all finite dimensional A-modules M and N,*

$$V_H(M,N) \subset V_H(M) \cap V_H(N).$$

Proof. This is true by the definitions, since the (left) action of H on the space $\mathrm{Ext}^*_A(M,N)$ factors through that on $\mathrm{Ext}^*_A(M,M)$, and by Theorem 1.6.3 is the same, up to a sign, as a right action factoring through that on $\mathrm{Ext}^*_A(N,N)$. □

Let $\mathrm{Irr}\, A$ denote a set of representatives of isomorphism classes of simple A-modules. The following lemma gives a relationship between the variety of a module and those of the simple A-modules, and a relationship among varieties for modules in a short exact sequence.

Proposition 7.3.4. *Let M, M_1, M_2, M_3 be finite dimensional A-modules. Then*

(i) $V_H(M) = \cup_{S\in \mathrm{Irr}(A)} V_H(M,S) = \cup_{S\in\mathrm{Irr}(A} V_H(S,M)$.

(ii) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence, then*

$$V_H(M_i) \subset V_H(M_j) \cup V_H(M_l)$$

whenever $\{i,j,l\} = \{1,2,3\}$.

(iii) $V_H(M) = V_H(M, A/\mathfrak{r}) = V_H(A/\mathfrak{r}, M)$.

Proof. If $0 \to U' \to U \to U'' \to 0$ is a short exact sequence of A-modules then $V_H(M,U) \subset V_H(M,U') \cup V_H(M,U'')$ since the annihilator in H of $\mathrm{Ext}^*_A(M,U)$ contains the product of the annihilators of $\mathrm{Ext}^*_A(M,U')$ and $\mathrm{Ext}^*_A(M,U'')$ in light of a long exact sequence for Ext (Theorem A.3.4). Therefore $V_H(M,N) \subset \cup_S V_H(M,S)$, the union over all simple modules in a composition series for M, and similarly $V_H(M,N) \subset \cup_S V_H(S,N)$. Thus (i) holds. For (ii), the above argument using Theorem A.3.4 shows that $V_H(M_i) \subset V_H(M_i,M_j) \cup V_H(M_i,M_l)$, and this is contained in $V_H(M_j) \cup V_H(M_l)$ by Lemma 7.3.3. By the above arguments, $V_H(M) \subset V_H(M, A/\mathfrak{r})$ and $V_H(M) \subset V_H(A/\mathfrak{r}, M)$. On the other hand, by Lemma 7.3.3, the reverse inclusions also are true. Thus (iii) holds. □

Definition 7.3.5. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of M, and view $\oplus_i P_i$ as a graded vector space. The *complexity* of M is $\mathrm{cx}_A(M) = \gamma(P_\bullet) = \gamma(\oplus_i P_i)$, the rate of growth of the resolution (see Definition 7.1.4).

Recall that the dimension of $V_H(M,N)$ is given by Definition 7.1.2, equivalently by Definition 7.1.3 as the Krull dimension of $H/I_H(M,N)$, equivalently by Definition 7.1.4 as the rate of growth of $H/I_H(M,N)$.

Proposition 7.3.6. *Let M be a finite dimensional A-module. Then*

$$\dim V_H(M) = \mathrm{cx}_A(M).$$

Proof. The proof is essentially the same as that of [**Ben91b**, Proposition 5.7.2], which is the case that A is a group algebra of a finite group: By Lemma 7.2.5(ii), $\mathrm{Ext}^*_A(M,M)$ is finitely generated as a module over $H/I_H(M)$. It follows that

$$\dim V_H(M) = \gamma(H/I_H(M)) = \gamma(\mathrm{Ext}^*_A(M,M)).$$

We will show that $\gamma(\mathrm{Ext}^*_A(M,M)) = \gamma(P_\bullet)$, where $P_\bullet$ is a minimal projective resolution of the A-module M. This is by definition $\mathrm{cx}_A(M)$.

For any simple A-module S, the multiplicity of its projective cover $P(S)$ as a direct summand of P_n is

$$\dim_k(\mathrm{Hom}_A(P_n, S)) = \dim_k(\mathrm{Ext}^n_A(M,S))$$

since $S = P(S)/\operatorname{rad}(P(S))$ and $\mathfrak{r} = \operatorname{rad}(A)$ acts trivially on S. So

$$\dim_k P_n = \sum_{S\in \mathrm{Irr}(A)} \dim_k P(S)\cdot \dim_k(\mathrm{Ext}^n_A(M,S)).$$

It follows that

$$\gamma(P_\bullet) \le \max\{\gamma(\mathrm{Ext}^*_A(M,S)) \mid S \in \mathrm{Irr}(A)\}.$$

Now for each simple A-module S, the action of H on $\mathrm{Ext}^*_A(M,S)$ factors through $\mathrm{Ext}^*_A(M,M)$, and both are finitely generated as H-modules. Therefore

$$\gamma(\mathrm{Ext}^*_A(M,S)) \le \gamma(\mathrm{Ext}^*_A(M,M))$$

for each S. It follows that $\gamma(P_\bullet) \le \gamma(\mathrm{Ext}^*_A(M,M))$. On the other hand, by definition of Ext,

$$\dim_k \mathrm{Ext}^n_A(M,M) \le \dim_k \mathrm{Hom}_k(P_n,M) = \dim_k(P_n)\dim_k(M)$$

for each n, and so $\gamma(\mathrm{Ext}^*_A(M,M)) \le \gamma(P_\bullet)$. Therefore $\gamma(\mathrm{Ext}^*_A(M,M)) = \gamma(P_\bullet)$, and this is by definition the complexity of M. □

For each A-module M, let $D(M) = \mathrm{Hom}_k(M,k)$, an A^{op}-module.

Proposition 7.3.7. *For all finite dimensional A-modules M, $V_H(M) = V_H(D(M))$.*

For a proof, see [**SS04**, Proposition 3.5].

7.4. Self-injective algebras and realization

Support varieties for self-injective algebras possess some particularly useful properties. For the rest of this chapter, assume A is a finite dimensional self-injective algebra satisfying condition (fg) from Section 7.2. Fix a subalgebra H of $\mathrm{HH}^*(A)$ satisfying (fg1) and (fg2), in accordance with Theorem 7.2.3. Under these assumptions, in Theorem 7.4.4 we state a tensor product property, one consequence being that any closed subset of $\mathrm{Max}(H)$ may be realized as the variety of an A-module.

Lemma 7.4.1. *Let M be a finite dimensional A-module. Then M is projective if and only if* $\dim V_H(M) = 0$.

Proof. If M is projective, then $0 \to M \to M \to 0$ is a projective resolution of M. By Proposition 7.3.6, $\dim V_H(M) = \mathrm{cx}_A(M) = 0$. Conversely, assume $\dim V_H(M) = 0$, that is, $\mathrm{cx}_A(M) = 0$, so that if $P_{\bullet}$ is a minimal projective resolution of M, then $P_n = 0$ for some n. That is,

$$0 \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution of M. Since A is self-injective, each P_i is also injective, so the sequence splits, and thus M is projective. □

The *Heller operator* $\Omega = \Omega_A$ of the following lemma is defined in Section A.1. We assume that it refers to the syzygy of a projective cover in this setting of self-injective algebras.

Lemma 7.4.2. *Let M be a finite dimensional indecomposable A-module. Then $V_H(M) = V_H(\Omega_A M)$.*

Proof. Let P_0 be a projective A-module with $P_0/\mathrm{rad}(P_0) \cong M/\mathrm{rad}(M)$. Since M is indecomposable, there is a unique primitive central idempotent e_j of A for which $e_jM = M$ and $e_jP_0 = P_0$. So $V_H(P_0)$ consists of precisely one point corresponding to e_j that is also in $V_H(M)$ and in $V_H(\Omega_A M)$. Now apply Proposition 7.3.4(ii) to the sequence $0 \to \Omega_A M \to P_0 \to M \to 0$. □

Next we give a realization result, namely that any closed homogeneous subvariety of $\mathrm{Max}(H)$ is the support variety of some A-module. This makes use of some bimodules attached to elements of H, analogous to Carlson's modules L_ζ in the group algebra case. We present here a variation on these bimodules, as originally defined in [**EHS$^+$04**], for self-injective algebras.

Let $\eta \in \mathrm{HH}^n(A)$, so that η is represented by an element $\hat{\eta}$ of the space $\mathrm{Hom}_{A^e}(\Omega^n_{A^e} A, A)$. Define the A^e-module N_η to be the kernel of $\hat{\eta}$, so that the following is a short exact sequence:

$$0 \to N_\eta \to \Omega^n_{A^e} A \xrightarrow{\hat{\eta}} A \to 0. \tag{7.4.3}$$

These A-bimodules N_η are related to the A-bimodules M_η and A-modules L_η of [**EHS$^+$04**], and have similar properties, but are not the same in general. We have made this choice as it is most similar to the original theory for finite group representations.

In the following theorem, by $V_H(\eta)$, we mean the variety of the ideal (η) generated by η, that is, all maximal ideals containing η.

Theorem 7.4.4. *Let M be an A-module and $\eta \in \mathrm{HH}^n(A)$. Then*

$$V_H(N_\eta \otimes_A M) = V_H(\eta) \cap V_H(M).$$

Moreover, for any homogeneous ideal I of H, there exists an A-module M such that $V_H(M) = V_H(I)$.

Proof. For the proof, we may assume that M is indecomposable so that it is a module for Ae_j for some primitive central idempotent e_j of A, and that $\eta \in \mathrm{HH}^*(e_jA)$. (See Section 1.2 for the decomposition of $\mathrm{HH}^*(A)$ into a direct sum of such components.)

By the definitions, a maximal ideal $\mathfrak{m}$ of H is in $V_H(M,N)$ if and only if $I_H(M,N) \subset \mathfrak{m}$ if and only if the localization $\mathrm{Ext}^*_A(M,N)_\mathfrak{m} \neq 0$.

By Proposition 7.3.4(i), $V_H(N_\eta \otimes_A M) = \cup_{S\in \mathrm{Irr}(A)} V_H(N_\eta \otimes_A M, S)$. We will first show that for each simple A-module S,

$$V_H(\eta) \cap V_H(M,S) \subset V_H(N_\eta \otimes_A M, S),$$

from which it will follow that $V_H(\eta) \cap V_H(M) \subset V_H(N_\eta \otimes_A M)$.

Let $\mathfrak{m}$ be a maximal ideal in $V_H(\eta) \cap V_H(M,S)$, that is $\mathfrak{m}$ contains (η) and $I_H(M,S)$. We want to show that $I_H(N_\eta \otimes_A M, S) \subset \mathfrak{m}$. Suppose this is not true. Then the localization $\mathrm{Ext}^*_A(N_\eta \otimes_A M, S)_\mathfrak{m} = 0$. Apply $- \otimes_A M$ to the short exact sequence (7.4.3). Since each A^e-module in the sequence is free as a right A-module, we obtain a short exact sequence of A-modules,

$$0 \to N_\eta \otimes_A M \to \Omega^n A \otimes_A M \to M \to 0.$$

For each simple A-module S, apply $\mathrm{Ext}^*_A(-,S)$ and consider the corresponding long exact sequence for Ext (Theorem A.3.5). By dimension shifting (Theorem A.2.3) and the observation that $\Omega^n_{A^e} A \otimes_A M$ and $\Omega^n_A M$ agree up to projective direct summands, it is:

$$\cdots \to \mathrm{Ext}^i_A(M,S) \xrightarrow{\tilde{\eta}} \mathrm{Ext}^{i+n}_A(M,S) \xrightarrow{\phi} \mathrm{Ext}^i_A(N_\eta \otimes_A M, S) \to \mathrm{Ext}^{i+1}_A(M,S) \to \cdots$$

where $\tilde{\eta}$ is the action of η on $\mathrm{Ext}^*_A(M,S)$. Let $z \in \mathrm{Ext}^{i+n}_A(M,S)$, and consider $\phi(z) \in \mathrm{Ext}^i_A(N_\eta \otimes_A M, S)$. Since $\mathrm{Ext}^*_A(N_\eta \otimes_A M, S)_\mathfrak{m} = 0$ by assumption, there is a homogeneous element $a \notin \mathfrak{m}$ such that $\phi(az) =$

$a\phi(z) = 0$. In light of the above long exact sequence, $az = \tilde{\eta}(y)$ for some $y \in \operatorname{Ext}_A^{|a|+i}(M, S)$. Upon localizing, we have $z = a^{-1}\tilde{\eta}(y)$, so

$$\operatorname{Ext}_A^*(M, S)_{\mathfrak{m}} = \tilde{\eta}(\operatorname{Ext}_A^*(M, S)_{\mathfrak{m}}).$$

Since $\eta \in \mathfrak{m}$ and $\operatorname{Ext}_A^*(M, S)$ is finitely generated over H, it now follows by Nakayama's Lemma that $\operatorname{Ext}_A^*(M, S)_{\mathfrak{m}} = 0$. This contradicts the assumption that $I_H(M, S) \subset \mathfrak{m}$. So $I_H(N_\eta \otimes_A M, S) \subset \mathfrak{m}$, and therefore $V_H(\eta) \cap V_H(M, S) \subset V_H(N_\eta \otimes_A M, S)$, as claimed. Thus $V_H(\eta) \cap V_H(M) \subset V_H(N_\eta \otimes_A M)$.

To prove the reverse inclusion $V_H(N_\eta \otimes_A M) \subset V_H(\eta) \cap V_H(M)$, we will first show that $V_H(N_\eta \otimes_A M) \subset V_H(\eta)$. By Proposition 7.3.4(i), it suffices to show that $V_H(S, N_\eta \otimes_A M) \subset V_H(\eta)$ for each simple A-module S. Equivalently, that $\operatorname{Ext}_A^*(S, N_\eta \otimes_A M)_{\mathfrak{m}} \neq 0$ for a maximal ideal $\mathfrak{m}$ of H implies $\eta \in \mathfrak{m}$. Suppose $\eta \notin \mathfrak{m}$. Then multiplication by η induces an isomorphism on $\operatorname{Ext}_A^*(S, N_\eta \otimes_A M)_{\mathfrak{m}}$ since it is invertible in $H_{\mathfrak{m}}$. As localization is exact, existence of the short exact sequence (7.4.3) implies that $\operatorname{Ext}_A^*(S, N_\eta \otimes_A M)_{\mathfrak{m}}$ is the kernel of the isomorphism $\eta : \operatorname{Ext}_A^*(S, M)_{\mathfrak{m}} \to \operatorname{Ext}_A^{*+n}(S, M)_{\mathfrak{m}}$. So $\operatorname{Ext}_A^*(S, N_\eta \otimes_A M)_{\mathfrak{m}} = 0$.

Finally we will show that $V_H(N_\eta \otimes_A M) \subset V_H(M)$. This is true by Proposition 7.3.4(ii) and Lemma 7.4.2 applied to

$$0 \to N_\eta \otimes_A M \to \Omega^n_{A^e} A \otimes_A M \to M \to 0$$

since $\Omega^n_{A^e} A \otimes_A M$ is the same as $\Omega^n_A M$ up to projective direct summands.

For the last statement, let I be an ideal of H. Since H is noetherian, I is finitely generated, say $I = (\eta_1, \dots, \eta_r)$. Let $M = N_{\eta_1} \otimes_A \cdots \otimes_A N_{\eta_r} \otimes_A A/\mathfrak{r}$. Then $V_H(M) = V_H(I)$. □

7.5. Self-injective algebras and indecomposable modules

We continue under the assumption that A is a finite dimensional self-injective algebra. We will show in Theorem 7.5.6 below that the variety of an indecomposable module is connected. For the proof, we will need some lemmas.

For each A-module M, let

$$M^{\#} = \operatorname{Hom}_A(M, A),$$

an A^{op}-module under the action $(af)(m) = f(am)$ for all $a \in A^{\mathrm{op}}$, $m \in M$, and $f \in \operatorname{Hom}_A(M, A)$. We may view $M^{\#}$ alternatively as a right A-module via $(fa)(m) = f(am)$, and consider the action of H on $\operatorname{Ext}_A^*(M^{\#}, M^{\#})$.

Lemma 7.5.1. *Let M be a finite dimensional A-module. Then $V_H(M) = V_H(M^{\#})$.*

The proof uses properties of the duality $(\ )^{\#}$ and adjoint functors, similarly to the proof of Lemma 7.2.4; see [**SS04**, Proposition 3.6].

Lemma 7.5.2. *Let η be a homogeneous element of H of positive degree n and let M be a finite dimensional A-module. Then $\eta \in I_H(M)$ if and only if $(\Omega_{A^e}^{-1} N_\eta \otimes_A M) \oplus P \cong M \oplus \Omega_A^{n-1} M \oplus P'$ for some projective A-modules P, P'.*

Proof. We may assume M is nonprojective, as the lemma is automatically true in case M is projective. Let $\eta \in H^n$, represented by an element of $\operatorname{Hom}_{A^e}(\Omega_{A^e}^n A, A) \cong \operatorname{Ext}_{A^e}^1(\Omega_{A^e}^{n-1} A, A)$, as in Theorem A.2.3. Apply the Snake Lemma (Lemma A.3.1) to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_\eta & \longrightarrow & \Omega_{A^e}^n A & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_{n-1} & \xrightarrow{=} & P_{n-1} & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

where $P_{\bullet} \to A$ is a minimal projective resolution of A as an A^e-module. We thus see that under the isomorphism $\operatorname{Hom}_{A^e}(\Omega_{A^e}^n A, A) \cong \operatorname{Ext}_{A^e}^1(\Omega_{A^e}^{n-1} A, A)$, η corresponds to an extension of $\Omega_{A^e}^{n-1} A$ by A as an A^e-module,

$$0 \to A \to \Omega_{A^e}^{-1} N_\eta \oplus P'' \to \Omega_{A^e}^{n-1} A \to 0, \tag{7.5.3}$$

for some projective A^e-module P''. Apply $- \otimes_A M$ to this short exact sequence, and since all modules are projective as right A-modules, we obtain a short exact sequence:

$$0 \to M \to (\Omega_{A^e}^{-1} N_\eta \otimes_A M) \oplus (P'' \otimes_A M) \to \Omega_{A^e}^{n-1} A \otimes_A M \to 0.$$

Now $\eta \in I_H(M)$ if and only if this sequence splits, that is if and only if

$$(\Omega_{A^e}^{-1} N_\eta \otimes_A M) \oplus (P'' \otimes_A M) \cong M \oplus (\Omega_{A^e}^{n-1} A \otimes_A M).$$

Note that $P'' \otimes_A M$ is projective as a left A-module, since P'' is projective as an A^e-module. The statement of the lemma now follows by the Krull-Schmidt Theorem since $\Omega_{A^e}^{n-1} A \otimes_A M$ and $\Omega_A^{n-1} M$ are the same up to projective direct summands. $\square$

Lemma 7.5.4. *Let M, N be finite dimensional A-modules. If*

$$\dim(V_H(M) \cap V_H(N)) = 0,$$

then $\operatorname{Ext}_A^i(M, N) = 0$ for all $i > 0$.

Proof. Since H is noetherian, $I_H(M)$ is finitely generated. Suppose $I_H(M) = (\eta_1, \dots, \eta_r)$. By Lemma 7.5.2 and induction on r, up to projective direct summands, the A-module M is a direct summand of

$$\Omega_{A^e}^{-1} N_{\eta_1} \otimes_A \cdots \otimes_A \Omega_{A^e}^{-1} N_{\eta_r} \otimes_A M.$$

So $\mathrm{Ext}^i_A(M,N)$ is a direct summand of

$$\mathrm{Ext}^i_A(\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r}\otimes_A M, N),$$

and this Ext space is isomorphic to

$$\mathrm{Ext}^i_A(M,(\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r})^{\#}\otimes_A N)$$

by an argument similar to the proof of Lemma 7.2.4. Also,

$$\begin{aligned}&\mathrm{Ext}^i_A(A/\mathfrak{r},(\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r})^{\#}\otimes_A N)\\ &\cong \mathrm{Ext}^i_A(\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r}\otimes_A A/\mathfrak{r}, N).\end{aligned}$$

So $V_H((\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r})^{\#}\otimes_A N)$ is contained in the intersection of the varieties $V_H(N)$ and $V_H((\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r})\otimes_A A/\mathfrak{r})$, by Lemma 7.3.3 and Proposition 7.3.4(iii). The latter is contained in $V_H(M)$ by Theorem 7.4.4, since $I_H(M)=(\eta_1,\ldots,\eta_r)$, so by hypothesis, the variety of $(\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r})^{\#}\otimes_A N$ has dimension 0. It follows that the module is projective by Lemma 7.4.1, and therefore it is injective, and so

$$\mathrm{Ext}^i_A(M,(\Omega^{-1}_{A^e}N_{\eta_1}\otimes_A\cdots\otimes_A\Omega^{-1}_{A^e}N_{\eta_r})^{\#}\otimes_A N)=0$$

for all $i>0$. The same is then true of the direct summand $\mathrm{Ext}^i_A(M,N)$. □

Lemma 7.5.5. *Let $\eta_1\in H^m$ and $\eta_2\in H^n$. There are projective A^e-modules P,Q for which there is a short exact sequence*

$$0\to\Omega^m_{A^e}N_{\eta_2}\oplus Q\to N_{\eta_1\eta_2}\oplus P\to N_{\eta_1}\to 0.$$

Proof. Consider the sequence $0\to N_{\eta_2}\to\Omega^n_{A^e}A\xrightarrow{\hat\eta_2}A\to 0$. Apply $\Omega^m_{A^e}A\otimes_A -$ to obtain

$$0\to\Omega^m_{A^e}A\otimes_A N_{\eta_2}\to\Omega^m_{A^e}A\otimes_A\Omega^n_{A^e}A\xrightarrow{\Omega^m(\hat\eta_2)}\Omega^m_{A^e}A\to 0,$$

which may be rewritten

$$0\to\Omega^m_{A^e}N_{\eta_2}\oplus Q\to\Omega^{m+n}_{A^e}A\oplus P\xrightarrow{\Omega^m(\hat\eta_2)}\Omega^m_{A^e}A\to 0$$

for some projective A^e-modules P, Q. Now $\widehat{\eta_1\eta_2} = \hat{\eta}_1\Omega^m(\hat{\eta}_2)$ by definition and so there is a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^m_{A^e} N_{\eta_2} \oplus Q & \longrightarrow & N_{\eta_1\eta_2} \oplus P & \longrightarrow & N_{\eta_1} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^m_{A^e} N_{\eta_2} \oplus Q & \longrightarrow & \Omega^{m+n}_{A^e} A \oplus P & \xrightarrow{\Omega^m(\hat{\eta}_2)} & \Omega^m_{A^e} A & \longrightarrow & 0 \\
 & & & & \downarrow{\widehat{\eta_1\eta_2}} & & \downarrow{\hat{\eta}_1} & & \\
 & & & & A & \xrightarrow{=} & A & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The desired sequence is the top row. $\square$

Theorem 7.5.6. *Let M be a finite dimensional A-module for which $V_H(M) = V_1 \cup V_2$ for some homogeneous varieties V_1 and V_2 with $\dim(V_1 \cap V_2) = 0$. Then there are A-modules M_1 and M_2 with $V_H(M_1) = V_1$ and $V_H(M_2) = V_2$ and*

$$M \cong M_1 \oplus M_2.$$

Thus the support variety of an indecomposable module is connected.

Proof. Let $m_1 = \dim V_1$ and $m_2 = \dim V_2$. We will prove the statement by induction on $m_1 + m_2$. If $m_1 = 0$ or $m_2 = 0$, the result is clear, so assume $m_1 > 0$ and $m_2 > 0$. Then there are homogeneous elements η_1 and η_2 of H such that $V_1 \subset V_H(\eta_1)$, $V_2 \subset V_H(\eta_2)$, and

$$\dim(V_2 \cap V_H(\eta_1)) = m_2 - 1, \quad \dim(V_1 \cap V_H(\eta_2)) = m_1 - 1.$$

Now $V_H(\eta_1\eta_2) = V_H(\eta_1) \cup V_H(\eta_2) \supset V_1 \cup V_2 = V_H(M)$, so $(\eta_1\eta_2)^s \in I_H(M)$ for some s. We may assume $\eta_1\eta_2 \in I_H(M)$ by replacing each η_i with η_i^s if $s > 1$. By Lemma 7.5.2,

$$\left(\Omega^{-1}_{A^e} N_{\eta_1\eta_2} \otimes_A M\right) \oplus P \cong M \oplus \Omega^{n-1}_A M \oplus P'$$

for some projective A-modules P, P'. Noting that Ω_A applied to an A-module is the same as applying $\Omega_{A^e} A \otimes_A -$ up to projective direct summands, and similarly for Ω^{-1}_A, $\Omega^{-1}_{A^e} A \otimes_A -$, we have

$$(N_{\eta_1\eta_2} \otimes_A M) \oplus P_1 \cong \Omega_A M \oplus \Omega^n_A M \oplus P_2$$

for some projective A-modules P_1, P_2. By Lemma 7.5.5, there is a short exact sequence

$$0 \to \Omega^m N_{\eta_2} \oplus Q \to N_{\eta_1\eta_2} \oplus Q' \to N_{\eta_1} \to 0$$

for some projective A^e-modules Q, Q'. Apply $-\otimes_A M$ to this sequence to obtain

$$0 \to (\Omega^m_{A^e} N_{\eta_2} \otimes_A M) \oplus (Q \otimes_A M) \to (N_{\eta_1\eta_2} \oplus Q') \otimes_A M \to N_{\eta_1} \otimes_A M \to 0.$$

Replacing $N_{\eta_1\eta_2} \otimes_A M$ via the above isomorphism, after taking a direct sum with P_1, in the second two of the three above modules, we have a short exact sequence

(7.5.7)

$$0 \to (\Omega^m_{A^e} N_{\eta_2} \otimes_A M) \oplus (Q \otimes_A M) \to \Omega_A M \oplus \Omega^n_A M \oplus P_2 \oplus (Q' \otimes_A M) \to (N_{\eta_1} \otimes_A M) \oplus P_1 \to 0.$$

Since $Q \otimes_A M$ is A-projective,

$$V_H((\Omega^m_{A^e} N_{\eta_2} \otimes_A M) \oplus (Q \otimes_A M)) = V_H(\Omega^m_A(N_{\eta_2} \otimes_A M)).$$

By Proposition 7.3.4(ii), up to a finite number of points, $V_H(\Omega^m_A(N_{\eta_2} \otimes_A M))$ is the same as $V_H(N_{\eta_2} \otimes_A M)$, and so these two varieties have the same dimension. Now by Theorem 7.4.4,

$$\begin{aligned} V_H(N_{\eta_2} \otimes_A M) &= V_H(\eta_2) \cap V_H(M) \\ &= V_H(\eta_2) \cap (V_1 \cup V_2) \\ &= (V_H(\eta_2) \cap V_1) \cup V_2, \end{aligned}$$

as $V_2 \subset V_H(\eta_2)$. The intersection of the varieties $V_H(\eta_2) \cap V_1$ and V_2 is contained in $V_1 \cap V_2$, and thus has dimension 0. Further, the sum of the dimensions of these two varieties is $m_1 - 1 + m_2$. By the induction hypothesis, $(\Omega^m_{A^e} N_{\eta_2} \otimes_A M) \oplus (Q \otimes_A M) \cong N_1 \oplus N_2$ for A-modules N_1, N_2 with $V_H(N_1) = V_H(\eta_2) \cap V_1$ and $V_H(N_2) = V_2$. We also find that by Theorem 7.4.4 similarly,

$$V_H((N_{\eta_1} \otimes_A M) \oplus P_1) = V_H(\eta_1) \cap V_H(M) = V_1 \cup (V_H(\eta_1) \cap V_2),$$

and so $(N_{\eta_1} \otimes_A M) \oplus P_1 \cong N_1' \oplus N_2'$ for A-modules N_1', N_2' with $V_H(N_1') = V_1$ and $V_H(N_2') = V_H(\eta_1) \cap V_2$. Thus the sequence (7.5.7) may be rewritten

$$0 \to N_1 \oplus N_2 \to X \to N_1' \oplus N_2' \to 0 \qquad (7.5.8)$$

for $X = \Omega_A M \oplus \Omega^n_A M \oplus P_2 \oplus (Q' \otimes_A M)$.

Now $V_H(N_1') \cap V_H(N_2) = V_1 \cap V_2$, which has dimension 0, and so by Lemma 7.5.4, $\mathrm{Ext}^i_A(N_1', N_2) = 0$ for all $i > 0$. Similarly, $V_H(N_1) \cap V_H(N_2') \subset V_1 \cap V_2$, and so has dimension 0, so $\mathrm{Ext}^i_A(N_2', N_1) = 0$ for all $i > 0$. Then

$$\mathrm{Ext}^1_A(N_1' \oplus N_2', N_1 \oplus N_2) \cong \mathrm{Ext}^1_A(N_1', N_1) \oplus \mathrm{Ext}^1_A(N_2', N_2)$$

and so (7.5.8) is a direct sum of two sequences $0 \to N_1 \to N_1'' \to N_1' \to 0$ and $0 \to N_2 \to N_2'' \to N_2' \to 0$. Moreover, $V_H(N_1'') \subset V_1$ and $V_H(N_2'') \subset V_2$. Rewriting X as a direct sum:

$$\Omega_A M \oplus \Omega^n_A M \oplus P_2 \oplus (Q' \otimes_A M) \cong N_1'' \oplus N_2''.$$

Now $Q' \otimes_A M$ is a projective A-module, so applying Ω_A^{-1} we find

$$M \oplus \Omega_A^{n-1} M \oplus Q'' \cong \Omega_A^{-1}(N_1'') \oplus \Omega_A^{-1}(N_2'') \oplus Q'''$$

for some projective A-modules Q'', Q'''. By the Krull-Schmidt Theorem, $M \cong M_1 \oplus M_2$ for some M_1 and M_2 with $V_H(M_1) \subset V_H(N_1'') \subset V_1$ and $V_H(M_2) \subset V_H(N_2'') \subset V_2$. By hypothesis, $V_H(M) = V_1 \cup V_2$, and this forces $V_H(M_1) = V_1$ and $V_H(M_2) = V_2$. □

As a final topic, we briefly discuss periodic indecomposable modules.

Definition 7.5.9. An A-module M is *periodic* if $\Omega_A^i M \cong M$ as A-modules for some i.

Example 7.5.10. The trivial module k for $A = k[x]/(x^n)$ is periodic, as can be seen from our work in Example 1.6.7. If $n = 2$, then the period is $i = 1$, and if $n > 2$, the period is $i = 2$.

We have the following result characterizing indecomposable periodic modules.

Theorem 7.5.11. *Let A be a finite dimensional self-injective algebra satisfying condition (fg). Let M be a finite dimensional indecomposable A-module. Then M is periodic if and only if* $\dim V_H(M) = 1$.

For a proof, see [**EHS**$^+$**04**, Proposition 4.2 and Theorem 4.3].

There are many further applications of support varieties. For example, all modules in a connected stable component of the Auslander-Reiten quiver have the same variety [**SS04**, Theorem 3.7]. There is a connection to representation type through complexity of modules [**EHS**$^+$**04**, Section 5]. For algebras A whose category of modules is a tensor category, such as when A is a Hopf algebra, in the best known settings the structure of the tensor category given by its tensor ideals can be understood using support variety theory.

Chapter 8

Hopf Algebras

Hopf algebras are an important class of algebras that include group algebras, universal enveloping algebras of Lie algebras, and quantum groups. In this chapter, we collect some techniques for understanding homological information about Hopf algebras. We introduce Hopf algebra cohomology and prove that it embeds as an algebra into Hochschild cohomology. Such embeddings have many applications, and we will give details about just a few here: They aid understanding of support varieties of modules for Hopf algebras as defined in Chapter 7. They can lead to formulas for cup products on Hochschild cohomology in terms of cup products on Hopf algebra cohomology; we present the case of a finite group algebra in Section 8.5. We more generally consider smash product algebras built from Hopf algebras and rings on which they act, and use an embedding in Section 8.6 to construct a spectral sequence relating the Hochschild cohomology of a smash product to cohomology of its components.

8.1. Hopf algebras and actions on rings

In this section we give a brief introduction to Hopf algebras and smash product algebras. For details, see [**Mon93, Rad12, Sch06**].

Let A be an algebra over the field k. Let $\pi : A \otimes A \to A$ denote its multiplication map and let $\eta : k \to A$ denote its unit map (that is, the embedding of the field k into A as scalar multiples of the multiplicative identity). We consider $A \otimes A$ to be an algebra with factorwise multiplication: $(a \otimes b)(c \otimes d) = ac \otimes bd$ for all $a, b, c, d \in A$. In the following definition, we canonically identify the spaces $k \otimes A$ and $A \otimes k$ with A.

Definition 8.1.1. A *Hopf algebra* is an algebra A over k together with algebra homomorphisms $\Delta : A \to A \otimes A$ (called *coproduct* or *comultiplication*) and $\varepsilon : A \to k$ (called *counit* or *augmentation*) and an algebra anti-homomorphism (that is, reversing order of multiplication) $S : A \to A$ (called *coinverse* or *antipode*) such that

$$\begin{aligned} (\Delta \otimes 1)\Delta &= (1 \otimes \Delta)\Delta, \\ (\varepsilon \otimes 1)\Delta &= 1 = (1 \otimes \varepsilon)\Delta, \\ \pi(S \otimes 1)\Delta &= \eta\varepsilon = \pi(1 \otimes S)\Delta, \end{aligned}$$

where 1 denotes the identity map on A. We say that A is *cocommutative* if $\tau\Delta = \Delta$, where $\tau : A \otimes A \to A \otimes A$ is the map that interchanges tensor factors, that is $\tau(a \otimes b) = b \otimes a$ for all $a, b \in A$.

It can be shown that S is an anti-coalgebra homomorphism, that is, $\Delta S = (S \otimes S)\tau\Delta$ and $\varepsilon S = \varepsilon$.

Some examples of Hopf algebras are the following.

Example 8.1.2. (Group algebras.) Let G be a group and $A = kG$, its group algebra. Let $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$. Then A is a cocommutative Hopf algebra.

Example 8.1.3. (Universal enveloping algebras of Lie algebras.) Let $\mathfrak{g}$ be a Lie algebra and let $A = U(\mathfrak{g})$, its universal enveloping algebra. That is,

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}).$$

Let $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$ for all $x \in \mathfrak{g}$. The maps Δ and ε are extended to be algebra homomorphisms, and S to be an algebra anti-homomorphism. Then A is a cocommutative Hopf algebra.

Example 8.1.4. (Quantum enveloping algebras.) Let $A = U_q(\mathfrak{g})$, a quantum enveloping algebra. See, for example, [**GK93**] for the definition in the general case. Here we give just one small example explicitly: Let q be a nonzero scalar, $q^2 \neq 1$. Let $U_q(\mathfrak{sl}_2)$ be the $\mathbb{C}$-algebra generated by E, F, K with $KE = q^2 EK$, $KF = q^{-2} FK$, and

$$EF = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$

Let $\Delta(E) = E \otimes 1 + K \otimes E$, $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$, $\Delta(K) = K \otimes K$, $\varepsilon(E) = 0$, $\varepsilon(F) = 0$, $\varepsilon(K) = 1$, $S(E) = -K^{-1}E$, $S(F) = -FK$, and $S(K) = K^{-1}$. Then A is a noncocommutative Hopf algebra. If q is a primitive complex nth root of unity, $n > 2$, we may consider the quotient of $U_q(\mathfrak{sl}_2)$ by the ideal generated by E^n, F^n, $K^n - 1$, denoted by $u_q(\mathfrak{sl}_2)$. This is a finite dimensional (that is, finite dimensional as a vector space) noncocommutative Hopf algebra, called a small quantum group.

Example 8.1.5. (Quantum elementary abelian groups.) Let m and n be positive integers, $n \geq 2$. Let q be a primitive complex nth root of unity, and let A be the $\mathbb{C}$-algebra generated by $x_1, \dots, x_m, g_1, \dots, g_m$ with relations $x_i^n = 0$, $g_i^n = 1$, $x_i x_j = x_j x_i$, $g_i g_j = g_j g_i$, and $g_i x_j = q^{\delta_{i,j}} x_j g_i$ for all i, j. Let $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$, $\Delta(g_i) = g_i \otimes g_i$, $\varepsilon(x_i) = 0$, $\varepsilon(g_i) = 1$, and $S(x_i) = -g_i^{-1} x_i$, $S(g_i) = g_i^{-1}$ for all i. Then A is a finite dimensional noncocommutative Hopf algebra.

We will use some standard notation for the coproduct, called *Sweedler notation*: Write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2,$$

or simply $\sum a_1 \otimes a_2$, where the notation a_1, a_2 for tensor factors is symbolic. Some authors dispense with the summation symbol, writing $a_1 \otimes a_2$ to denote this sum.

If A is a finite dimensional Hopf algebra, then $A^* = \mathrm{Hom}_k(A, k)$ is also a Hopf algebra under the duals of the defining maps of A. That is, identifying $(A \otimes A)^*$ canonically with $A^* \otimes A^*$, multiplication on A^* is Δ^*, comultiplication is μ^*, the unit map is ε^*, the counit map is η^*, and the antipode is S^*. If A is an infinite dimensional Hopf algebra, there are meaningful dual Hopf algebras such as the finite dual (see [**Mon93**]). If A and B are Hopf algebras, then $A \otimes B$ is a Hopf algebra with factorwise product, coproduct, and other maps, and A^{op} is a Hopf algebra with the same coproduct, counit, and antipode as A. In this way, $A^e = A \otimes A^{\mathrm{op}}$ is also a Hopf algebra.

An element h of a Hopf algebra A is a *left integral* of A if $a \cdot h = \varepsilon(a) h$ for all $a \in A$. A right integral is defined similarly. It can be shown that the spaces of left and right integrals of a finite dimensional Hopf algebra are one dimensional and are interchanged by the antipode. Maschke's Theorem for Hopf algebras states that a finite dimensional Hopf algebra A is semisimple if, and only if, $\varepsilon(h) \neq 0$ for a nonzero left (respectively, right) integral h.

A finite dimensional Hopf algebra is a Frobenius algebra, and therefore self-injective. Specifically, a nonzero left integral λ in the dual Hopf algebra A^* is a Frobenius homomorphism of A. In particular, the left A-module A is isomorphic to the left A-module given by the dual of the right A-module A, via the map that takes a to ϕ_a where $\phi_a(b) = \lambda(ba)$ for all $a, b \in A$.

The quantum elementary abelian groups in Example 8.1.5 are examples of skew group rings as defined in Section 2.5. We generalize skew group rings by defining smash products next.

Definition 8.1.6. Let A be a Hopf algebra. An A-*module algebra* is an algebra R over k that is also an A-module for which

$$\begin{aligned} a \cdot 1 &= \varepsilon(a)1, \\ a \cdot (rr') &= \sum (a_1 \cdot r)(a_2 \cdot r') \end{aligned}$$

for all $a \in A$ and $r, r' \in R$. The *smash product* $R\#A$ of A with R is an algebra that is $R \otimes A$ as a vector space, with multiplication defined by

$$(r \otimes a)(r' \otimes a') = \sum r(a_1 \cdot r') \otimes a_2 a'$$

for all $a, a' \in A$ and $r, r' \in R$.

By definition, R and A are both subalgebras of the smash product $R\#A$. So it should cause no confusion if we sometimes abuse notation and write r for $r \otimes 1$ and a for $1 \otimes a$ as elements of $R\#A$, for $r \in R$ and $a \in A$.

A more general construction than the smash product is a crossed product involving cocycles, and both are examples of the yet more general Hopf Galois extensions. We will not need these more general notions.

We rewrite the action of A on R as an internal action within the smash product $R\#A$: For each $a \in A$ and $r \in R$,

$$\begin{aligned} (a \cdot r) \otimes 1 &= \sum ((a_1 \varepsilon(a_2)) \cdot r) \otimes 1 &= \sum a_1 \cdot r \otimes \varepsilon(a_2) \\ & &= \sum a_1 \cdot r \otimes a_2 S(a_3) \\ & &= \sum (1 \otimes a_1)(r \otimes 1)(1 \otimes S(a_2)). \end{aligned}$$

We call this an *adjoint action* of A, by analogy to the adjoint action of a Lie algebra.

The quantum elementary abelian groups of Example 8.1.5 are smash products of $R = k[x_1, \dots, x_m]/(x_1^n, \dots, x_m^n)$ with $A = kG$ where $G = \langle g_1, \dots, g_m \rangle$ is a direct product of cyclic groups $\langle g_i \rangle$, each of order n, acting by $g_i \cdot x_j = q^{\delta_{ij}} x_j$. The skew group algebras of Section 2.5 are all smash product algebras. For an example in which the Hopf algebra is not a group algebra, take $A = U_q(\mathfrak{sl}_2)$ from Example 8.1.4, $R = k\langle x, y \rangle/(xy - qyx)$, and $E \cdot x = 0$, $E \cdot y = x$, $F \cdot x = y$, $F \cdot y = 0$, $K^{\pm 1} \cdot u = q^{\pm 1} u$, $K^{\pm 1} \cdot v = q^{\mp 1} v$. Then R is an A-module algebra, and we may form the smash product algebra $A\#R$.

8.2. Modules for Hopf algebras

Modules for Hopf algebras have some important properties: Tensor products (over k) of modules are modules, and Homs (over k) of modules are modules. These structures make categories of A-modules into tensor categories [**EGNO15**]. In this section, we define these module structures and relationships among them.

We work primarily with left modules as before, and the term module will mean left module if not otherwise specified. However, we will also need right modules in Section 8.6, and we include some discussion in this section of the properties of right modules that we will need there.

Given a Hopf algebra A, the field k is an A-module via the counit ε, sometimes called the *trivial module*, that is, $a \cdot c = \varepsilon(a)c$ for all $a \in A$, $c \in k$. For any two A-modules V and W, their vector space tensor product $V \otimes W$ is an A-module via the coproduct Δ:

$$a \cdot (v \otimes w) = \sum (a_1 \cdot v) \otimes (a_2 \cdot w)$$

for all $a \in A$, $v \in V$, and $w \in W$. Similarly, $\mathrm{Hom}_k(V, W)$ is an A-module via

$$(a \cdot f)(v) = \sum a_1 f(S(a_2)v)$$

for all $a \in A$, $f \in \mathrm{Hom}_k(V, W)$, and $v \in V$. In particular, if V is an A-module, its dual vector space $V^* = \mathrm{Hom}_k(V, k)$ has an A-module structure given by

$$(a \cdot f)(v) = f(S(a)v)$$

for all $a \in A$, $v \in V$, and $f \in V^*$. Note that the tensor product of the trivial module k with V is isomorphic to V: $k \otimes V \cong V$ and $V \otimes k \cong V$ due to the properties of the maps Δ, ε.

For any A-module V, let V^A denote the A-submodule of V on which A acts via ε, that is,

$$V^A = \{v \in V \mid a \cdot v = \varepsilon(a)v \text{ for all } a \in A\},$$

called the submodule of A-*invariants* of V. We will use the same notation for A-invariants of a right A-module.

Lemma 8.2.1. *Let V, W be A-modules. There is an isomorphism of vector spaces*

$$\mathrm{Hom}_A(V, W) \cong (\mathrm{Hom}_k(V, W))^A.$$

Proof. This is [**Sch06**, Lemma 4.1]. Let $f \in \mathrm{Hom}_A(V, W)$. Then

$$(a \cdot f)(v) \;=\; \sum a_1 f(S(a_2)v) \;=\; \sum a_1 S(a_2) f(v) \;=\; \varepsilon(a) f(v)$$

for all $a \in A$, $v \in V$. So $f \in (\mathrm{Hom}_k(V,W))^A$. Conversely, let $f \in (\mathrm{Hom}_k(V,W))^A$. Then

$$\begin{aligned} f(a\cdot v) &= \sum f(\varepsilon(a_1)a_2\cdot v) &= \sum \varepsilon(a_1)f(a_2\cdot v) \\ & &= \sum a_1 f(S(a_2)\cdot(a_3\cdot v)) \\ & &= \sum a_1 f(\varepsilon(a_2)v) \\ & &= \sum a_1\varepsilon(a_2)f(v) \quad = \quad a\cdot(f(v)) \end{aligned}$$

for all $a \in A$, $v \in V$. So $f \in \mathrm{Hom}_A(V,W)$. □

For a Hopf algebra having a bijective antipode, there is an alternative action on Hom, as we describe in the next remark. All finite dimensional Hopf algebras have bijective antipodes, as well as many infinite dimensional Hopf algebras.

Remark 8.2.2. Under the assumption that the antipode S is bijective with inverse map $\overline{S}$, we may alternatively define a dual module to V as ${}^*V = \mathrm{Hom}_k(V,k)$ with action $(a\cdot f)(v) = f(\overline{S}(a)v)$ for all $a \in A$, $v \in V$, and $f \in {}^*V$. Similarly give $\mathrm{Hom}_k(V,W)$ the alternative A-module structure $(a\cdot f)(v) = \sum a_2 f(\overline{S}(a_1)v)$ for all $a \in A$, $f \in \mathrm{Hom}_k(V,W)$, $v \in V$. Lemma 8.2.1 holds under this alternative action, that is, the subspace of A-homomorphisms in $\mathrm{Hom}_k(V,W)$ is also equal to the A-invariant subspace of $\mathrm{Hom}_k(V,W)$ under this alternative action. To see this, apply S to $\sum a_2\overline{S}(a_1)$ to obtain $\sum a_1 S(a_2) = \varepsilon(a)$, implying that $\sum a_2\overline{S}(a_1) = \varepsilon(a)$ for all $a \in A$.

There are right module versions of all of these actions as well:

Remark 8.2.3. If V, W are right A-modules, then $\mathrm{Hom}_k(V,W)$ is a right A-module via $(f\cdot a)(v) = \sum f(vS(a_1))a_2$ for all $f \in \mathrm{Hom}_A(V,W)$, $a \in A$, $v \in V$. There is a right module version of Lemma 8.2.1. If S is bijective, then $\mathrm{Hom}_k(V,W)$ is also a right A-module via $(f\cdot a)(v) = \sum f(v\overline{S}(a_2))a_1$ for all $f \in \mathrm{Hom}_k(V,W)$, $a \in A$, $v \in V$, and again there is a corresponding version of Lemma 8.2.1.

We next establish relations among the A-modules obtained by taking Hom, $\otimes$, and duals.

Lemma 8.2.4. *Let U, V, and W be A-modules. There is a natural isomorphism of A-modules*

$$\mathrm{Hom}_k(U\otimes V, W) \cong \mathrm{Hom}_k(U, \mathrm{Hom}_k(V,W)),$$

and a natural isomorphism of vector spaces

$$\mathrm{Hom}_A(U\otimes V, W) \cong \mathrm{Hom}_A(U, \mathrm{Hom}_k(V,W)).$$

Proof. Define functions $\phi : \mathrm{Hom}_k(U \otimes V, W) \to \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$ by

$$(\phi(f)(u))(v) = f(u \otimes v),$$

and $\psi : \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \to \mathrm{Hom}_k(U \otimes V, W)$ by

$$(\psi(g))(u \otimes v) = (g(u))(v).$$

By its definition, ψ is inverse to ϕ. We check that ϕ is an A-module homomorphism. Let $a \in A$ and $f \in \mathrm{Hom}_k(U \otimes V, W)$. Then, as S reverses the order of comultiplication, for all $u \in U$ and $v \in V$,

$$\begin{aligned}(\phi(a \cdot f)(u))(v) &= (a \cdot f)(u \otimes v)\\ &= \sum a_1(f(S(a_2) \cdot (u \otimes v)))\\ &= \sum a_1(f(S(a_3)u \otimes S(a_2)v)).\end{aligned}$$

On the other hand,

$$\begin{aligned}(a \cdot \phi(f))(u)(v) &= \sum (a_1(\phi(f)(S(a_2)u)))(v)\\ &= \sum a_1((\phi(f))(S(a_3)u)(S(a_2)v)\\ &= \sum a_1(f(S(a_3)u \otimes S(a_2)v)).\end{aligned}$$

Therefore $\phi(a \cdot f) = a \cdot \phi(f)$.

The second statement now follows by Lemma 8.2.1. □

Remark 8.2.5. A similar proof shows that if U, V, W are right A-modules, then there is a right A-module isomorphism

$$\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W))$$

under the first action defined in Remark 8.2.3. Taking A-invariants, we obtain an isomorphism

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(V, \mathrm{Hom}_k(U, W)).$$

Lemma 8.2.6. *Let V, W be A-modules. If V is finite dimensional as a vector space over k, there is an A-module isomorphism*

$$\mathrm{Hom}_k(V, W) \cong W \otimes V^*.$$

Proof. Let $\phi : W \otimes V^* \to \mathrm{Hom}_k(V, W)$ and $\psi : \mathrm{Hom}_k(V, W) \to W \otimes V^*$ be defined by $(\phi(w \otimes f))(v) = f(v)w$ and $\psi(f) = \sum_i f(v_i) \otimes v_i^*$, where $\{v_i\}, \{v_i^*\}$ are dual bases for V, V^*, for all $v \in V$, $w \in W$, and $f \in V^*$. Let $a \in A$. Then

$$\begin{aligned}\phi(a \cdot (w \otimes f))(v) &= \sum \phi(a_1 w \otimes (a_2 \cdot f))(v)\\ &= \sum ((a_2 \cdot f)(v))(a_1 w) \quad = \quad \sum f(S(a_2)v) a_1 w.\end{aligned}$$

On the other hand,

$$\begin{aligned}(a\cdot(\phi(w\otimes f)))(v) &= \sum a_1(\phi(w\otimes f)(S(a_2)v))\\ &= \sum a_1(f(S(a_2)v)w) \;=\; \sum f(S(a_2)v)a_1w.\end{aligned}$$

Therefore $\phi(a\cdot(w\otimes f)) = a\cdot(\phi(w\otimes f))$. One may check that ϕ is inverse to ψ. □

Remark 8.2.7. Under the alternative A-module structure described in Remark 8.2.2, there are A-module isomorphisms

$$\mathrm{Hom}_k(V,W)\cong {}^*V\otimes W$$

and

$$\mathrm{Hom}_k(U\otimes V,W)\cong \mathrm{Hom}_k(V,\mathrm{Hom}_k(U,W)).$$

Now consider only finite dimensional A-modules. This observation combined with Lemma 8.2.6 implies that V^* is a left dual of V in the category of finite dimensional A-modules and ${}^*V$ is a right dual of V in this category (see [**EGNO15**]). It follows that ${}^*(V^*)\cong V$ and $({}^*V)^*\cong V$ for all V in this category; these isomorphisms can also be deduced directly from the definitions.

Lemma 8.2.8. *Let P be a projective A-module, and V any A-module. Then $P\otimes V$ is a projective A-module. If the antipode S is bijective, then $V\otimes P$ is a projective A-module. Similar statements hold for right modules, as well as when "projective" is replaced by "flat".*

Proof. We give two proofs of the first two statements. The first proof is essentially that of [**Ben91a**, Proposition 3.1.5]: The projective module P is a direct summand of a free module, so it suffices to prove that $A\otimes V$ and $V\otimes A$ are both free as A-modules. There is an isomorphism $A\otimes V \xrightarrow{\sim} A\otimes V_{tr}$, where V_{tr} is the underlying vector space of V, but with the trivial A-module structure (via ε). This isomorphism is similar to one in [**Mon93**, Theorem 1.9.4], and is given by $a\otimes v\mapsto \sum a_1\otimes S(a_2)v$, the inverse function by $a\otimes v\mapsto\sum a_1\otimes a_2v$, for all $v\in V$, $a\in A$. Now V_{tr} is a direct sum of copies of the trivial module k, and so $A\otimes V_{tr}$ is a free A-module. Similarly, there is an isomorphism of left A-modules, $V\otimes A\xrightarrow{\sim} V_{tr}\otimes A$, via the A-module homomorphism $v\otimes a\mapsto\sum \overline{S}(a_1)v\otimes a_2$ whose inverse is $v\otimes a\mapsto\sum a_1v\otimes a_2$, and so $V\otimes A$ is a free A-module.

The second proof uses properties of functors: As V is projective over the field k and P is projective over A, $\mathrm{Hom}_A(P,\mathrm{Hom}_k(V,-))$ is an exact functor. By Lemma 8.2.4, this is the same as $\mathrm{Hom}_A(P\otimes V,-)$. Therefore $P\otimes V$ is projective. A similar argument applies to $V\otimes P$, using Remark 8.2.7 and the alternative A-action on Hom given in Remark 8.2.2.

The last statement, for right modules, may be proven similarly, and the statement about flat modules follows since flat modules are direct limits of finitely generated free modules. See also [**EGNO15**]. □

8.3. Hopf algebra cohomology and actions on Ext

In this section we define Hopf algebra cohomology and derive many properties of Ext for modules of a Hopf algebra A. We will connect these ideas with Hochschild cohomology in the next section.

Lemma 8.3.1. *Let U, V, W be A-modules. There is an isomorphism of graded vector spaces,*

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(U, W \otimes V^*).$$

Proof. Let $P_\bullet$ be a projective resolution of the A-module U. By Lemma 8.2.8, $P_\bullet \otimes V$ is a projective resolution of $U \otimes V$. The natural isomorphisms of Lemmas 8.2.4 and 8.2.6 yield a chain homotopy equivalence between $\mathrm{Hom}_A(P_\bullet, W \otimes V^*)$ and $\mathrm{Hom}_A(P_\bullet \otimes V, W)$, and thus an isomorphism on Ext as claimed. □

Remark 8.3.2. Using the alternative A-action on Hom described in Remark 8.2.2, we similarly find that

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(V, {}^*U \otimes W).$$

Now let M, M', N, N' be A-modules. We will define a cup product for each $i, j \geq 0$ as in [**Ben91a**, §3.2],

$$\smile : \mathrm{Ext}^i_A(M, M') \times \mathrm{Ext}^j_A(N, N') \longrightarrow \mathrm{Ext}^{i+j}_A(M \otimes N, M' \otimes N').$$

Let $P_\bullet$ be a projective resolution of M, and let $Q_\bullet$ be a projective resolution of N. Consider the total complex of the tensor product complex $P_\bullet \otimes Q_\bullet$, with action of A on each $P_i \otimes Q_j$ given by the coproduct Δ as described in Section 8.2. Note that the differentials are A-module homomorphisms since the differentials on $P_\bullet$ and on $Q_\bullet$ are A-module homomorphisms. By Lemma 8.2.8, each module in this tensor product complex is projective. By the Künneth Theorem (Theorem A.4.1), since the tensor product is over the field k and Tor^k_1 is 0, $P_\bullet \otimes Q_\bullet$ is a projective resolution of the A-module $M \otimes N$.

Let $f \in \mathrm{Hom}_A(P_i, M')$, $g \in \mathrm{Hom}_A(Q_j, N')$ represent elements in the spaces $\mathrm{Ext}^i_A(M, M')$, $\mathrm{Ext}^j_A(N, N')$, respectively. Then

$$f \otimes g \in \mathrm{Hom}_A(P_i \otimes Q_j, M' \otimes N')$$

and this function may be extended to an element of

$$\operatorname{Hom}_A\left(\bigoplus_{r+s=i+j}(P_r \otimes Q_s), M' \otimes N'\right)$$

by defining it to be the zero map on all components other than $P_i \otimes Q_j$. By definition, $d(f \otimes g) = d(f) \otimes g + (-1)^{|f|} f \otimes d(g)$. It follows that if f and g are cocycles, then their tensor product $f \otimes g$ is again a cocycle, and the tensor product of a cocycle with a coboundary is a coboundary. Therefore this induces a well-defined product on cohomology, the cup product $\smile$.

We will need the following result, stated as [**Ben91a**, Proposition 3.2.1].

Lemma 8.3.3. *If M, M', N, N' are left A-modules and $\zeta \in \operatorname{Ext}_A^m(M, M')$, $\eta \in \operatorname{Ext}_A^n(N, N')$, then the cup product*

$$\zeta \smile \eta \in \operatorname{Ext}_A^{m+n}(M \otimes N, M' \otimes N')$$

is equal to the Yoneda composite of

$$\zeta \otimes 1_{N'} \quad \text{and} \quad 1_M \otimes \eta$$

in $\operatorname{Ext}_A^m(M \otimes N', M' \otimes N')$ and $\operatorname{Ext}_A^n(M \otimes N, M \otimes N')$, respectively.

Proof. Let

$$E: \qquad 0 \longrightarrow M' \longrightarrow M_{m-1} \longrightarrow \cdots M_0 \longrightarrow M \longrightarrow 0,$$

$$F: \qquad 0 \longrightarrow N' \longrightarrow N_{n-1} \longrightarrow \cdots N_0 \longrightarrow N \longrightarrow 0$$

be an m- and an n-extension representing ζ and η, respectively. Their cup product is represented by the total complex of the following tensor product, which is an $(m+n)$-extension of $M \otimes N$ by $M' \otimes N'$, as a consequence of the Künneth Theorem (Theorem A.4.1), since the tensor product is over the field k:

$$\begin{array}{ccccccc}
M_0 \otimes N' & \longleftarrow & M_1 \otimes N' & \longleftarrow \cdots \longleftarrow & M_{m-1} \otimes N' & \longleftarrow & M' \otimes N' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_0 \otimes N_{n-1} & \longleftarrow & M_1 \otimes N_{n-1} & \longleftarrow \cdots \longleftarrow & M_{m-1} \otimes N_{n-1} & \longleftarrow & M' \otimes N_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_0 \otimes N_1 & \longleftarrow & M_1 \otimes N_1 & \longleftarrow \cdots \longleftarrow & M_{m-1} \otimes N_1 & \longleftarrow & M' \otimes N_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_0 \otimes N_0 & \longleftarrow & M_1 \otimes N_0 & \longleftarrow \cdots \longleftarrow & M_{m-1} \otimes N_0 & \longleftarrow & M' \otimes N_0
\end{array}$$

There is a map from the total complex of this bicomplex to the Yoneda composite of $E \otimes N'$ with $M \otimes F$: Project the bicomplex onto the leftmost column followed by the map $M_0 \to M$ and onto the top row. Specifically, in degrees $i+j$ with $0 \le i+j \le n-1$, send $M_i \otimes N_j$ to 0 if $i > 0$ and to $M \otimes N_j$ as a projection from $M_0 \otimes N_j$ if $i = 0$. In degrees $i+j$ with $n \le i+j \le m+n-1$, send $M_i \otimes N_j$ to 0 if $j < n$ and to $M_{i+j-n} \otimes N'$ if $j = n$. One checks that this is a chain map. □

Definition 8.3.4. The *Hopf algebra cohomology* of the Hopf algebra A over the field k is

$$\mathrm{H}^*(A,k) = \mathrm{Ext}^*_A(k,k).$$

More generally, we write $\mathrm{H}^*(A,M) = \mathrm{Ext}^*_A(k,M)$ for any A-module M.

We could equally well have defined Hopf algebra cohomology via right A-modules, and we will use this right module version in Section 8.6. Left and right modules are interchanged by applying the antipode, since it is an algebra anti-homomorphism.

By Lemma 8.2.1 and the definitions, in degree 0 we have

$$\mathrm{H}^0(A,M) \cong \mathrm{Hom}_A(k,M) \cong (\mathrm{Hom}_k(k,M))^A \cong M^A.$$

Remark 8.3.5. Letting $M = M' = N = N' = k$ in Lemma 8.3.3, since $k \otimes k \cong k$, the cup product (equivalently, Yoneda composition) gives $\mathrm{H}^*(A,k)$ the structure of a graded ring. Another proof that the cup product is the same as the Yoneda product is the Eckmann-Hilton argument, which simultaneously shows that the product is graded-commutative, that is

$$\alpha \smile \beta = (-1)^{|\alpha||\beta|} \beta \smile \alpha$$

for all $\alpha, \beta \in \mathrm{H}^*(A,k)$. See Suarez-Alvarez [**SA04**] for a general context for this type of argument, which does not require cocommutativity of A (cf. [**Ben91a**, Cor. 3.2.2]). More generally, when $M = N = k$ and $M' = N' = B$ is an A-module algebra, we may compose the cup product with the map induced by multiplication $B \otimes B \to B$ to obtain a ring structure on $\mathrm{H}^*(A,B)$. In the next section, we will let B be the algebra A itself, under the adjoint action of A, as defined there.

Example 8.3.6. Let G be a group and let $A = kG$, the group algebra. Then $\mathrm{H}^*(kG,k) = \mathrm{Ext}^*_{kG}(k,k)$ is the group cohomology of G with coefficients in k, and the cup product gives it the structure of a graded commutative ring. As a small example, let p be a prime, let k be a field of characteristic p, and $G = \mathbb{Z}/p\mathbb{Z}$. Then the group cohomology of G with coefficients in k is

$$\mathrm{H}^*(G,k) \cong \begin{cases} k[y], & \text{if } p = 2 \\ k[y,z]/(y^2), & \text{if } p > 2 \end{cases}$$

where $|y| = 1$, $|z| = 2$. (Note that $kG \cong k[x]/(x^p)$ since $\mathrm{char}(k) = p$, which can be seen by taking $x = g - 1$. Then see Example 1.6.7.)

Let $M = M' = k$ and $N' = N$ in Lemma 8.3.3. By composing with the isomorphism $k \otimes N \cong N$, we obtain an action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(N,N)$, via $-\otimes N$ followed by Yoneda composition. On the other hand, we have an action by Yoneda composition of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(k, N \otimes N^*)$.

In the following statement, we apply Lemma 8.3.1 with $U = k$, $V = N$:

Theorem 8.3.7. *Let N be a left A-module. The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(N,N)$, given by $-\otimes N$ followed by Yoneda composition, corresponds to that on $\mathrm{Ext}^*_A(k, N\otimes N^*)$, given by Yoneda composition, under the isomorphism*

$$\mathrm{Ext}^*_A(N,N) \cong \mathrm{Ext}^*_A(k, N \otimes N^*).$$

Proof. Let $P_\bullet$ be a projective resolution of k, so that $P_\bullet \otimes N$ is a projective resolution of $k \otimes N \cong N$. We must check that the following diagram commutes for each m,n, where ϕ_m, ϕ_{m+n} are the isomorphisms given by Lemma 8.2.4 with $V = N$ and $U = P_m, P_{m+n}$, respectively, and the horizontal maps are the chain level maps corresponding to the cup product of Lemma 8.3.3.

$$\begin{array}{ccc} \mathrm{Hom}_A(P_m,k)\otimes\mathrm{Hom}_A(P_n\otimes N,N) & \longrightarrow & \mathrm{Hom}_A(P_{m+n}\otimes N,N) \\ \downarrow{\scriptstyle 1\otimes\phi_n} & & \downarrow{\scriptstyle \phi_{m+n}} \\ \mathrm{Hom}_A(P_m,k)\otimes\mathrm{Hom}_A(P_n,\mathrm{Hom}_k(N,N)) & \longrightarrow & \mathrm{Hom}_A(P_{m+n},\mathrm{Hom}_k(N,N)) \end{array}$$

Let $\zeta \in \mathrm{Ext}^m_A(k,k)$ and $\eta \in \mathrm{Ext}^n_A(N,N)$, represented by $f \in \mathrm{Hom}_A(P_m,k)$ and $g \in \mathrm{Hom}_A(P_n \otimes N, N)$, respectively. Identify f with the corresponding function from an mth syzygy module $\Omega^m(k)$ to k, and extend to a chain map $f_{\bullet}$ with $f_i \in \mathrm{Hom}_A(P_{m+i}, P_i)$. The top horizontal map takes $f \otimes g$ to $g(f_n \otimes 1)$, and applying ϕ_{m+n} we have

$$\phi_{m+n}(g(f_n \otimes 1))(x)(v) = g(f_n(x) \otimes v)$$

for all $x \in P_{m+n}$, $v \in N$. On the other hand, $(1 \otimes \phi_n)(f \otimes g) = f \otimes \phi_n(g)$, and applying the bottom horizontal map we find

$$(\phi_n(g) f_n)(x)(v) = g(f_n(x) \otimes v).$$

Therefore the diagram commutes. □

There is another action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$, given by $M \otimes -$ followed by Yoneda composition. In case A is cocommutative (or more generally quasitriangular), this action is the same as that given by $- \otimes M$. In general it will not be the same; see, for example, [**BW14**]. We state next the counterpart of Theorem 8.3.7 for this action under the assumption that the antipode S is bijective. Let $\mathrm{Hom}'_k(V,W)$ denote the A-module that is $\mathrm{Hom}_k(V,W)$ as a vector space, but with action as described in Remark 8.2.2. Let ${}^*V = \mathrm{Hom}_k(V,k)$, with A-module structure as described in Remark 8.2.2. Then, by Remark 8.2.7, there are isomorphisms of A-modules:

$$\mathrm{Hom}'_k(U \otimes V, W) \cong \mathrm{Hom}'_k(V, \mathrm{Hom}'_k(U,W)) \cong \mathrm{Hom}'_k(V, {}^*U \otimes W).$$

It follows that

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(V, {}^*U \otimes W), \tag{8.3.8}$$

and consequently

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(V, {}^*U \otimes W).$$

Theorem 8.3.9. *Assume the antipode S of A is bijective and let M be an A-module. The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$, given by $M \otimes -$ followed by Yoneda composition, corresponds to that on $\mathrm{Ext}^*_A(k, {}^*M \otimes M)$, given by Yoneda composition, under the isomorphism*

$$\mathrm{Ext}^*_A(M,M) \cong \mathrm{Ext}^*_A(k, {}^*M \otimes M).$$

Proof. Let $P_{\bullet}$ be a projective resolution of k, so that $M \otimes P_{\bullet}$ is a projective resolution of $M \otimes k \cong M$. We must check that the following diagram commutes for each m, n, where ϕ_m, ϕ_{m+n} are the isomorphisms given in (8.3.8), with $U = M$ and $V = P_n, P_{m+n}$, respectively, and the horizontal maps are the chain level maps corresponding to the cup product of Lemma 8.3.3.

$$\begin{array}{ccc}
\mathrm{Hom}_A(P_m,k) \otimes \mathrm{Hom}_A(M \otimes P_n, M) & \longrightarrow & \mathrm{Hom}_A(M \otimes P_{m+n}, M) \\
\downarrow{\scriptstyle 1 \otimes \phi_n} & & \downarrow{\scriptstyle \phi_{m+n}} \\
\mathrm{Hom}_A(P_m,k) \otimes \mathrm{Hom}_A(P_n, \mathrm{Hom}'_k(M,M)) & \longrightarrow & \mathrm{Hom}_A(P_{m+n}, \mathrm{Hom}'_k(M,M))
\end{array}$$

Let $\zeta \in \operatorname{Ext}_A^m(k,k)$ and $\eta \in \operatorname{Ext}_A^n(M,M)$, represented by $f \in \operatorname{Hom}_A(P_m, k)$ and $g \in \operatorname{Hom}_A(M \otimes P_n, M)$, respectively. Identify f with the corresponding function from an mth syzygy module $\Omega^m(k)$ to k, and extend to a chain map $f_{\bullet}$ with $f_i \in \operatorname{Hom}_A(P_{m+i}, P_i)$. The top horizontal map takes $f \otimes g$ to $g(1_M \otimes f_n)$, and applying ϕ_{m+n} we have

$$\phi_{m+n}(g(1_M \otimes f_n))(x)(v) = g(v \otimes f_n(x))$$

for all $x \in P_{m+n}$, $v \in M$. On the other hand, $(1 \otimes \phi_n)(f \otimes g) = f \otimes \phi_n(g)$, and applying the bottom horizontal map we find

$$(\phi_n(g) f_n)(x)(v) = g(v \otimes f_n(x)).$$

Therefore the diagram commutes. □

As noted in Remark 8.2.7, for any A-modules M, N, there are A-module isomorphisms ${}^*(N^*) \cong N$ and $({}^*M)^* \cong M$. This provides further relationships among various actions: Set $M = N^*$, so that ${}^*M \cong N$. Applying the isomorphisms in the statements of Theorems 8.3.7 and 8.3.9, we see that

$$\operatorname{Ext}_A^*(N,N) \cong \operatorname{Ext}_A^*(k, N \otimes M) \cong \operatorname{Ext}_A^*(M,M).$$

In this way, the second described action in each part of Theorem 8.3.10 below makes sense.

Theorem 8.3.10. *Assume the antipode S of A is bijective.*

(i) *Let N be a finite dimensional A-module. The action of* $\mathrm{H}^*(A,k)$ *on* $\operatorname{Ext}_A^*(N,N)$*, given by $-\otimes N$ followed by Yoneda composition, corresponds to the action given by $N^* \otimes -$ followed by Yoneda composition.*

(ii) *Let M be a finite dimensional A-module. The action of* $\mathrm{H}^*(A,k)$ *on* $\operatorname{Ext}_A^*(M,M)$*, given by $M \otimes -$ followed by Yoneda composition, corresponds to the action given by $- \otimes {}^*M$ followed by Yoneda composition.*

Proof. For (i), let $M = N^*$, and so ${}^*M \cong N$, as noted above. Apply Theorems 8.3.7 and 8.3.9. The proof of (ii) is similar. □

8.4. Bimodules and Hochschild cohomology

In this section, we describe connections between Hochshild cohomology and Hopf algebra cohomology and their actions on Ext. We begin by giving some relations among the Hopf algebras A and A^e and their modules.

Lemma 8.4.1. *Let $\delta : A \to A^e$ be the function defined by*

$$\delta(a) = \sum a_1 \otimes S(a_2)$$

for all $a \in A$. Then δ is an injective algebra homomorphism.

Proof. First note that $\delta(1) = 1 \otimes 1$, the identity in A^e. Let $a, b \in A$. Since S is an algebra anti-homomorphism,

$$\begin{aligned}\delta(ab) &= \sum a_1 b_1 \otimes S(a_2 b_2) \\ &= \sum a_1 b_1 \otimes S(b_2) S(a_2) \\ &= (\sum a_1 \otimes S(a_2))(\sum b_1 \otimes S(b_2)) = \delta(a)\delta(b),\end{aligned}$$

as multiplication in the second factor is opposite that in A.

To see that δ is injective, compose with the function $\pi : A^e \to A$ defined by $\pi(a \otimes b) = a\varepsilon(b)$. We have, for all $a \in A$,

$$\pi\delta(a) = \pi(\sum a_1 \otimes S(a_2)) = \sum a_1 \varepsilon(S(a_2)) = \sum a_1 \varepsilon(a_2) = a,$$

that is $\pi\delta$ is the identity map on A. This implies that δ is injective. □

We will identify A with the subalgebra $\delta(A)$ of A^e. This will allow us to induce modules from A to A^e, using tensor products: Let M be an A-module, and consider A^e to be a right A-module via right multiplication by elements of $\delta(A)$. Then the vector space $A^e \otimes_A M$ is a A^e-module, the action given by left multiplication on the factor A^e.

Lemma 8.4.2. *There is an isomorphism of A^e-modules*

$$A \cong A^e \otimes_A k,$$

where $A^e \otimes_A k$ is the A^e-module induced from the trivial A-module k via the embedding of A into A^e given by the map δ of Lemma 8.4.1.

Proof. Let $f : A \to A^e \otimes_A k$ be the function defined by $f(a) = a \otimes 1 \otimes 1$, and let $g : A^e \otimes_A k \to A$ be the function defined by $g(a \otimes b \otimes 1) = ab$ for all $a, b \in A$. We will check that f is an A^e-module homomorphism, with inverse function g.

Let $a, b, c \in A$. Then, since $c = \sum c_1 \varepsilon(c_2)$, we have

$$\begin{aligned}f((b \otimes c)(a)) = f(bac) &= bac \otimes 1 \otimes 1 \\ &= \sum bac_1 \otimes \varepsilon(c_2) \otimes 1 \\ &= \sum bac_1 \otimes S(c_2) c_3 \otimes 1.\end{aligned}$$

Now identifying A with $\delta(A) \subset A^e$, since the rightmost factor is in k with action of A given by ε, and the tensor product is over A, we may rewrite this as

$$\begin{aligned}\sum ba \otimes c_2 \otimes \varepsilon(c_1) &= \sum ba \otimes \varepsilon(c_1)c_2 \otimes 1\\ &= ba \otimes c \otimes 1\\ &= (b\otimes c)(a\otimes 1 \otimes 1) = (b \otimes c) f(a).\end{aligned}$$

Therefore f is an A^e-module homomorphism.

Now let $a, b \in A$. We have

$$\begin{aligned} gf(a) &= g(a\otimes 1\otimes 1) = a,\\ \text{and } fg(a\otimes b\otimes 1) &= f(ab) = ab \otimes 1 \otimes 1\\ &= \sum ab_1\varepsilon(b_2)\otimes 1 \otimes 1\\ &= \sum ab_1 \otimes \varepsilon(b_2)\otimes 1\\ &= \sum ab_1 \otimes S(b_2)b_3 \otimes 1\\ &= \sum a \otimes b_2 \otimes \varepsilon(b_1)\\ &= \sum a \otimes \varepsilon(b_1)b_2 \otimes 1 = a \otimes b \otimes 1.\end{aligned}$$

Therefore f and g are inverse functions. □

Remark 8.4.3. We will need the following related properties of modules in Section 8.6. Let R be an A-module algebra, and let $R^e\#\delta(A)$ denote the subalgebra of $(R\#A)^e$ generated by R^e and $\delta(A)$. Note that R is an $R^e\#\delta(A)$-module via the usual R^e-module and A-module structures on R. A similar proof to that of Lemma 8.4.2 shows more generally that

$$R\#A \cong (R\#A)^e \otimes_{R^e\#\delta(A)} R$$

as $(R\#A)^e$-modules. There is also a right module version under the assumption that the antipode S is bijective with inverse $\overline{S}$: Let $\delta' : A \to A^{\mathrm{op}} \otimes A$ be the function defined by

$$\delta'(a) = \sum \overline{S}(a_2) \otimes a_1$$

for all $a \in A$. Then δ' is an injective algebra homomorphism, by a similar proof to that of Lemma 8.4.1. There is an isomorphism of right $A^{\mathrm{op}} \otimes A$-modules (equivalently, left A^e-modules)

$$A \cong k \otimes_{\delta'(A)} (A^{\mathrm{op}} \otimes A),$$

by a proof similar to that of Lemma 8.4.2. More generally, we may let $(R^{\mathrm{op}} \otimes R)\#\delta'(A)$ be the subalgebra of $(R\#A)^{\mathrm{op}} \otimes (R\#A)$ generated by $R^{\mathrm{op}} \otimes R$ and $\delta'(A)$. Note that R is a right $\delta'(A)$-module: Set

$$r \cdot (\sum \overline{S}(a_2) \otimes a_1) = \overline{S}(a) \cdot r,$$

that is the right $\delta'(A)$-module structure on R is given by the left A-module structure and $\overline{S}$. This action is compatible with the right $R^{\mathrm{op}} \otimes R$-module structure of R, and so R is a right $(R^{\mathrm{op}} \otimes R)\#\delta'(A)$-module in this way. Then as a left $(R\#A)^e$-module (equivalently as a right $(R\#A)^{\mathrm{op}} \otimes (R\#A)$-module),

$$R\#A \cong R \otimes_{(R^{\mathrm{op}} \otimes R)\#\delta'(A)} ((R\#A)^{\mathrm{op}} \otimes (R\#A)).$$

Lemma 8.4.4. *Assume the antipode S is bijective. Then the right A-module A^e, where A acts by right multiplication by $\delta(A)$, is a projective A-module.*

Proof. We claim that $S : A \to A^{\mathrm{op}}$ is an isomorphism of right A-modules, where A acts on the right by multiplication on A and by multiplication by $S(A)$ on A^{op}. We need only check S is an A-module map: $S(a \cdot b) = S(b)S(a) = S(a) \cdot S(b)$, in A^{op}, for all $a, b \in A$. This yields an isomorphism of right A-modules $A \otimes A \to A \otimes A^{\mathrm{op}} = A^e$. Now $A \otimes A$ is projective as a right A-module by Lemma 8.2.8. Thus A^e is a projective right A-module, the action of A being precisely multiplication by $\delta(A)$. □

We will consider A to be an A-module by the left adjoint action, that is for all $a, b \in A$,

$$a \cdot b = \sum a_1 b S(a_2).$$

Denote this A-module by A^{ad}. More generally, if M is any A-bimodule, denote by M^{ad} the A-module with action given by $a \cdot m = \sum a_1 m S(a_2)$ for all $a \in A$, $m \in M$.

The following theorem is due to Ginzburg and Kumar [**GK93**]; our proof is from [**PW09, SW99**].

Theorem 8.4.5. *Let A be a Hopf algebra over k with bijective antipode S. There is an isomorphism of algebras*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, A^{ad}).$$

Proof. By Lemma 8.2.8, A^e is a flat right A-module. Thus we may apply the Eckmann-Shapiro Lemma (Lemma A.5.2) with $B = A^e$, $M = k$, and $N = A$ and Lemma 8.4.2 to obtain an isomorphism of vector spaces,

$$\mathrm{Ext}^*_{A^e}(A, A) \cong \mathrm{Ext}^*_A(k, A^{ad}).$$

It remains to prove that cup products are preserved by this isomorphism. This follows from the proof of [**SW99**, Proposition 3.1], valid more generally in this context, as we explain next.

Let $P_{\bullet}$ denote an A-projective resolution of k. Let $X_{\bullet} = A^e \otimes_A P_{\bullet}$, an A^e-projective resolution of $A^e \otimes_A k \cong A$.

There is an A-chain map $\iota : P_{\bullet} \to X_{\bullet}$ defined by $\iota(p) = (1 \otimes 1) \otimes p$ for all $p \in P_i$. Let $f \in \mathrm{Hom}_{A^e}(X_i, A)$ be a cocycle representing a cohomology class in $\mathrm{Ext}^*_{A^e}(A, A)$. The corresponding cohomology class in $\mathrm{Ext}^*_A(k, A^{ad})$ is represented by $f \circ \iota$.

Let $D : P_\bullet \to P_\bullet \otimes P_\bullet$ be a chain map. Such a map exists and is unique up to homotopy by the Comparison Theorem (Theorem A.1.6). Since $k \otimes k \cong k$, the Künneth Theorem (Theorem A.4.1) implies that $P_\bullet \otimes P_\bullet$ is also a projective resolution of k as an A-module, via similar arguments to those given before. Therefore D induces an isomorphism on cohomology. The map D also induces a chain map $D' : X_\bullet \to X_\bullet \otimes_A X_\bullet$ as follows. There is a map of A^e-chain complexes $\theta : A^e \otimes_A (P_\bullet \otimes P_\bullet) \to X_\bullet \otimes_A X_\bullet$, given by

$$\theta((a \otimes b) \otimes (p \otimes q)) = ((a \otimes 1) \otimes p) \otimes ((1 \otimes b) \otimes q).$$

Also note that D induces a map from $A^e \otimes_A P_\bullet$ to $A^e \otimes_A (P_\bullet \otimes P_\bullet)$. Let D' be the composition of this map with θ. Again D' is unique up to homotopy.

Now let $f \in \mathrm{Hom}_{A^e}(X_i, A)$, $g \in \mathrm{Hom}_{A^e}(X_j, A)$ be cocycles. The above observations imply that the following diagram commutes:

$$\begin{array}{ccccccc}
X_\bullet & \xrightarrow{D'} & X_\bullet \otimes_A X_\bullet & \xrightarrow{f \otimes g} & A \otimes_A A & \xrightarrow{\sim} & A \\
\iota \uparrow & & & & & & \| \\
P_\bullet & \xrightarrow{D} & P_\bullet \otimes P_\bullet & \xrightarrow{f\iota \otimes g\iota} & A \otimes A & \xrightarrow{\pi} & A
\end{array}$$

where π is multiplication. The top row yields the product in $\mathrm{Ext}^*_{A^e}(A, A)$ and the bottom row yields the product in $\mathrm{Ext}^*_A(k, A^{ad})$. □

The following consequence is due to Linckelmann [**Lin00**], and may instead be proven directly.

Corollary 8.4.6. *Let A be a commutative Hopf algebra. There is an isomorphism of algebras*

$$\mathrm{HH}^*(A) \cong A \otimes \mathrm{H}^*(A, k).$$

Proof. Since A is commutative, it acts trivially on the A-module A^{ad}, that is, $(A^{ad})^A = A^{ad}$. By the Universal Coefficients Theorem (Theorem A.4.3), Theorem 8.4.5, and analysis of the cup products, the statement holds. □

Another consequence of Theorem 8.4.5 is the following.

Corollary 8.4.7. *Let $I = \mathrm{Ker}(\varepsilon)$, the augmentation ideal of the Hopf algebra A. Then as an algebra,*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, k) \oplus \mathrm{H}^*(A, I^{ad}),$$

a direct sum of the subalgebra $\mathrm{H}^*(A, k)$ and the ideal $\mathrm{H}^*(A, I^{ad})$.

We may thus view $\mathrm{H}^*(A, k)$ as both a quotient and a subalgebra of $\mathrm{HH}^*(A)$.

Proof. Under the left adjoint action of A on itself, the trivial module k is isomorphic to the submodule of A^{ad} given by all scalar multiples of the identity 1. In fact k is a direct summand of A^{ad}, its complement being $I = \mathrm{Ker}(\varepsilon)$. As $\mathrm{Ext}^*_A(k, -)$ is additive, the result follows. □

Remark 8.4.8. There is a Tor version that is easier: $\mathrm{HH}_i(A) \cong \mathrm{H}_i(A, A^{ad})$ as abelian groups. It follows that $\mathrm{H}_i(A, k)$ is a direct summand of $\mathrm{HH}_i(A)$.

There is a connection between the actions of Hopf algebra cohomology and of Hochschild cohomology on Ext, as noted in [**PW09**]:

Proposition 8.4.9. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^*(A,k) & & \\ {\scriptstyle A^e\otimes_A -}\downarrow & \searrow^{-\otimes_k M} & \\ \mathrm{HH}^*(A) & \xrightarrow{-\otimes_A M} & \mathrm{Ext}^*_A(M,M) \end{array}$$

Thus the action of $\mathrm{H}^*(A,k)$ *on* $\mathrm{Ext}^*_A(M,M)$*, given by* $-\otimes_k M$ *followed by Yoneda composition, factors through the action of* $\mathrm{HH}^*(A)$ *on* $\mathrm{Ext}^*_A(M,M)$*.*

Proof. Let $0 \to k \to P_{n-1} \to \cdots \to P_0 \to k \to 0$ be an n-extension of A-modules. For any A-module X, consider the function

$$f_X : (A^e \otimes_A X) \otimes_A M \to X \otimes_k M$$

given by

$$f_X(a \otimes b \otimes x \otimes m) = \sum a_1 x \otimes a_2 b m$$

for all $a, b \in A$, $x \in X$, $m \in M$. By construction, the action of A on $(A^e \otimes_A X) \otimes_A M$ is on the leftmost factor of A only. One checks that f_X is a functorial A-module isomorphism, and thus the actions are equivalent as stated. □

One consequence of the proposition, under some finiteness assumptions, is that in the support variety theory of Chapter 7, if one chooses H to be the subalgebra of $\mathrm{HH}^*(A)$ generated by $\mathrm{H}^*(A,k)$ and $\mathrm{HH}^0(A) \cong Z(A)$, the varieties for modules given by Hochschild cohomology and by Hopf algebra cohomology are closely related.

8.5. Finite group algebras

In this section, we apply Theorem 8.4.5 to the special case $A = kG$, a group algebra of a finite group G. This leads to a decomposition of Hochschild cohomology $\mathrm{HH}^*(kG)$ into a vector space direct sum of group cohomology rings of centralizer subgroups, as first shown by Burghelea for Hochschild homology [**Bur85**]. Techniques from group cohomology then yield a description of the product on Hochschild cohomology $\mathrm{HH}^*(kG)$ in terms of products on group cohomology. We briefly introduce the needed techniques from group cohomology here; for details, see [**Ben91a, Ben91b, Eve61**].

Definition 8.5.1. The *group cohomology* of G with coefficients in k is

$$\mathrm{H}^*(G,k) = \mathrm{H}^*(kG,k) = \mathrm{Ext}^*_{kG}(k,k).$$

Similarly, for any kG-module M, we will write

$$\mathrm{H}^*(G,M) = \mathrm{H}^*(kG,M) = \mathrm{Ext}^*_{kG}(k,M).$$

By Theorem 8.4.5,

$$\mathrm{HH}^*(kG) \cong \mathrm{H}^*(G, kG^{ad}). \tag{8.5.2}$$

We interpret the kG-module kG^{ad}: Let $g,h \in G$. The action of g on the element h viewed as being in kG^{ad} is given by

$$g \cdot h = ghS(g) = ghg^{-1},$$

since $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$. Thus the kG-module kG^{ad} has vector space basis G on which elements of G act by conjugation. It then decomposes into a direct sum of kG-submodules corresponding to conjugacy classes. Let $g_1, \ldots, g_r$ be a set of conjugacy class representatives. For each i, the G-set that is the conjugacy class of g_i may be written $G \cdot g_i \cong G/C(g_i)$, where $C(g_i)$ is the centralizer in G of g_i. The kG-submodule of kG^{ad} having basis the conjugacy class of g_i may thus be written

$$kG \cdot g_i \cong kG \otimes_{kC(g_i)} k, \tag{8.5.3}$$

that is, this module is the trivial module k for $kC(g_i)$ induced to kG. For finite group algebras, induction and coinduction yield isomorphic modules since kG is isomorphic to $(kG)^*$ as a kG-module. So by the Eckmann-Shapiro Lemma (Lemma A.5.2),

$$\begin{aligned} \mathrm{H}^*(G, kG \otimes_{kC(g_i)} k) &= \mathrm{Ext}^*_{kG}(k, kG \otimes_{kC(g_i)} k) \\ &\cong \mathrm{Ext}^*_{kC(g_i)}(k,k) = \mathrm{H}^*(C(g_i),k). \end{aligned}$$

We may thus rewrite Hochschild cohomology $\mathrm{HH}^*(kG)$, first applying the isomorphisms (8.5.2) and (8.5.3):

$$\mathrm{HH}^*(kG) \cong \oplus_{i=1}^r \mathrm{H}^*(G, kG \cdot g_i) \cong \oplus_{i=1}^r \mathrm{H}^*(C(g_i),k). \tag{8.5.4}$$

That is, the Hochschild cohomology of kG is isomorphic, as a graded vector space, to the direct sum of group cohomology rings for centralizer subgroups, one summand for each element in a set of conjugacy class representatives. Analyzing the two applications of the Eckmann-Shapiro Lemma used in this description, we see that we may write this isomorphism explicitly as follows. We identify elements in $\mathrm{HH}^*(kG)$ with elements in $\mathrm{H}^*(G, kG^{ad})$, at the chain level, by restriction from $(kG)^e$ to kG under the map δ of Lemma 8.4.1. Our description of the product will be for such elements in $\mathrm{H}^*(G, kG^{ad})$. For any subgroup H of G, and any kG-module M, there is the structure of a kH-module on M by restriction. Since kG is free as a kH-module, restriction takes a kG-projective resolution of k to a kH-projective resolution of k, thus inducing a well-defined map

$$\operatorname{res}^G_H : \mathrm{H}^*(G, M) \to \mathrm{H}^*(H, M).$$

Let $\pi_i : \mathrm{H}^*(C(g_i), kG \otimes_{kC(g_i)} k) \to \mathrm{H}^*(C(g_i), k)$ denote the map induced by the projection of $kG \otimes_{kC(g_i)} k$ onto the $kC(g_i)$-direct summand $k \otimes_{kC(g_i)} k \cong k$. Then the isomorphism above is given in one direction by

$$\begin{array}{ccc} \mathrm{H}^*(G, kG^{ad}) & \xrightarrow{\sim} & \oplus_{i=1}^r \mathrm{H}^*(C(g_i), k) \\ \zeta & \mapsto & (\pi_i \operatorname{res}^G_{C(g_i)} \zeta)_i. \end{array}$$

In order to describe products, we will also need an expression for the inverse isomorphism. This will be expressed in terms of corestriction maps on group cohomology that we define next: The map

$$\operatorname{cores}^G_H : \mathrm{H}^*(H, M) \to \mathrm{H}^*(G, M)$$

is defined at the chain level as follows. Let P be a projective resolution of k as a kG-module, and let $f \in \operatorname{Hom}_{kH}(P_n, M)$. Then

$$\operatorname{cores}^G_H(f)(x) = \sum_{g \in [G/H]} g \cdot f(g^{-1} \cdot x)$$

for all $x \in P_n$, where $[G/H]$ denotes a set of coset representatives of H in G. Since f is a kH-module homomorphism, the values of this function do not depend on choice of coset representatives. This map also commutes with the differential and so induces a well-defined map on cohomology, for which we use the same notation.

Now let $\iota : \mathrm{H}^*(C(g_i), k) \to \mathrm{H}^*(C(g_i), kG \otimes_{kC(g_i)} k)$ denote the map induced by embedding k into $kG \otimes_{kC(g_i)} k$ as $k \otimes_{kC(g_i)} k$. Then the desired inverse isomorphism may be described as follows:

$$\begin{array}{ccc} \mathrm{H}^*(G, kG^{ad}) & \xleftarrow{\sim} & \oplus_{i=1}^r \mathrm{H}^*(C(g_i), k) \\ \displaystyle\sum_{i=1}^r \operatorname{cores}^G_{C(g_i)} \iota(\alpha_i) & \leftarrow\!\shortmid & (\alpha_i)_i. \end{array}$$

For notational convenience, set

$$\gamma_i(\alpha_i) = \operatorname{cores}^G_{C(g_i)} \iota(\alpha_i).$$

We need one more map on group cohomology: If H is a subgroup of G and $g \in G$, write ${}^gH = \{ghg^{-1} \mid h \in H\}$ for the conjugate subgroup. There is a map $g^* : \mathrm{H}^*(H, k) \to \mathrm{H}^*({}^gH, k)$ given at the chain level by

$$g^*(f)(x) = g \cdot (f(g^{-1} \cdot x))$$

for all $x \in P_n$ and $f \in \operatorname{Hom}_{kH}(P_n, k)$.

The following is a special case of [**SW99**, Theorem 5.1], and gives the product on Hochschild cohomology $\mathrm{HH}^*(kG)$ in terms of the vector space direct sum (8.5.4).

Theorem 8.5.5. *Let* $\alpha \in \mathrm{H}^*(C(g_i)), k)$ *and* $\beta \in \mathrm{H}^*(C(g_j), k)$. *Then*

$$\gamma_i(\alpha) \smile \gamma_j(\beta) = \sum_{x\in D} \gamma_l(\mathrm{cores}^{C(g_k)}_{W(x)}(\mathrm{res}^{{}^yC(g_i)}_{W(x)}\, y^*\alpha \smile \mathrm{res}^{{}^{yx}C(g_j)}_{W(x)}(yx)^*\beta))$$

where D *is a set of double coset representatives for* $C(g_i)\backslash G/C(g_j)$, *the integer* $l = l(x)$ *and the group element* $y = y(x)$ *are chosen so that* $g_l = ({}^yg_i)({}^{yx}g_j)$, *and* $W(x) = {}^yC(g_i) \cap {}^{yx}C(g_j)$.

For a proof, see [**SW99**]. The main idea of the proof is that group cohomology, together with the maps restriction, corestriction, and conjugation, is a Green functor [**Eve61**]. The formula in the theorem is precisely the product formula for a Green functor arising from a Mackey decomposition, interpreted in this notation and setting of Hochschild cohomology.

The following corollary, due to Cibils and Solotar [**CS97**], is a special case of Corollary 8.4.6, and can also be proven directly.

Corollary 8.5.6. *Let* G *be a finite abelian group. The Hochschild cohomology of* kG *is isomorphic, as an algebra, to the tensor product of* kG *and group cohomology* $\mathrm{H}^*(G,k)$:

$$\mathrm{H}^*(kG) \cong kG \otimes \mathrm{H}^*(G,k).$$

8.6. A spectral sequence for a smash product

In this section, we give a spectral sequence relating Hochschild cohomology of a smash product $R\#A$ with that of R and Hopf algebra cohomology of A. This spectral sequence and a more general version for Hopf Galois extensions is due to Ştefan [** Şte95**]. In Section 8.1 we introduced smash products arising from a left A-action on R, and it turns out that as a result we will need to work with right A-module cohomology here, and we will assume that the antipode S of A is bijective. First we need a lemma about some special types of right A-modules.

For any $(R\#A)^e$-module N, the space $\mathrm{Hom}_{R^e}(R,N)$ is a right A-module under the action

$$(f\cdot a)(r) = \sum r\overline{S}(a_2)f(1)a_1 \qquad (8.6.1)$$

for all $a \in A$, $f \in \mathrm{Hom}_{R^e}(R,N)$, and $r \in R$. To see this, note that an element of $\mathrm{Hom}_{R^e}(R,N)$ is determined by its value on 1, which must be an element x of N such that $rx = xr$ for all $r \in R$. One checks that for any $a \in A$, the element $\sum \overline{S}(a_2)xa_1$ also then has this property. This action generalizes the Miyashita-Ulbrich action from groups to Hopf algebras.

Lemma 8.6.2. *Let A be a Hopf algebra, let R be an A-module algebra, and let M be an $(R\#A)^e$-module. Then there is an isomorphism of right A-modules,*

$$\mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, M)) \cong \mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), M).$$

The $(R\#A)^e$-module $\mathrm{Hom}_k((R\#A)^e, M)$ in the lemma is the coinduced module (see Section A.5) of M from k to $(R\#A)^e$. Similarly we view the A-module $\mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), M)$ as a coinduced module from R to $A^{\mathrm{op}} \otimes (R\#A)$, and A acts via the embedding $\delta'(A)$ of Remark 8.4.3.

Proof. Let $f \in \mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, M))$ and define an element $\phi(f)$ of $\mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), M)$ by

$$\phi(f)(a \otimes ra') = f(1)(a \otimes ra')$$

for all $a, a' \in A$ and $r \in R$. Then ϕ is an A-module homomorphism: The function f is determined by $f(1)$, which has the property that $rf(1) = f(1)r$. Now $\mathrm{Hom}_k((R\#A)^e, M)$ is the module M coinduced from k to $(R\#A)^e$, so this property is equivalent to the property that $f(1)(xr \otimes y) = f(1)(x \otimes ry)$ for all $r \in R$ and $x, y \in R\#A$. It follows that for all $a, a', a'' \in A$ and $r \in R$,

$$\begin{aligned} (\phi(f \cdot a))(a' \otimes ra'') &= (f \cdot a)(1)(a' \otimes ra'') \\ &= \overline{S}(a_2) f(1) a_1 (a' \otimes ra'') \\ &= f(1)(\overline{S}(a_2)a' \otimes ra''a_1) \end{aligned}$$

and

$$\begin{aligned} (\phi(f) \cdot a)(a' \otimes ra'') &= \phi(f)(\overline{S}(a_2)a' \otimes ra''a_1) \\ &= f(1)(\overline{S}(a_2)a' \otimes ra''a_1), \end{aligned}$$

and so ϕ is an A-module homomorphism.

We next show that ϕ is bijective: Let $g \in \mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), M)$ and define an element $\psi(g)$ of $\mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((A\#R)^e, M))$ by

$$\psi(g)(1)(ar \otimes r'a') = g(a \otimes rr'a').$$

One checks that ψ is an inverse map to ϕ. □

Now we are ready to construct the spectral sequence for Hochschild cohomology of bimodules over a smash product. The Hopf algebra cohomology in the next theorem is that of right A-modules. If M is an $(R\#A)^e$-module, the Hochschild cohomology $\mathrm{HH}^*(R, M)$ is a right A-module under an action induced by that of equation (8.6.1): Take an injective resolution $Q_{\bullet}$ of M, take the right A-action on each $\mathrm{Hom}_{R^e}(R, Q_q)$ given by (8.6.1), and check that this action of A commutes with the differentials and thus induces an action on Hochschild cohomology as claimed.

Theorem 8.6.3. *Let A be a Hopf algebra, let R be an A-module algebra, and let M be an $(R\#A)^e$-module. There is a spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(A, \mathrm{HH}^q(R, M)) \implies \mathrm{HH}^{p+q}(R\#A, M).$$

We give a direct proof of the theorem in our setting. A proof in the more general setting of Hopf Galois extensions is in [**Şte95**].

Proof. Let $P_{\bullet}$ be a projective resolution of k as a right A-module.

Let $Q_{\bullet}$ be an injective resolution of M as an $(R\#A)^e$-module. By restricting to the subalgebra R^e of $(R\#A)^e$, since $(R\#A)^e$ is free over R^e, $Q_{\bullet}$ becomes an R^e-injective resolution of M. (By the Nakayama relations (Lemma A.5.1), $\mathrm{Hom}_{R^e}(U, I) \cong \mathrm{Hom}_{(R\#A)^e}((R\#A)^e \otimes_{R^e} U, I)$ for all R^e-modules U and $(R\#A)^e$-modules I, so $\mathrm{Hom}_{R^e}(-, I)$ is an exact functor when I is an injective $(R\#A)^e$-module, implying that the restriction of I to R^e is also injective.)

Let

$$C^{p,q} = \mathrm{Hom}_A(P_p, \mathrm{Hom}_{R^e}(R, Q_q)),$$

a double complex that we may view as:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_A(P_0, \mathrm{Hom}_{R^e}(R, Q_2)) & \longrightarrow & \mathrm{Hom}_A(P_1, \mathrm{Hom}_{R^e}(R, Q_2)) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_A(P_0, \mathrm{Hom}_{R^e}(R, Q_1) & \longrightarrow & \mathrm{Hom}_A(P_1, \mathrm{Hom}_{R^e}(R, Q_1)) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_A(P_0, \mathrm{Hom}_{R^e}(R, Q_0)) & \longrightarrow & \mathrm{Hom}_A(P_1, \mathrm{Hom}_{R^e}(R, Q_0)) & \longrightarrow & \cdots
\end{array}$$

We will analyze the two first quadrant spectral sequences associated to this double complex in the notation of Section A.6. They each converge to $\mathrm{H}^{\bullet}(C)$ by Theorem A.6.6. Since each P_i is projective as an A-module, $\mathrm{Hom}_A(P_i, -)$ is exact, and we have

$$\mathrm{H}''(C) = \mathrm{Hom}_A(P_{\bullet}, \mathrm{HH}^{\bullet}(R, M))$$

and so

$$\mathrm{H}'(\mathrm{H}''(C)) = \mathrm{H}^{\bullet}(A, \mathrm{HH}^{\bullet}(R, M)).$$

On the other hand,

$$\mathrm{H}'(C) \cong \mathrm{H}^{\bullet}(A, \mathrm{Hom}_{R^e}(R, Q_{\bullet})).$$

We claim that for each q, $\mathrm{H}^p(A, \mathrm{Hom}_{R^e}(R, Q_q)) = 0$ whenever $p > 0$. This is a special case of [**Şte95**, Proposition 3.2]. To see this in our setting, first note that there is an injective $(R\#A)^e$-module homomorphism

$$i : Q_q \to \mathrm{Hom}_k((R\#A)^e, Q_q)$$

given by $i(m)(x \otimes y) = xmy$ for all $m \in Q_q$ and $x, y \in R\#A$. Since Q_q is an injective $(R\#A)^e$-module, this map splits, and so Q_q is a direct summand of $\mathrm{Hom}_k((R\#A)^e, Q_q)$ as an $(R\#A)^e$-module. This implies that $\mathrm{Hom}_{R^e}(R, Q_q)$ is a direct summand of $\mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, Q_q))$ as a right A-module. So to prove our claim, it suffices to show that

$$\mathrm{H}^p(A, \mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, Q_q)) = 0$$

whenever $p > 0$. Now this is the degree p cohomology of the complex

$$\mathrm{Hom}_A(P_{\bullet}, \mathrm{Hom}_{R^e}(R, \mathrm{Hom}_k((R\#A)^e, Q_q))),$$

and by Lemma 8.6.2, this is isomorphic to

$$\mathrm{Hom}_A(P_{\bullet}, \mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), Q_q)).$$

In turn, this is isomorphic to $\mathrm{Hom}_k(P_{\bullet} \otimes_A (A^{\mathrm{op}} \otimes (R\#A)), Q_q)$: If f is an element of $\mathrm{Hom}_A(P_p, \mathrm{Hom}_k(A^{\mathrm{op}} \otimes (R\#A), Q_q))$ for some p, let $\phi(f)$ be the element of $\mathrm{Hom}_k(P_p \otimes (R\#A), Q_q)$ such that $\phi(f)(x \otimes y) = f(x)(y)$ for all $x \in P_p$, $y \in R\#A$. An inverse map is given by taking $g \in \mathrm{Hom}_k(P_p \otimes_A (A^{\mathrm{op}} \otimes (R\#A)), Q_q)$ to $\psi(g)$ where $\psi(g)(x)(y) = g(x \otimes y)$. One checks that $\psi(g)$ is a homomorphism of right A-modules, and that ϕ, ψ are inverse maps. Now $\mathrm{Hom}_k(P_{\bullet} \otimes_A (A^{\mathrm{op}} \otimes (R\#A)), Q_q)$ has homology 0 in positive degrees, since $- \otimes_A (A^{\mathrm{op}} \otimes (R\#A))$ and $\mathrm{Hom}_k(-, Q_q)$ are exact functors.

As a result, the cohomology $\mathrm{H}'(C)$ is concentrated in the leftmost column. In the qth position, by Remarks 8.2.3 and 8.4.3, it is

$$\begin{aligned} \mathrm{H}^0(A, \mathrm{Hom}_{R^e}(R, Q_q)) &\cong \mathrm{Hom}_A(k, \mathrm{Hom}_{R^e}(R, Q_q)) \\ &\cong \mathrm{Hom}_{R^e}(R, Q_q)^A \\ &\cong \mathrm{Hom}_{(R^{\mathrm{op}} \otimes R)\#\delta'(A)}(R, Q_q) \\ &\cong \mathrm{Hom}_{(R\#A)^e}(R\#A, Q_q). \end{aligned}$$

For the last two isomorphisms, recall that the map $\delta' : A \to A^e$ is given by $\delta'(a) = \sum \overline{S}(a_2) \otimes a_1$ for all $a \in A$, and thus A-invariants of $\mathrm{Hom}_{R^e}(R, Q_q)$ are precisely the right $(R^{\mathrm{op}} \otimes R)\#\delta'(A)$-homomorphisms from R to Q_q. Remark 8.4.3, together with the Nakayama relations (Lemma A.5.1), yields the last isomorphism. It follows that $\mathrm{H}''(\mathrm{H}'(C)) = \mathrm{HH}^{\bullet}(R\#A, M)$, as claimed. □

Corollary 8.6.4. *Let A be a semisimple Hopf algebra, let R be an A-module algebra, and let M be an $(R\#A)^e$-module. Then*

$$\mathrm{HH}^*(R\#A, M) \cong (\mathrm{HH}^*(R, M))^A.$$

Proof. Since A is semisimple, the spectral sequence of Theorem 8.6.3 collapses: The E_2 page is concentrated in the leftmost column, which is thus $\mathrm{H}^0(A, \mathrm{HH}^*(R, M)) \cong (\mathrm{HH}^*(R, M))^A$. □

Appendix A

Homological Algebra Background

In this appendix, we collect terminology and results from homological algebra that we will use throughout the book. Proofs and other details may be found in standard homological algebra texts such as [**HS71, Wei94**].

A.1. Complexes, resolutions, dimensions

Let R be a ring. We always assume that R has multiplicative identity 1 and all modules are unital modules, that is, 1 acts as the identity map. Modules will be left modules unless otherwise specified. We will often take $R = A \otimes A^{\mathrm{op}}$ where A is an algebra over a field k and $\otimes = \otimes_k$.

A *complex* $C_\bullet$ of R-modules, also written $(C_\bullet, d_\bullet)$, is a sequence of R-modules and R-module homomorphisms, called *differentials*,

$$C_\bullet : \qquad \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

where $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The *degree* (or *dimension*) of an element x of C_n is n, and we write $|x| = n$. For each n, the kernel $\mathrm{Ker}(d_n)$ is the R-submodule of C_n consisting of *n-cycles*, the image $\mathrm{Im}(d_{n+1})$ is the R-submodule of C_n consisting of *n-boundaries*, and $\mathrm{H}_n(C_\bullet) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n+1})$ is the *nth homology* of $C_\bullet$.

We will often leave off the subscript on $C_\bullet$, writing C instead, when it is clear from context that this notation refers to the whole complex.

We take a *chain complex* to be a complex for which $C_n = 0$ for $n < 0$, and a *cochain complex* to be a complex for which $C_n = 0$ for $n > 0$. Some authors use these terms more generally to refer to complexes. Some complexes may be indexed differently, replacing n by $-n$ in $C_\bullet$ above, with the maps still oriented as shown so that the indexing agrees with the left to right ordering of integers on a standard number line. A cochain complex then has differential of degree $+1$, and we may choose to write the index as a superscript:

$$C^\bullet : \qquad 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

For each n, elements in the kernel of d_n are n-*cocycles*, and elements in the image of d_{n-1} are n-*coboundaries*. We write $\mathrm{H}^n(C^\bullet) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1})$ and refer to this as the *cohomology* of the cochain complex $C_\bullet$. The following definitions and statements may be rephrased in terms of this other indexing choice.

We say that $C_\bullet$ is *acyclic*, or *exact*, if $\mathrm{H}_n(C_\bullet) = 0$ for all n. A *short exact sequence* is an exact complex of the form $0 \to U \to V \to W \to 0$.

Let (C, d) be a complex and $n \in \mathbb{Z}$. The *shifted* (or *translated*) *complex* $C[n]$ has

$$C[n]_i = C_{i+n}$$

and differentials $(-1)^n d$, so for example, $C[n]_0 = C_n$. (For a cochain complex, we take instead $C[n]^i = C^{i-n}$.)

Let (C, d) and (C', d') be complexes. A *chain map* $f_\bullet : C_\bullet \to C'_\bullet$ consists of an R-module homomorphism $f_n : C_n \to C'_n$, for each n, for which $f_{n-1}d_n = d'_n f_n$ for each n, that is, the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\
\cdots & \longrightarrow & C'_1 & \xrightarrow{d'_1} & C'_0 & \xrightarrow{d'_0} & C'_{-1} & \longrightarrow & \cdots
\end{array}$$

A chain map induces a map on homology, and is a *quasi-isomorphism* if this induced map is an isomorphism.

Two chain maps $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ are *chain homotopic* if there are R-module homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n \qquad \text{(A.1.1)}$$

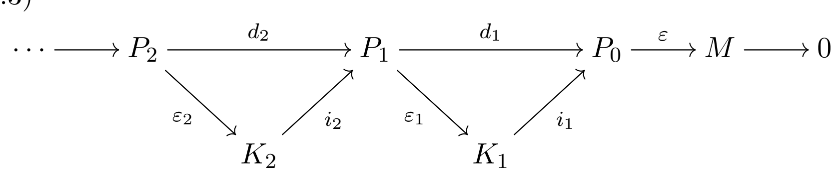
for all n. We call $s_\bullet$ a *homotopy* for $f_\bullet - g_\bullet$. Chain homotopy is an equivalence relation, and two chain homotopic maps induce the same maps on (co)homology. If $g_\bullet$ is the zero map, we call $s_\bullet$ a *chain contraction* of $f_\bullet$. If there is a chain contraction of the identity map on $C_\bullet$, it is also sometimes called a *contracting homotopy*, and it follows that $C_\bullet$ is acyclic. This conclusion holds under the weaker hypothesis that there are set maps $s_n : C_n \to C'_{n+1}$ (not necessarily R-module homomorphisms) satisfying equation (A.1.1). Sometimes when R is a k-algebra one may find k-linear maps s_n satisfying this hypothesis that are not R-module homomorphisms.

Let M be an R-module. A *projective resolution* of M is a chain complex $P_{\bullet}$ consisting of projective R-modules P_n $(n \geq 0)$ for which $\mathrm{H}_0(P_{\bullet}) \cong M$ and $\mathrm{H}_n(P_{\bullet}) = 0$ for all $n \neq 0$. Thus $P_{\bullet}$ is quasi-isomorphic to the complex that is M concentrated in degree 0 and 0 elsewhere, with all maps 0. Another consequence is that $M \cong P_0/\mathrm{Im}(d_1)$ and the following sequence is exact:

(A.1.2) $$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where ε is the quotient map from P_0 to $P_0/\mathrm{Im}(d_1)$ followed by an isomorphism to M. We refer to the complex (A.1.2) as the *augmented complex* of $P_{\bullet}$. Sometimes this augmented complex is abbreviated $P_{\bullet} \to M$ and referred to as the projective resolution of M, when it is clear from context what is meant. Note that projective resolutions always exist: Every R-module M is a homomorphic image of a projective R-module, for example, the free module on a set of generators of M. One may use this fact to build projective resolutions as follows. Let P_0 be a projective R-module having M as a homomorphic image under a map ε. Let $K_1 = \mathrm{Ker}(\varepsilon)$. Then K_1 is a homomorphic image of a projective R-module P_1 via some map $\varepsilon_1 : P_1 \to K_1$. Denote by i_1 the inclusion map $i_1 : K_1 \to P_0$ and set $d_1 = i_1\varepsilon_1$. Let $K_2 = \mathrm{Ker}(d_1)$ and continue.

(A.1.3)



We call K_i an *ith syzygy module* of M. It depends on some choices, and Lemma A.1.5 below is a precise statement about these choices. It follows from Schanuel's Lemma:

Lemma A.1.4 (Schanuel's Lemma)**.** *Let* $0 \to K \to P \xrightarrow{\varepsilon} M \to 0$ *and* $0 \to K' \to P' \xrightarrow{\varepsilon'} M \to 0$ *be two short exact sequences of* R*-modules with* P, P' *projective. Then* $K \oplus P' \cong K' \oplus P$.

Schanuel's Lemma immediately implies the following.

Lemma A.1.5. *If* K_i *and* K_i' *are ith syzygy modules of the* R*-module* M*, then there are projective* R*-modules* P, P' *with* $K_i \oplus P \cong K_i' \oplus P'$.

Another way to state Lemma A.1.5 is to say that K_i and K_i' are equivalent under an equivalence relation: Two R-modules U and V are equivalent if $U \oplus P \cong V \oplus P'$ for some projective R-modules P, P'.

The *Heller operator* Ω is defined by $\Omega(M) = K_1$, understood to take values in an equivalence class of R-modules. Sometimes we write $\Omega_R = \Omega$ to emphasize the choice of ring R. This operator is often used in settings where projective direct summands do not matter, such as Theorem A.2.3 below. In some contexts, $\Omega(M)$ may instead be defined uniquely up to isomorphism, for example in settings where there is a notion of a minimal resolution. It should be clear from context which is meant.

The next theorem in particular implies a relation among projective resolutions.

Theorem A.1.6 (Comparison Theorem). *Let $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ be chain complexes with $M = \mathrm{H}_0(P_\bullet)$, $N = \mathrm{H}_0(Q_\bullet)$, and let $\varepsilon : P_0 \to M$ and $\varepsilon' : Q_0 \to N$ be corresponding augmentation maps. Assume that the augmented complex $\cdots \to Q_1 \to Q_0 \to N \to 0$ is exact and that P_i is projective for each i. If $f : M \to N$ is an R-module homomorphism, then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ for which $f\varepsilon = \varepsilon' f_0$. This chain map is unique up to chain homotopy.*

In particular, if $P_\bullet, Q_\bullet$ are projective resolutions of M, N, respectively, the Comparison Theorem states that there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ *lifting* $f : M \to N$.

An *injective resolution* of an R-module M is a cochain complex $I_\bullet$ consisting of injective R-modules I_n for which $\mathrm{H}_0(I_\bullet) \cong M$ and $\mathrm{H}_n(I_\bullet) = 0$ for all $n \neq 0$. In other words, $M \cong \mathrm{Ker}(d_0)$ and the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\ \iota\ } I_0 \xrightarrow{\ d_0\ } I_1 \xrightarrow{\ d_1\ } \cdots, \tag{A.1.7}$$

where ι is an isomorphism from M to $\mathrm{Ker}(d_0)$ followed by inclusion into I_0. We refer to the complex (A.1.7) as the *augmented complex* of $I_\bullet$, and sometimes as the injective resolution of M when it is clear from context what is intended. Note that injective resolutions always exist: Baer's Theorem states that every R-module can be embedded in an injective R-module, and one builds an injective resolution in a similar fashion to that described for a projective resolution above: Let $L_1 = \mathrm{Coker}(\iota) = I_0/\mathrm{Im}(\iota)$, then embed L_1 into an injective module I_1 via $\iota_1 : L_1 \to I_1$, set $\delta_0 = \iota_1\pi_0$ where $\pi_0 : I_0 \to L_1$ is the quotient map, and so on.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{\ \iota\ } & I_0 & \xrightarrow{\qquad \delta_0 \qquad} & I_1 & \xrightarrow{\qquad \delta_1 \qquad} & I_2 & \longrightarrow & \cdots \\ & & & & & \searrow^{\pi_0} \quad \nearrow^{\iota_1} & & \searrow^{\pi_1} \quad \nearrow^{\iota_2} & & & \\ & & & & & L_1 & & L_2 & & & \end{array} \tag{A.1.8}$$

Again, the module L_1 is unique up to injective direct summands, due to a dual version of Schanuel's Lemma, which states that if $0 \to N \to I \to L \to 0$

and $0 \to N \to I' \to L' \to 0$ are exact sequences with I, I' injective, then there is an isomorphism $L \oplus I' \cong L' \oplus I$.

We define the operator Ω^{-1} to be $\Omega^{-1}(M) = L_1$, understood to take values in an equivalence class of modules, and call it a *first cosyzygy module* of M. Similarly, $L_i = \Omega^{-i}M$ is an *ith cosyzygy module* of M. This notation is chosen with the following setting in mind: Assume R is a *self-injective* algebra over a field k, that is, R is injective as an R-module (under left multiplication). Then projective A-modules are also injective, and vice versa. Combining diagrams (A.1.3) and (A.1.8) we obtain

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P_1 & & \xrightarrow{\quad d_1 \quad} & & P_0 & & \xrightarrow{\quad \iota\varepsilon \quad} & & I_0 & & \xrightarrow{\quad \delta_0 \quad} & & I_1 \longrightarrow \cdots \\
& & {\scriptstyle \varepsilon_1}\searrow & & \nearrow{\scriptstyle i_1} & & {\scriptstyle \varepsilon}\searrow & & \nearrow{\scriptstyle \iota} & & {\scriptstyle \pi_0}\searrow & & \nearrow{\scriptstyle \iota_1} & \\
& & & K_1 & & & & M & & & & L_1 & &
\end{array}$$

The modules in the top row are all projective and injective and may be viewed alternately as terms in projective and injective resolutions of the modules in the bottom row. It follows that $\Omega^{-1}(\Omega(M))$ is equivalent to M: There are projective modules P, P' such that

$$\Omega^{-1}(\Omega(M)) \oplus P \cong M \oplus P'.$$

Other types of resolutions may be defined similarly, for example, flat resolutions.

The *projective dimension* $\operatorname{pdim}_R(M)$ of an R-module M is the smallest integer n such that there is a projective resolution of M:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If no such n exists we write $\operatorname{pdim}_R(M) = \infty$. We similarly define injective dimension and flat dimension. Note that $\operatorname{pdim}_R(M)$ is the smallest integer n such that an nth syzygy module of M is projective. Similarly we see that $\operatorname{pdim}_R(\Omega M) = \operatorname{pdim}_R(M) - 1$ if M is not projective.

The *left global dimension* of R is

$$\operatorname{gldim} R = \sup\{\operatorname{pdim}_R(M) \mid M \text{ is a left } R\text{-module}\}.$$

We may similarly define right global dimension. Authors often distinguish between the two by writing gldim_l or gldim_r.

An important class of examples is provided by the following theorem.

Theorem A.1.9 (Hilbert's Syzygy Theorem)**.** *Let k be a field. Then*

$$\operatorname{gldim} k[x_1, \ldots, x_n] = n.$$

A ring R is left *hereditary* if every left ideal is projective, equivalently if $\operatorname{gldim}_l R \leq 1$. Thus for example, $k[x]$ is hereditary.

Let A, B, Y be R-modules and let $\alpha : Y \to A$, $\beta : Y \to B$ be R-module homomorphisms. A *pushout* of α, β is an R-module X together with R-module homomorphisms $\phi : A \to X$, $\psi : B \to X$ such that $\phi\alpha = \psi\beta$ and for any R-module Z and R-homomorphisms $\tilde{\phi} : A \to Z$, $\tilde{\psi} : B \to Z$ for which $\tilde{\phi}\alpha = \tilde{\psi}\beta$, there is a unique R-module homomorphism $\eta : X \to Z$ such that $\tilde{\phi} = \eta\phi$, $\tilde{\psi} = \eta\psi$:

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow{\psi} & X \end{array}$$

Note that we may take

$$X = A \oplus B / \{(-\alpha(y), \beta(y)) \mid y \in Y\}$$

and ϕ, ψ to be the maps induced by inclusion into $A \oplus B$.

Let A, B, X be R-modules and let $\phi : A \to X$, $\psi : B \to X$ be R-module homomorphisms. A *pullback* of ϕ, ψ is an R-module Y together with R-module homomorphisms $\alpha : Y \to A$, $\beta : Y \to B$ such that $\phi\alpha = \psi\beta$ and for any R-module Z and R-module homomorphisms $\tilde{\alpha} : Z \to A$, $\tilde{\beta} : Z \to B$ for which $\phi\tilde{\alpha} = \psi\tilde{\beta}$, there is a unique R-module homomorphism $\eta : Z \to Y$ such that $\tilde{\alpha} = \alpha\eta$, $\tilde{\beta} = \beta\eta$. Note that we may take

$$Y = \{(a, b) \in A \oplus B \mid \phi(a) = \psi(b)\}$$

and α, β to be the maps induced by projection from $A \oplus B$.

A.2. Ext and Tor

Let M and N be R-modules. Let $P \xrightarrow{\varepsilon} M$ be a projective resolution of M. Applying $\mathrm{Hom}_R(-, N)$ to the sequence $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$, we obtain

(A.2.1) $$0 \longrightarrow \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2^*} \cdots$$

where $d_i^*(f) = fd_i$ for all i and $f \in \mathrm{Hom}_R(P_i, N)$. We set $d_0^* = 0$. Note that $d_{i+1}^* d_i^* = 0$ since $d_i d_{i+1} = 0$, so the sequence (A.2.1) is a (cochain) complex of abelian groups (that is, $\mathbb{Z}$-modules). If R is commutative, it is a complex of R-modules. If R is an algebra over a field k, it is a complex of k-vector spaces. For consistency in indexing, we may sometimes set $\partial_n = d_{n+1}^*$ for all $n \geq -1$. We define $\mathrm{Ext}_R^*(M, N)$ to be the cohomology of this complex:

$$\begin{aligned} \mathrm{Ext}_R^n(M, N) &= H^n(\mathrm{Hom}_R(P_{\bullet}, N)) \\ &= \mathrm{Ker}(d_{n+1}^*)/\mathrm{Im}(d_n^*) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n-1}) \end{aligned}$$

for $n \geq 0$, and $\mathrm{Ext}_R^*(M, N) = \oplus_{n \geq 0} \mathrm{Ext}_R^n(M, N)$. An application of the Comparison Theorem (Theorem A.1.6) shows that $\mathrm{Ext}_R^*(M, N)$ does not depend on choice of projective resolution of M. Note that

$$\operatorname{Ext}^0_R(M,N) \cong \operatorname{Hom}_R(M,N).$$

By construction, if M is itself a projective R-module, then $\operatorname{Ext}^n_R(M,N) = 0$ for all $n > 0$.

Equivalently, we may define $\operatorname{Ext}^*_R(M,N)$ via an injective resolution: Let $N \xrightarrow{\iota} I$ be an injective resolution of N. Apply $\operatorname{Hom}_R(M,-)$ to the sequence $0 \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$ to obtain a sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,I_0) \xrightarrow{(d_0)_*} \operatorname{Hom}_R(M,I_1) \xrightarrow{(d_1)_*} \cdots \tag{A.2.2}$$

where $(d_i)_*(f) = d_i f$ for all i and $f \in \operatorname{Hom}_R(M,I_i)$. Set $(d_{-1})_* = 0$. Then $(d_{i+1})_*(d_i)_* = 0$ for all $i \geq -1$, so the sequence (A.2.2) is a (cochain) complex of abelian groups. It can be shown that

$$\operatorname{Ext}^n_R(M,N) \cong \operatorname{H}^n(\operatorname{Hom}_R(M,I_\bullet)) = \operatorname{Ker}((d_n)_*)/\operatorname{Im}((d_{n-1})_*).$$

If N is an injective R-module, we now see that $\operatorname{Ext}^n_R(M,N) = 0$ for all $n > 0$.

For each n, the group $\operatorname{Ext}^n_R(M,N)$ has an interpretation in terms of exact sequences: An *n-extension* of M by N is an exact sequence of R-modules

$$U_\bullet : \qquad 0 \longrightarrow N \longrightarrow U_{n-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M \longrightarrow 0.$$

If $V_\bullet$ is another n-extension of M by N, a map from $U_\bullet$ to $V_\bullet$ is a chain map which on either M or N is an identity map (denoted 1):

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & N & \longrightarrow & V_{n-1} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Maps of n-extensions generate an equivalence relation. There is a well-defined binary operation on equivalence classes, called the Baer sum, under which the set of equivalence classes of n-extensions is an abelian group. The group $\operatorname{Ext}^n_R(M,N)$ is isomorphic to the group of equivalence classes of n-extensions of M by N. We outline this one-to-one correspondence next.

Let $0 \to N \to U_{n-1} \to \cdots \to U_1 \to U_0 \to M \to 0$ be an n-extension of M by N. We will define an element of $\operatorname{Ext}^n_R(M,N)$ corresponding to it. Let $P \to M$ be a projective resolution of M. By the Comparison Theorem (Theorem A.1.6), there is a chain map

$$\begin{array}{ccccccccccccccc}
P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & N & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Then $f_n \in \operatorname{Hom}_R(P_n, N)$ and $f_n d_{n+1} = 0$, so f_n is a cocycle. Since $f_\bullet$ is unique up to chain homotopy, any two such maps represent the same element of $\operatorname{Ext}^n_R(M, N)$.

Now let $f \in \operatorname{Hom}_R(P_n, N)$ for which $fd_{n+1} = 0$. We will define an n-extension of M by N corresponding to f. Let X be a pushout of $P_n \xrightarrow{d_n} P_{n-1}$ and $P_n \xrightarrow{f} N$; we may simply take

$$X = (P_{n-1} \oplus N)/\{(-d_n(x), f(x)) \mid x \in P_n\}.$$

Then the following diagram commutes:

$$\begin{array}{ccccccccccccc}
P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & P_{n-2} & \longrightarrow \cdots \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow f & & \downarrow \binom{1}{0} & & \downarrow = & & \downarrow = & & \downarrow = & & \\
0 & \longrightarrow & N & \xrightarrow{\binom{0}{1}} & X & \xrightarrow{(1,0)} & P_{n-2} & \xrightarrow{d_{n-2}} \cdots \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0
\end{array}$$

(Equivalently, we may replace P_n by $K_n = \operatorname{Ker}(d_{n-1})$ in the pushout diagram.) The lower sequence is an n-extension of M by N.

The following theorem is called dimension shifting since it allows any Ext^n group to be expressed as an Ext^1 group (shifting degree, or dimension, from n to 1). It follows from close inspection of diagrams (A.1.3) and (A.1.8).

Theorem A.2.3 (Dimension shifting)**.** *Let $\Omega^i M$ denote an ith syzygy module, and $\Omega^{-i}M$ an ith cosyzygy module of M. Then*

$$\begin{aligned}
\operatorname{Ext}^n_R(M, N) &\cong \operatorname{Ext}^1_R(\Omega^{n-1}M, N), \\
\operatorname{Ext}^n_R(M, N) &\cong \operatorname{Ext}^1_R(M, \Omega^{1-n}N)
\end{aligned}$$

for all $n \geq 2$.

As observed in the last section, if A is self-injective, then $\Omega^{-1}\Omega N$ is equivalent to N (that is, isomorphic up to projective direct summands). Thus we have the following corollary.

Corollary A.2.4. *If A is a self-injective algebra, then*

$$\operatorname{Ext}^n_A(M, N) \cong \operatorname{Ext}^n_A(\Omega M, \Omega N)$$

for all A-modules M, N and $n \geq 1$.

For a finite dimensional algebra generally: We may choose a minimal projective resolution, that is one in which $P_n/\operatorname{rad}(P_n) \cong K_n/\operatorname{rad}(K_n)$ for all n. Defining $\Omega^n M = K_n$ in this way, we have $\operatorname{Ext}^n_A(M, N) \cong \operatorname{Hom}_A(\Omega^n M, N)$.

Now suppose M is a right R-module and N is a left R-module. Let $P \to M$ be a (right R-module) projective resolution of M. Apply $- \otimes_R N$ to obtain a sequence of $\mathbb{Z}$-modules:

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_R N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_R N \longrightarrow 0.$$

Here, as elsewhere, in order to minimize notational clutter, we suppress the subscript R on the tensor symbol $\otimes$, for maps and elements, when it is clear from context that they involve tensor products over R. We set $d_0 = 0$. This is a chain complex and we define $\operatorname{Tor}_n^R(M, N)$ to be its homology:

$$\operatorname{Tor}_n^R(M, N) = \mathrm{H}_n(P_\bullet \otimes_R N) = \operatorname{Ker}(d_n \otimes 1_N)/\operatorname{Im}(d_{n+1} \otimes 1_N).$$

By the Comparison Theorem (Theorem A.1.6), $\operatorname{Tor}_n^R(M, N)$ does not depend on choice of projective resolution of M. Note that $\operatorname{Tor}_0^R(M, N) \cong M \otimes_R N$.

Equivalently, we may define $\operatorname{Tor}_n^R(M, N)$ via a (left R-module) projective resolution of N: Let $Q \to N$ be a projective resolution of N and apply $M \otimes_R -$ to obtain a sequence

$$\cdots \longrightarrow M \otimes_R Q_2 \xrightarrow{1_M \otimes d_2} M \otimes_R Q_1 \xrightarrow{1_M \otimes d_1} M \otimes_R Q_0 \longrightarrow 0.$$

It can be shown that $\operatorname{Tor}_n^R(M, N) \cong \mathrm{H}_n(M \otimes_R Q_\bullet)$. By construction then, if either M or N is flat as an R-module, then $\operatorname{Tor}_n^R(M, N) = 0$ for all $n > 0$.

A.3. Long exact sequences

The following lemma is called the Snake Lemma since the diagram in the statement can be extended to include the indicated homomorphism ∂ and drawn with a snake curving from the top right to the bottom left.

Lemma A.3.1 (Snake Lemma)**.** *Let U, U', V, V', W, W' be R-modules for which there is a commuting diagram with exact rows:*

$$\begin{array}{ccccccc} & & U' & \longrightarrow & V' & \xrightarrow{p} & W' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & U & \xrightarrow{i} & V & \longrightarrow & W & & \end{array}$$

There is an exact sequence

$$\operatorname{Ker}(f) \to \operatorname{Ker}(g) \to \operatorname{Ker}(h) \xrightarrow{\partial} \operatorname{Coker}(f) \to \operatorname{Coker}(g) \to \operatorname{Coker}(h)$$

where $\partial = i^{-1}gp^{-1}(w')$ for all $w' \in \operatorname{Ker}(h)$. If the map $U' \to V'$ is injective, then $\operatorname{Ker}(f) \to \operatorname{Ker}(g)$ is injective, and if $V \to W$ is surjective, then $\operatorname{Coker}(g) \to \operatorname{Coker}(h)$ is surjective.

By the notation $p^{-1}(w')$ in the lemma, we mean any element in the inverse image of w'. Its value under $i^{-1}g$ followed by projection to $\operatorname{Coker}(f)$ will not depend on this choice.

A consequence of the Snake Lemma (Lemma A.3.1) is the following theorem.

Theorem A.3.2. *Let* $0 \to U_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} W_\bullet \to 0$ *be a short exact sequence of complexes. For each* n*, there is an abelian group homomorphism* $\partial_n : \mathrm{H}_n(W_\bullet) \to \mathrm{H}_{n-1}(U_\bullet)$ *such that*

$$\cdots \longrightarrow \mathrm{H}_{n+1}(W) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(U) \xrightarrow{\overline{f}_n} \mathrm{H}_n(V) \xrightarrow{\overline{g}_n} \mathrm{H}_n(W) \xrightarrow{\partial_n} \cdots$$

is an exact sequence, where $\overline{f}_n, \overline{g}_n$ *denote the maps induced by* f_n, g_n.

The homomorphisms ∂_n in the theorem are called *connecting homomorphisms.*

The following lemma is called the Horseshoe Lemma due to the shape of the diagram in the statement.

Lemma A.3.3 (Horseshoe Lemma)**.** *Let* U', U, U'' *be* R*-modules for which there is an exact sequence* $0 \to U' \to U \to U'' \to 0$*, and let* $P'_\bullet \to U'$*,* $P''_\bullet \to U''$ *be projective resolutions of* U' *and* U''*:*

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & U' & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & U & & \\
 & & & & & & \downarrow & & \\
\cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & U'' & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

For each n*, let* $P_n = P'_n \oplus P''_n$*. Then there are differentials* d_i *such that* $\cdots \to P_1 \xrightarrow{d_1} P_0 \to U \to 0$ *is a projective resolution of* U*, and the right column lifts to an exact sequence of complexes* $0 \to P'_\bullet \xrightarrow{i_\bullet} P_\bullet \xrightarrow{\pi_\bullet} P''_\bullet \to 0$ *with* $i_\bullet, \pi_\bullet$ *the standard inclusion and projection maps, respectively.*

The Horseshoe Lemma (Lemma A.3.3) is used in conjunction with Theorem A.3.2 to obtain the following four long exact sequences.

Theorem A.3.4 (First long exact sequence for Ext)**.** *Let* U *be an* R*-module and let* $0 \to V' \to V \to V'' \to 0$ *be an exact sequence of* R*-modules. There* *is an exact sequence*

$$0 \to \mathrm{Hom}_R(U,V') \longrightarrow \mathrm{Hom}_R(U,V) \longrightarrow \mathrm{Hom}_R(U,V'') \xrightarrow{\ \partial\ }$$

$$\mathrm{Ext}^1_R(U,V') \longrightarrow \mathrm{Ext}^1_R(U,V) \longrightarrow \mathrm{Ext}^1_R(U,V'') \xrightarrow{\partial} \mathrm{Ext}^2_R(U,V') \cdots$$

Theorem A.3.5 (Second long exact sequence for Ext)**.** *Let V be an R-module and let $0 \to U' \to U \to U'' \to 0$ be an exact sequence. There is an exact sequence*

$$0 \to \mathrm{Hom}_R(U'',V) \longrightarrow \mathrm{Hom}_R(U,V) \longrightarrow \mathrm{Hom}_R(U',V) \xrightarrow{\ \partial\ }$$

$$\mathrm{Ext}^1_R(U'',V) \longrightarrow \mathrm{Ext}^1_R(U,V) \longrightarrow \mathrm{Ext}^1_R(U',V) \xrightarrow{\partial} \mathrm{Ext}^2_R(U'',V) \cdots$$

Theorem A.3.6 (First long exact sequence for Tor)**.** *Let V be a left R-module and let $0 \to U' \to U \to U'' \to 0$ be an exact sequence of right R-modules. Then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(U'',V) \xrightarrow{\partial} \mathrm{Tor}^R_1(U',V) \longrightarrow \mathrm{Tor}^R_1(U,V) \longrightarrow$$

$$\mathrm{Tor}^R_1(U'',V) \longrightarrow U' \otimes_R V \longrightarrow U \otimes_R V \longrightarrow U'' \otimes_R V \longrightarrow 0.$$

Theorem A.3.7 (Second long exact sequence for Tor)**.** *Let U be a right R-module and let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of left R-modules. Then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(U,V'') \xrightarrow{\partial} \mathrm{Tor}^R_1(U,V') \longrightarrow \mathrm{Tor}^R_1(U,V) \longrightarrow$$

$$\mathrm{Tor}^R_1(U,V'') \longrightarrow U \otimes_R V' \longrightarrow U \otimes_R V \longrightarrow U \otimes_R V'' \longrightarrow 0.$$

A.4. Bicomplexes

Let R be a ring. Sometimes we will take $R = \mathbb{Z}$ or $R = k$ (a field).

A *bicomplex* (or *double complex*) of R-modules is a set $B = \{B_{i,j}\}_{i,j\in\mathbb{Z}}$ of R-modules B_{ij} with maps

$$d^h_{i,j} : B_{i,j} \to B_{i-1,j} \quad \text{and} \quad d^v_{i,j} : B_{i,j} \to B_{i,j-1}$$

(called *horizontal* and *vertical* differentials, respectively) such that $d^h d^h = 0$, $d^v d^v = 0$, and $d^v d^h + d^h d^v = 0$. We say that B is *bounded* if for each n, there are finitely many $B_{i,j}$ with $i+j=n$ that are nonzero. The *total complex* of the bicomplex B is

$$\mathrm{Tot}(B)_n = \oplus_{i+j=n} B_{i,j}$$

with differential $d = d^h + d^v$. (To be more precise, we may write $\mathrm{Tot}^{\oplus}(B)_n = \oplus_{i+j=n} B_{i,j}$ and $\mathrm{Tot}^{\prod}(B)_n = \prod_{i+j=n} B_{ij}$. We will generally work with bounded complexes, in which case there is no distinction.) When we refer to the homology of the bicomplex B, we mean the homology of its total complex.

Important examples of bicomplexes are tensor product complexes and Hom complexes, as we define next.

Let $(C_\bullet, d_\bullet^C)$, $(D_\bullet, d_\bullet^D)$ be complexes of right and left R-modules, respectively. Let $B_{i,j} = C_i \otimes_R D_j$ with

$$d_{i,j}^h = d_i^C \otimes 1_D \quad \text{and} \quad d_{i,j}^v = (-1)^i 1_C \otimes d_j^D.$$

Then $B_{\bullet,\bullet}$ is a bicomplex of $\mathbb{Z}$-modules. (If R is commutative, then B is a bicomplex of R-modules.)

Theorem A.4.1 (Künneth Theorem). *Let $C_\bullet$ and $D_\bullet$ be complexes of right and left R-modules, respectively, for which C_n and $d(C_n)$ are flat R-modules for all $n \in \mathbb{Z}$. Then for all $n \in \mathbb{Z}$, there is a short exact sequence:*

$$0 \longrightarrow \oplus_{i+j=n} \mathrm{H}_i(C) \otimes_R \mathrm{H}_j(D) \longrightarrow \mathrm{H}_n(C \otimes_R D) \longrightarrow \oplus_{i+j=n} \mathrm{Tor}_1^R(\mathrm{H}_i(C), \mathrm{H}_j(D)) \longrightarrow 0.$$

Remark A.4.2. The hypothesis that C_n and $d(C_n)$ are flat can be replaced by the hypothesis that D_n and $d(D_n)$ are flat as left R-modules.

Viewing a module as a complex concentrated in degree 0, with differentials all 0, we obtain the following corollary.

Theorem A.4.3 (Universal Coefficients Theorem). *Let C be a complex of right R-modules in which all C_n, $d(C_n)$ are flat, and let M be a left R-module. There is a short exact sequence*

$$0 \longrightarrow \mathrm{H}_n(C) \otimes_R M \longrightarrow \mathrm{H}_n(C \otimes_R M) \longrightarrow \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C), M) \longrightarrow 0.$$

If C is quasi-isomorphic to C' and D is quasi-isomorphic to D', then $C \otimes_R D$ is quasi-isomorphic to $C' \otimes_R D'$, via tensor product maps.

Let $(C_\bullet, D_\bullet^C)$, $(D_\bullet, d_\bullet^D)$ be complexes of left R-modules. Let $B_{i,j} = \mathrm{Hom}_R(C_i, D_j)$ with

$$d_{i,j}^h(f) = (-1)^{i-j} f d_{i+1}^C \quad \text{and} \quad d_{i,j}^v(f) = d_j^D f$$

for all $f \in \mathrm{Hom}_R(C_i, D_j)$. Then $B_{i,j}$ is a bicomplex of $\mathbb{Z}$-modules. It is common instead first to reindex so that either C or D becomes a cocomplex, but this choice will suffice for our purposes.

If C is quasi-isomorphic to C' and D is quasi-isomorphic to D', then $\mathrm{Hom}_R(C, D)$ is quasi-isomorphic to $\mathrm{Hom}_R(C', D')$.

Theorem A.4.4 (Acyclic Assembly Lemma). *Let B be a bounded bicomplex of R-modules. Then $\mathrm{Tot}(B)$ is acyclic if one of the following four conditions holds: $B_{i,j} = 0$ for all $j < 0$ and B has exact columns or exact rows, or $B_{i,j} = 0$ for all $i < 0$ and B has exact columns or rows.*

See [**Wei94**, Lemma 2.7.3] for a precise statement for unbounded complexes.

A.5. Categories, functors, derived functors

A *category* $\mathcal{C}$ is a collection of *objects* $\mathrm{Obj}(\mathcal{C})$ together with a set of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for each pair of objects A, B of $\mathcal{C}$, including an *identity morphism* $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for each object A and a binary operation called *composition* $\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ for every triple A, B, C of objects of $\mathcal{C}$, such that

$$(hg)f = h(gf) \quad \text{and} \quad 1_B f = f 1_A$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ and objects A, B, C, D of $\mathcal{C}$. (Here, as elsewhere, we have written gf in place of $g \circ f$ to denote composition.)

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an *isomorphism* if there is a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$.

In this book, we work primarily with categories of left or right modules or bimodules for a ring. The morphisms are module homomorphisms, the identity morphism is the identity homomorphism, and composition is function composition. If R is a ring, we will use the notation R-Mod (respectively, R-mod) to denote the categories of all left R-modules (respectively, all finitely generated left R-modules). The notation Mod-R (respectively, mod-R) denotes similar categories of right R-modules. We abbreviate $\mathrm{Hom}_{R\text{-Mod}}$ (respectively, $\mathrm{Hom}_{R\text{-mod}}$, $\mathrm{Hom}_{\text{Mod-}R}$, $\mathrm{Hom}_{\text{mod-}R}$) by Hom_R in all these cases. Note that for any pair of R-modules A, B, the set $\mathrm{Hom}_R(A, B)$ is in fact an abelian group under addition of functions.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns an object $F(A)$ of $\mathcal{D}$ to each object A of $\mathcal{C}$, and a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for each pair of objects A, B of $\mathcal{C}$ in such a way that $F(1_A) = 1_{F(A)}$ for all A and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms f, g that can be composed. The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is given by $1_{\mathcal{C}}(A) = A$ and $1_{\mathcal{C}}(f) = f$ for all objects A and morphisms f of $\mathcal{C}$. We have in fact defined a *covariant functor*, to be more precise. A *contravariant functor* similarly assigns to each object A of $\mathcal{C}$ an object

$F(A)$ of $\mathcal{D}$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ such that $F(1_A) = 1_{F(A)}$ and $F(g \circ f) = F(f) \circ F(g)$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \to G$ assigns a morphism $\eta_A : F(A) \to G(A)$ to each object A of $\mathcal{C}$ in such a way that $G(f) \circ \eta_A = \eta_B \circ F(f)$ for all objects A, B of $\mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, that is the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

If η_A is an isomorphism for each object A, we say that η is a *natural isomorphism*, and write $F \cong G$.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to D$, $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $FG \cong 1_{\mathcal{D}}$, $GF \cong 1_{\mathcal{C}}$.

Let R and S be rings. A functor $F : R\text{-Mod} \to S\text{-Mod}$ is *additive* if F induces homomorphisms of abelian groups $\mathrm{Hom}_R(A, B) \cong \mathrm{Hom}_S(F(A), F(B))$ for all R-modules A, B. The rings R and S are *Morita equivalent* if R-Mod and S-Mod are equivalent categories via additive functors $F : R\text{-Mod} \to S\text{-Mod}$ and $G : S\text{-Mod} \to R\text{-Mod}$. Note that the original definition of Morita equivalence requires F and G to have a particular form; this definition is equivalent.

Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

for each object X of $\mathcal{C}$ and each object Y of $\mathcal{D}$. Then we say that F is a *left adjoint* to G, and that G is a *right adjoint* to F.

Examples of adjoint functors are provided by induction and coinduction of modules: Let A be a k-algebra and let B be a k-subalgebra of A. Let N be a B-module. The *induced* (also called *tensor induced*) A-module is $A \otimes_B N$ with action of A given by multiplication on the left factor A. The *coinduced* A-module is $\mathrm{Hom}_B(A, N)$ with action of A given by

$$(a \cdot f)(a') = f(a'a)$$

for all $a, a' \in A$ and $f \in \mathrm{Hom}_B(A, N)$.

The following lemma is a statement about adjoint functors. See [**Ben91a**, Proposition 2.8.3].

Lemma A.5.1 (Nakayama relations). *Let A be a k-algebra and let B be a k-subalgebra of A. Let M be an A-module and let N be a B-module. Then*

$$\begin{aligned}\mathrm{Hom}_B(N, M) &\cong \mathrm{Hom}_A(A \otimes_B N, M),\\ \mathrm{Hom}_B(M, N) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)).\end{aligned}$$

That is, restriction from A to B has a left adjoint given by induction and a right adjoint given by coinduction.

The following lemma is a consequence of Lemma A.5.1; see [**Ben91a**, Corollary 2.8.4].

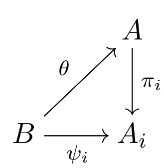
Lemma A.5.2 (Eckmann-Shapiro Lemma). *Let A be a k-algebra and let B be a k-subalgebra of A such that A is projective as a right B-module. Let M be an A-module and let N be a B-module. Then*

$$\begin{aligned}\mathrm{Ext}^n_B(N, M) &\cong \mathrm{Ext}^n_A(A \otimes_B N, M),\\ \mathrm{Ext}^n_B(M, N) &\cong \mathrm{Ext}^n_A(M, \mathrm{Hom}_B(A, N)).\end{aligned}$$

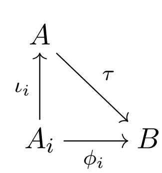
More generally we will be interested in additive functors on abelian categories, a generalization of categories of modules that retains enough structure for homological algebra. We define these next after some other needed definitions.

A *zero object* of a category $\mathcal{C}$ is an object A such that $|\mathrm{Hom}_{\mathcal{C}}(A, B)| = 1$ and $|\mathrm{Hom}_{\mathcal{C}}(B, A)| = 1$ for all objects B of $\mathcal{C}$ (or in other words, A is both an *initial* and a *terminal* object). We often write 0 instead of A.

Let $\{A_i\}_{i\in I}$ be a set of objects A_i of $\mathcal{C}$ indexed by some set I. A *product* $\prod_{i\in I} A_i$ is an object A, together with morphisms $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i)$ for all $i \in I$ satisfying the following universal property: If B is an object of $\mathcal{C}$ and $\psi_i \in \mathrm{Hom}_{\mathcal{C}}(B, A_i)$ for all $i \in I$ then there is a unique $\theta \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that the following diagram commutes for all $i \in I$:



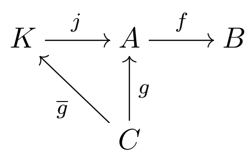
A *coproduct* $\coprod_{i\in I} A_i$ is an object A together with morphisms $\iota_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, A)$ satisfying: If B is an object of $\mathcal{C}$ and $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B)$ for all $i \in I$, then there is a unique $\tau \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that the following diagram commutes for all $i \in I$:



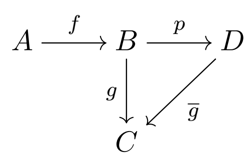
For categories of modules, product is direct product and coproduct is direct sum.

A category $\mathcal{C}$ is *additive* if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group for every object A, B in $\mathcal{C}$, composition of morphisms is $\mathbb{Z}$-bilinear, and $\mathcal{C}$ has a zero object, finite products and coproducts.

Let $\mathcal{C}$ be an additive category and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for objects A, B of $\mathcal{C}$. A *kernel* of f is an object K in $\mathcal{C}$ and a morphism $j \in \mathrm{Hom}_{\mathcal{C}}(K, A)$ such that $fj = 0$, and whenever C is an object and $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ satisfies $fg = 0$, there is a unique $\overline{g} \in \mathrm{Hom}_{\mathcal{C}}(C, K)$ such that $j\overline{g} = g$. That is, the following diagram commutes:



A *cokernel* of f is an object D in $\mathcal{C}$ and a morphism $p \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ such that $pf = 0$, and whenever C is an object and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ satisfies $gf = 0$, there is a unique $\overline{g} \in \mathrm{Hom}_{\mathcal{C}}(D, C)$ such that $\overline{g}p = g$. That is, the following diagram commutes:



Let $\mathcal{C}$ be a category. Let A, B be objects of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then f is a *monomorphism* if whenever C is an object of $\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, if $fg = fh$ then $g = h$. The morphism f is an *epimorphism* if whenever C is an object of $\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, if $gf = hg$ then $g = h$.

A category $\mathcal{C}$ is *abelian* if it is additive, every morphism has both a kernel and a cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel. Categories of R-modules are abelian.

Let $\mathcal{C}$ be an abelian category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms f, g in $\mathcal{C}$ is *exact* if f is a kernel of g and g is a cokernel of f. Projective and injective objects of $\mathcal{C}$ can be defined via standard diagrams, as well as projective and injective resolutions (which do not always exist in general). Many of the standard homological constructions and properties of the previous sections make sense in any abelian category.

Let $\mathcal{C}, \mathcal{D}$ be abelian categories. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* (respectively, *right exact*) if for every exact sequence $0 \to A \to B \to C$

(respectively, $A \to B \to C \to 0$), the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact (respectively, $F(A) \to F(B) \to F(C) \to 0$ is exact). For example, $\mathrm{Hom}_R(D, -)$ is left exact. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* (respectively, *right exact*) if for every exact sequence $A \to B \to C \to 0$ (respectively, $0 \to A \to B \to C$), the sequence

$$0 \to F(C) \to F(B) \to F(A)$$

is exact (respectively, $F(C) \to F(B) \to F(A) \to 0$ is exact). For example, $\mathrm{Hom}_R(-, D)$ is left exact. In either case, F is *exact* if it is both left and right exact.

Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$ an additive (covariant) functor. Assume $\mathcal{C}$ *has enough projectives*, that is, assume that for each object A of $\mathcal{C}$, there is an epimorphism from a projective object in $\mathcal{C}$ to A. For each object A in $\mathcal{C}$ choose a projective resolution $P_\bullet$ of A, which exists since $\mathcal{C}$ has enough projectives. Apply the functor F:

$$F(P_\bullet): \qquad \cdots \longrightarrow F(P_2) \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \longrightarrow 0.$$

Then $F(P_\bullet)$ is a complex and we define the *left derived functor* of F to be $L_\bullet F$ where $L_n F(A) = \mathrm{H}_n(F(P_\bullet))$. Note that if F is right exact, then $L_0 F(A) \cong F(A)$. (The adjective "left" here indicates the objects are on the left with 0 at the end.) A typical example is: Let R be a ring, $\mathcal{C} = \text{Mod-}R$, $\mathcal{D} = \mathbb{Z}\text{-Mod}$, B an object of R-Mod, and F the functor $- \otimes_R B$. Then

$$L_n F(A) = \mathrm{Tor}_n^R(A, B).$$

Assume $\mathcal{D}$ *has enough injectives*, that is, assume that for each object A of $\mathcal{C}$, there is a monomorphism from A to an injective object. For each object B in $\mathcal{C}$ choose an injective resolution $I_\bullet$ of B. Apply the functor F:

$$F(I_\bullet): \qquad 0 \longrightarrow F(I_0) \longrightarrow F(I_1) \longrightarrow F(I_2) \longrightarrow \cdots$$

Then $F(I_\bullet)$ is a complex and we define the *right derived functor* of F to be $R^\bullet F$ where $R^n F(B) = \mathrm{H}^n(F(I_\bullet))$. Note that if F is left exact, then $R^0 F(B) \cong F(B)$. (The adjective "right" here indicates the objects are on the right with 0 at the beginning.) A typical example is: Let R be a ring, $\mathcal{C} = R\text{-Mod}$, $\mathcal{D} - \mathbb{Z}\text{-Mod}$, A an object of $\mathcal{C}$, and $F(B) = \mathrm{Hom}_R(A, B)$. Then

$$R^n F(B) = \mathrm{Ext}_R^n(A, B).$$

Similarly one defines derived functors of contravariant functors: For a right derived functor, use a projective resolution, and for a left derived functor, use an injective resolution.

A.6. Spectral sequences

We will define cohomology spectral sequences here; homology spectral sequences are similar but with arrows reversed.

Definition A.6.1. A *cohomology spectral sequence* in an abelian category $\mathcal{C}$ is a set $\{E_r^{pq} \mid p, q, r \in \mathbb{Z},\ r \geq 0\}$ of objects in $\mathcal{C}$, together with morphisms

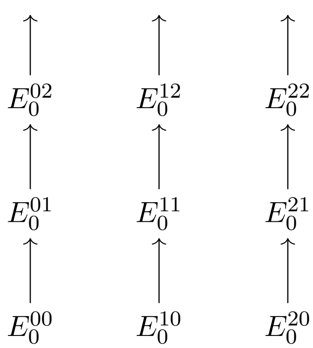
$$d_r^{pq} : E_r^{pq} \to E_r^{p+r, q-r+1}$$

for which $d_r^2 = 0$ and $E_{r+1} \cong \mathrm{H}^\bullet(E_r)$, that is,

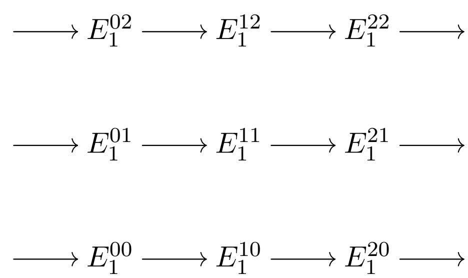
$$E_{r+1}^{p,q} \cong \mathrm{Ker}(d_r^{pq}) / \mathrm{Im}(d_r^{p-r, q+r-1}).$$

For each r, the set E_r of objects E_r^{pq} together with the morphisms d_r^{pq} is the *rth page* of the spectral sequence. A page can be visualized in a plane, such as pages E_0, E_1, E_2 below.

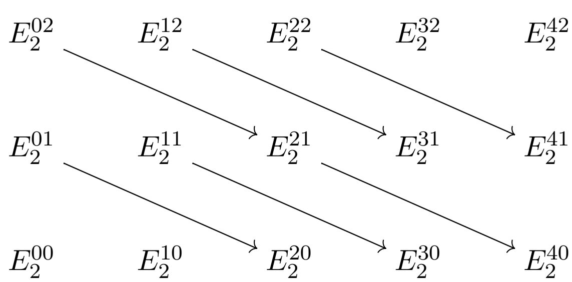
$\underline{E_0}$



$\underline{E_1}$



$\underline{E_2}$



Definition A.6.2. A spectral sequence (E, d) is *bounded* if for each $n \in \mathbb{Z}$ there are finitely may nonzero E_0^{pq} with $p + q = n$.

By its definition, in case (E, d) is bounded, for each pair p, q, there is an r_0 (depending on p, q) such that $E_r^{pq} \cong E_{r_0}^{pq}$ for all $r \geq r_0$. In this case, we write

$$E_\infty^{pq} = E_{r_0}^{pq}.$$

Definition A.6.3. A bounded spectral sequence (E, d) *converges* if there is a family $\{H^n \mid n \in \mathbb{Z}\}$ of objects of $\mathcal{C}$, each having a finite filtration

$$0 = F^t H^n \subset \cdots \subset F^{p+1} H^n \subset F^p H^n \subset F^{p-1} H^n \subset \cdots \subset F^s H^n = H^n,$$

and isomorphisms $E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$ for all p, q.

In this book we will use spectral sequences associated to double complexes, defined next. In this context, common notation for a double complex is $B = (B^{pq}, d', d'')$ where d' and d'' are the horizontal and vertical differentials. For each n, write $B^n = \operatorname{Tot}(B)_n = \oplus_{p+q=n} B^{p,q}$, a complex with $d = d' + d''$. The notation B will then sometimes refer to this complex, when no confusion will arise.

For each p, n, let

$$F^p B^n = \oplus_{p' \geq p} B^{p', n-p'}. \tag{A.6.4}$$

That is, we truncate the double complex at the pth column, replacing all objects to the left of this column by 0, and then sum over diagonal lines whose indices sum to a fixed value n.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{p,2} & \longrightarrow & B^{p+1,2} & \longrightarrow & B^{p+2,2} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{p,1} & \longrightarrow & B^{p+1,1} & \longrightarrow & B^{p+2,1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{p,0} & \longrightarrow & B^{p+1,0} & \longrightarrow & B^{p+2,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

If B is bounded, then for each n, this yields a finite filtration of B^n. If B is a first quadrant double complex (that is, $B^{pq} = 0$ whenever $p < 0$ or $q < 0$), then $F^0 B^n = B^n$ and $F^p B^n = 0$ for all $p > n$.

For each p, q, r, let

$$C_r^{pq} = \{x \in F^p B^{p+q} \mid d(x) \in F^{p+r} B^{p+q+1}\}.$$

In particular, $C_0^{pq} = F^p B^{p+q}$ by definition. Also by definition, if $x \in C_r^{pq}$, then $d(x)$ has component 0 within the band $p \le p' \le p + r$, in order that $d(x)$ be in $F^{p+r} B^{p+q+1}$ as specified. Let $E_0^{pq} = C_0^{pq}$ and

$$E_r^{pq} = \frac{C_r^{pq} + F^{p+1} B^{p+q}}{d(C_{r-1}^{p-r+1,q+r-2}) + F^{p+1} B^{p+q}} \tag{A.6.5}$$

for each $r > 0$ and $p, q \in \mathbb{Z}$. By the definitions there are induced morphisms

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$$

for which $d_r^2 = 0$. By the definitions,

$$\mathrm{H}^\bullet(E_r) \cong E_{r+1}.$$

Note that $E_1^{p,q}$ is often written $\mathrm{H}''(B)^{p,q}$, that is the cohomology of B with vertical differentials only, and E_2^{pq} is often written $\mathrm{H}'\,\mathrm{H}''(B)^{p,q}$, the cohomology of $\mathrm{H}''(B)^{pq}$ with respect to the differential induced by the horizontal differential on B only.

We have assumed B is bounded, and so for each p, q, there is an r_0 for which the differentials $d_{r_0}^{pq}$ as well as $d_{r_0}^{p-r_0,q+r_0-1}$ (that is, those starting and ending at position p, q) are zero maps. So $E_\infty^{pq} = E_{r_0}^{pq}$.

Let $p, q \in \mathbb{Z}$, $n = p + q$, and let $x \in B^{p,q}$ be a cocycle such that $x \notin F^{p+1} B^n$. Then x determines an element of E_r^{pq} for all $r \ge 1$ and d_r is 0 on the corresponding element of E_∞^{pq}. This describes a morphism from $F^p\,\mathrm{H}^{p+q}(B)$, which is the image of $\mathrm{H}^{p+q}(F^p B)$ in $\mathrm{H}^{p+q}(B)$, to E_∞^{pq}. Moreover, this is an epimorphism since $E_\infty^{pq} = E_{r_0}^{pq}$ for some r_0. The kernel of the epimorphism is $F^{p+1}\,\mathrm{H}^{p+q}(B)$ by the definitions. As a consequence, $\mathrm{H}^\bullet(B)$ is filtered with filtration given by $F^p\,\mathrm{H}^\bullet(B)$ and

$$F^p\,\mathrm{H}^n(B)/F^{p+1}\,\mathrm{H}^n(B) \cong E_\infty^{pq}(B)$$

for fixed $p + q = n$. That is, E_r converges to $\mathrm{H}^\bullet(B)$:

Theorem A.6.6. *Let B be a bounded bicomplex. Give B the filtration (A.6.4) and let E be the corresponding spectral sequence given by (A.6.5). Then E_r converges to* $\mathrm{H}^*(B)$.

The spectral sequence E is *multiplicative* if E_0 has a bigraded product

$$E_0^{p,q} \times E_0^{p',q'} \to E_0^{p+p',q+q'}$$

satisfying the Leibniz relations:

$$d(xy) = d(x)y + (-1)^p x d(y)$$

for $x \in E_0^{p,q}, y \in E_0^{p',q'}$. It follows that for all r, E_r has a bigraded product such that the Leibniz relations hold. In the context of Theorem A.6.6, if E is multiplicative, then it converges to the associated graded algebra of $\mathrm{H}^*(B)$.

Note that we could have chosen to filter the complex instead by truncating rows, resulting in another spectral sequence. Comparison of these two spectral sequences can be useful.

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