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THE REPRESENTATION RING OF THE QUANTUM DOUBLE OF A FINITE GROUP

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CHAPTER 1

INTRODUCTION

The quantum double of a Hopf algebra, also called the Drinfel'd double, was defined by Drinfel'd in the context of finding solutions to the quantum Yang-Baxter equation of statistical mechanics [10]. The special case of the quantum double $D(G)$ of a finite group G lends itself to study by group-theoretic methods, and has implications for group theory. The representations of $D(G)$ over the complex numbers arise in the study of holomorphic orbifold models in conformal field theory [9, 12], as well as in applications to moonshine, that is the connection between the representations of finite groups and modular functions. An explicit description of representations for $D(G)$ is anticipated in a paper of Lusztig on characters of Hecke algebras [11]. Bantay developed the complex character theory, similar to that of a finite group, of a more general algebra than the quantum double $D(G)$ which has application to generalized Thompson series and moonshine [3]. In this thesis, we present some results about the representations of the quantum double over algebraically closed fields of arbitrary characteristic.

The quantum double $D(G)$ is a skew group algebra. It is the smash product of the group algebra kG, where k is a field, with its Hopf algebraic dual $(kG)^*$. This construction is analogous to a semidirect product of groups; it provides an algebraic structure on the vector space $(kG)^* \otimes kG$. The algebra $D(G)$ may be given the tensor product coalgebra structure, and in this way it becomes a Hopf algebra.

In Chapter 2 we give details and state various properties of the quantum double $D(G)$ of a finite group. For example, we show that $D(G)$ is a symmetric algebra. We prove that $D(G)$ is semisimple if and only if the order of G is invertible in k; this was proved in [12] by a different approach. We also make a simple observation which shows that the cohomology groups of $D(G)$ are isomorphic to those of kG.

As $D(G)$ is a Hopf algebra, it has a representation ring $R(D(G))$. This is the $\mathbb{C}$-algebra generated by isomorphism classes of finite dimensional $D(G)$-modules with direct sum for addition and tensor product for multiplication. In fact, this ring is commutative, as we show in Chapter 2.

In Chapter 3, we consider G-equivariant k-vector bundles on finite G-sets; these are analogous to the G-equivariant $\mathbb{C}$-vector bundles appearing in [11]. We conclude an equivalence of the category of $D(G)$-modules with the category of G-equivariant vector bundles on the G-set G (under the conjugation action) from a more general equivalence given in Theorem 3.2.2. We deduce the well-known result that the indecomposable $D(G)$-modules are indexed by pairs (U, g) where g is a representative of a conjugacy class of G and U is an indecomposable $kC(g)$-module. Here $C(g) = C_G(g)$ denotes the centralizer of g in G. Different approaches to this result appear in [7, 12]; see also [9] for the special case $k = \mathbb{C}$. The equivalence of categories also leads to an isomorphism of representation rings, providing an alternate description of the representation ring $R(D(G))$ analogous to a ring considered by Lusztig when $k = \mathbb{C}$ [11]. This description facilitates proofs in Chapters 4 and 5.

In Chapter 4, we examine the representation ring $R(D(G))$ of the quantum double using an approach similar to that taken by Benson and Parker in examining the representation ring $R(G)$, or Green ring, of the group algebra kG [5]. They developed a theory of characters of the Green ring $R(G)$, that is algebra homomorphisms from $R(G)$ to $\mathbb{C}$. These characters are called "species" in order to distinguish them from characters of the group. Species provide an extension of the concept of Brauer characters of the group; there are certain species which correspond to the Brauer characters, and in general there are other species as well.

We prove two main results in Chapter 4 concerning the characters of the representation ring $R(D(G))$ of the quantum double. The first, Theorem 4.2.4, is a formula for characters of $R(D(G))$ given characters, or species, of the Green ring $R(G)$. For the second, Theorem 4.3.2, we assume the characteristic of k is prime. This result uses the characters obtained in Theorem 4.2.4 from the Brauer characters of the group to prove that the Grothendieck ring, which is the quotient of the representation ring $R(D(G))$ by the ideal of short exact sequences, is semisimple.

Theorem 4.3.2 is analogous to a theorem of Brauer which essentially states that the Grothendieck ring of kG-modules is semisimple, with a complete set of characters given by the columns of the Brauer character table [4]. The values in this table characterize kG-modules up to composition factors. Similarly, Theorem 4.3.2 allows us to write down a character table that characterizes $D(G)$-modules by their composition factors.

Theorem 4.2.4, while used as a lemma for Theorem 4.3.2, is also of more general interest. If the characteristic p of k is a prime dividing the order of the group G, there are in general species other than those corresponding to the Brauer characters [5]. Thus species are a finer invariant than are the Brauer characters for kG-modules, and Theorem 4.2.4 provides for a correspondingly finer invariant of $D(G)$-modules than that indicated by Theorem 4.3.2.

In Chapter 5, we show that there are certain Hopf subalgebras of $D(G)$, indexed by the subgroups of G, such that the collection of their representation rings constitutes a Green functor. Thévenaz' twin functor construction [16] then leads to a direct sum decomposition of the representation ring $R(D(G))$ in Theorem 5.2.3 analogous to a similar decomposition of the Green ring. This decomposition provides an immediate proof of an induction theorem (Corollary 5.2.4), and a connection between the questions of semisimplicity of $R(D(G))$ and questions of semisimplicity of the Green ring $R(G)$ (Corollary 5.2.5). Finally, the decomposition in Theorem 5.2.3 yields a proof that the characters of the representation ring $R(D(G))$ described in Chapter 4 are all of its characters (Theorem 5.3.2).

We refer the reader to [1] for basic results about representations of finite groups and finite dimensional algebras, to [13] or [15] for standard definitions and results regarding Hopf algebras, and to [4] for basic results about Brauer characters and group cohomology. Results about Green functors (or algebra G-functors) may be found in [16]. The term "Green functor" appears to be the one accepted today. Results about species may be found in [5].

CHAPTER 2

DEFINITIONS AND PRELIMINARY RESULTS

Let G be a finite group and k an algebraically closed field of characteristic p. All modules will be right modules, finite dimensional over k. Tensor products will be over k unless otherwise indicated.

2.1. Definitions of the quantum double $D(G)$

We shall give two definitions of the *quantum double* $D(G)$ of the group G. Let $(kG)^* = \mathrm{Hom}_k(kG, k)$ be the k-algebra on the space dual to kG with multiplication pointwise on group elements, that is $(ff')(g) = f(g)f'(g)$ for all $f, f' \in (kG)^*$ and $g \in G$. We give the vector space $(kG)^* \otimes kG$ an algebra structure as follows. The group G acts as automorphisms of $(kG)^*$ by

$$f^g(x) = f(gxg^{-1})$$

for all $g, x \in G$ and $f \in (kG)^*$. Multiplication on $(kG)^* \otimes kG$ may then be defined by

$$(f \otimes g)(f' \otimes g') = ff'^{g^{-1}} \otimes gg',$$

for all $f, f' \in (kG)^*$ and $g, g' \in G$. The resulting algebra is the quantum double $D(G)$ of G. Both kG and $(kG)^*$ are naturally embedded as subalgebras of $D(G)$.

The group algebra kG is a Hopf algebra [13, 15] with coproduct $\Delta : kG \to kG \otimes kG$ defined by

$$\Delta(g) = g \otimes g$$

(that is, g is a group-like element), counit $\epsilon : kG \to k$ by $\epsilon(g) = 1$, and coinverse $s : kG \to kG$ by $s(g) = g^{-1}$, for all elements g of G. We give $(kG)^*$ the dual

Hopf algebra structure. Then $D(G)$ becomes a Hopf algebra with the tensor product coalgebra structure, and algebra structure as defined above.

We next exhibit the Hopf algebraic structure of $D(G)$ explicitly on the natural basis; the following may be taken as an alternative definition of the quantum double $D(G)$ of the group G. If $\{\phi_g\}_{g\in G}$ is the basis of $(kG)^*$ dual to $\{g\}_{g\in G}$, then $D(G)$ has as a basis all elements $\phi_g \otimes h$, which we write more simply as $\phi_g h$, for $g, h \in G$. On this basis, the product is defined by $\phi_g h \phi_{g'} h' = \phi_g \phi_{hg'h^{-1}} hh'$, which is nonzero if and only if $g = hg'h^{-1}$. Thus the identity is $1_{D(G)} = \sum_{g\in G} \phi_g 1$, where 1 is the identity for G. The coproduct is given by

$$\Delta(\phi_g h) = \sum_{x\in G} \phi_x h \otimes \phi_{x^{-1}g} h,$$

the counit by $\epsilon(\phi_g h) = \delta_{1,g}$, and the coinverse by $s(\phi_g h) = h^{-1}\phi_{g^{-1}} = \phi_{h^{-1}g^{-1}h} h^{-1}$.

2.2. Representation rings

We next discuss representation rings and fix notation. If A is any Hopf algebra, the tensor product $U \otimes V$ of two right A-modules U and V is given the structure of a right A-module by restricting the natural action of $A \otimes A$ on $U \otimes V$ via the coproduct $\Delta : A \to A \otimes A$. This is a right A-module action since Δ is an algebra homomorphism. The field k is given the structure of a right A-module by restriction of the action of k on itself by right multiplication to A via the counit $\epsilon : A \to k$. In other words, an element $a \in A$ acts on k as multiplication by $\epsilon(a)$. This is called the *trivial* module. Up to isomorphism, the trivial module is a multiplicative identity with respect to tensor product of modules; this follows from the counit property of Hopf algebras.

Let $r(A)$ denote the ring generated by isomorphism classes of A-modules with direct sum for addition and tensor product for multiplication, that is

$$[U] + [V] = [U \oplus V] \quad \text{and} \quad [U] \cdot [V] = [U \otimes V],$$

where $[U]$ denotes the isomorphism class of the A-module U. Then $r(A)$ is a ring with identity given by the isomorphism class of the trivial module. Associativity of

$r(A)$ follows from coassociativity of the coproduct for A. We shall refer to both $r(A)$ and $R(A) = r(A) \otimes_{\mathbb{Z}} \mathbb{C}$ as *representation rings.* We shall work primarily with $R(A)$, as our main interest is in characters. By abuse of language and notation, we shall consider A-modules to be elements of the representation rings, when we really mean their isomorphism classes.

We next define the "ideal of short exact sequences" $R_0(A)$ of the representation ring $R(A)$. Let $R_0(A)$ be the ideal of $R(A)$ generated by elements of the form $U - U' - U''$ where

$$0 \to U' \to U \to U'' \to 0$$

is a short exact sequence of A-modules. The *Grothendieck ring* of A is the quotient $\mathcal{R}(A) = R(A)/R_0(A)$.

Returning to the situation of the quantum double, let H be a subgroup of the finite group G. We note that the natural embedding of the group algebra kH as a subalgebra of $D(G)$ is also an embedding of Hopf algebras. Thus we have the following lemma, where $R(H) = R(kH)$ is the representation ring, or *Green ring*, of the group algebra kH.

Lemma 2.2.1 *Restriction of action from $D(G)$ to kH defines an algebra homomorphism on the corresponding representation rings,*

$$\mathrm{res}_H^{D(G)} : R(D(G)) \to R(H). \ \square$$

We shall sometimes consider a $D(G)$-module to be a kH-module via $\mathrm{res}_H^{D(G)}$, and may not explicitly state that we are using the restriction map.

The quantum double $D(A)$ of any Hopf algebra A is *quasitriangular* [10, 13]. That is, there exists an invertible element R of $D(A) \otimes D(A)$ such that $R\Delta(x)R^{-1} = \sigma(\Delta(x))$ for all $x \in D(A)$, where Δ is the coproduct of $D(A)$, and σ is the twist automorphism interchanging two factors. For the quantum double $D(G)$ of a finite group G, we have $R = \sum_{g \in G} \phi_g \otimes g$, with $R^{-1} = \sum_{g \in G} \phi_g \otimes g^{-1}$. The equation $R\Delta R^{-1} = \sigma\Delta$ results in an isomorphism between $U \otimes V$ and $V \otimes U$, for any two modules U and V, given by the twist σ followed by the action of R. We prove this isomorphism for the special case $D(G)$ in the following lemma.

Lemma 2.2.2 *Given any two $D(G)$-modules U and V, the $D(G)$-modules $U \otimes V$ and $V \otimes U$ are isomorphic. In particular, the representation ring $R(D(G))$ is commutative.*

Proof: Let f be the map from $U \otimes V$ to $V \otimes U$ given by the twist σ followed by the natural action of $R = \sum_{g \in G} \phi_g \otimes g$ on $V \otimes U$. As both operations are invertible, f is a bijection. It remains to check that f is a $D(G)$-map. Let $u \in U$, $v \in V$, and $g, h \in G$. Then

$$\begin{aligned} f(u \otimes v)\phi_g h &= \left(\sum_{z \in G} v\phi_z \otimes uz \right) \phi_g h \\ &= \sum_{z,y \in G} v\phi_z\phi_y h \otimes uz\phi_{y^{-1}g} h \\ &= \sum_{z \in G} v\phi_z h \otimes uz\phi_{z^{-1}g} h \\ &= \sum_{z \in G} v\phi_z h \otimes u\phi_{gz^{-1}} zh. \end{aligned}$$

On the other hand,

$$\begin{aligned} f\left((u \otimes v)\phi_g h\right) &= \sum_{x \in G} f(u\phi_x h \otimes v\phi_{x^{-1}g} h) \\ &= \sum_{x,y \in G} v\phi_{x^{-1}g} h\phi_y \otimes u\phi_x hy \\ &= \sum_{x,y \in G} v\phi_{x^{-1}g}\phi_{hyh^{-1}} h \otimes u\phi_x hy \\ &= \sum_{y \in G} v\phi_{hyh^{-1}} h \otimes u\phi_{ghy^{-1}h^{-1}} hy. \end{aligned}$$

Resumming over $z = hyh^{-1}$, we find that

$$f\left((u \otimes v)\phi_g h\right) = \sum_{z \in G} v\phi_z h \otimes u\phi_{gz^{-1}} zh = f(u \otimes v)\phi_g h. \quad \square$$

2.3. Properties of $D(G)$

We next discuss various properties of the algebra $D(G)$, some of which will be used in the sequel.

The algebra $D(G)$ is a skew group algebra, and thus it is a fully group-graded algebra (or strongly group-graded algebra) [14]. This fact will be useful in Chapter 5. In particular, for each subgroup H of G, there is a subalgebra

$$D_G(H) = \sum_{g \in G, h \in H} k\phi_g h.$$

We choose this notation in accordance with that in the next chapter; note that $D_G(G) = D(G)$. The coalgebra structure of $D(G)$ restricts to a coalgebra structure on each subalgebra $D_G(H)$, so that $D_G(H)$ is in fact a Hopf subalgebra of $D(G)$. It is useful to notice that for any two subgroups $L \leq H$ of G, we have an isomorphism of left $D_G(L)$-modules

$$D_G(H) \simeq \sum_{h \in L \backslash H} D_G(L)h,$$

with $D_G(L)h \simeq D_G(L)$ as a left $D_G(L)$-module for each right coset representative h of $L \backslash H$. That is, $D_G(H)$ is a free left $D_G(L)$-module.

We caution that while the representation ring $R(D(G))$ is commutative by Lemma 2.2.2, the representation ring $R(D_G(H))$, for H a proper subgroup of G, may not be commutative. In particular, for the identity subgroup $H = 1$, the Hopf subalgebra $D_G(H)$ is simply $(kG)^*$. The indecomposable $(kG)^*$-modules are all the spaces $k\phi_g$ for $g \in G$, the tensor product of two such modules corresponding to the product in G.

Larson and Sweedler proved that all finite dimensional Hopf algebras are Frobenius algebras (see [13], Theorem 2.1.3). The bilinear form provided in the proof of that theorem for the case of the quantum double $D(G)$ turns out to be symmetric as well. In the proof of the next lemma, we define this bilinear form without reference to the background in [13]. The lemma will not be needed in this thesis.

Lemma 2.3.1 *The quantum double $D(G)$ of a finite group G is a symmetric algebra.*

Proof: Define a linear map $\tau : D(G) \to k$ by requiring that $\tau(\phi_g h) = \delta_{1,h}$. Define a pairing by

$$(x, y) = \tau(xy),$$

for all $x, y \in D(G)$. This pairing is clearly bilinear. On the standard basis elements of $D(G)$, we have

$$(\phi_g h, \phi_{g'} h') = \tau(\phi_g h \phi_{g'} h') = \delta_{g, hg'h^{-1}} \delta_{1, hh'}.$$

If we let $\widetilde{\phi_g h} = \phi_{h^{-1}gh} h^{-1}$, then $\widetilde{\phi_g h}$ is the unique standard basis element whose pairing with $\phi_g h$ is nonzero. Thus the form is nondegenerate. It is symmetric since $\sim$ is an involution. The definition of the pairing as the composition of the product map with τ results in its associativity, that is $(x, yz) = (xy, z)$ for all $x, y, z \in D(G)$. Thus we have a nondegenerate symmetric associative bilinear form on $D(G)$, and $D(G)$ is a symmetric algebra. $\square$

In case $k = \mathbb{C}$, we point out that the $\mathbb{Z}$-span of $\{\phi_g h\}_{g,h \in G}$ in $D(G)$ is a *based ring* [11], with $\tau : \sum_{g,h \in G} \mathbb{Z} \phi_g h \to \mathbb{Z}$ given by $\tau(\phi_g h) = \delta_{1,h}$ and involution $\widetilde{\phi_g h} = h^{-1} \phi_g = \phi_{h^{-1}gh} h^{-1}$, as in the lemma. In this case it follows that $D(G)$ is semisimple [11], as was noted in [9]. Theorem 2.3.3 below gives a criterion for semisimplicity in the general case. We shall use the following lemma.

Lemma 2.3.2 *Let U be a $D(G)$-module and V a $D(G)$-submodule of U which is a direct summand of U as a kG-module. Then V is a $D(G)$-direct summand of U.*

Proof: Let $\pi : U \to V$ denote a kG-module projection of U onto V. Define $\overline{\pi} : U \to V$ by $\overline{\pi}(u) = \sum_{g \in G} \pi(u\phi_g)\phi_g$ for all $u \in U$.

If $v \in V$, then $\overline{\pi}(v) = \sum_{g \in G} \pi(v\phi_g)\phi_g = \sum_{g \in G} v\phi_g\phi_g = v$, as V is a $D(G)$-module and $\sum_{g \in G} \phi_g = 1$. Thus $\overline{\pi}$ is the identity on V.

It remains to show that $\overline{\pi}$ is a $D(G)$-module map. If $u \in U$ and $g', h' \in G$, then

$$\overline{\pi}(u)\phi_{g'} h' = \sum_{g \in G} \pi(u\phi_g)\phi_g\phi_{g'} h'$$

$$
\begin{aligned}
&= \pi(u\phi_{g'})\phi_{g'}h' \\
&= \pi(u\phi_{g'}h')\phi_{h'^{-1}g'h'} \\
&= \sum_{g\in G} \pi(u\phi_{g'}\phi_{h'gh'^{-1}}h')\phi_g \\
&= \sum_{g\in G} \pi(u\phi_{g'}h'\phi_g)\phi_g \\
&= \overline{\pi}(u\phi_{g'}h'). \ \square
\end{aligned}
$$

In the next result, we shall make use of the trivial $D(G)$-module k, on which a basis element $\phi_g h$ of $D(G)$ acts as multiplication by $\epsilon(\phi_g h) = \delta_{1,g}$. Its proof is adapted from the proof of Maschke's Theorem given in [1, p. 12]. The "if" part also follows from Maschke's Theorem for crossed products [14].

Theorem 2.3.3 *The algebra $D(G)$ is semisimple if and only if the characteristic p of k does not divide the order $|G|$ of G.*

Proof: Assume first that p does not divide $|G|$. Let U be a $D(G)$-module, and V a $D(G)$-submodule of U. It suffices to prove that V is a direct summand of U. Restricting to kG, we see that V is a kG-submodule of U. As p does not divide $|G|$, Maschke's Theorem states that kG is semisimple, and so V is a kG-direct summand of U. By Lemma 2.3.2, V is then a $D(G)$-direct summand of U.

Conversely, suppose that p does divide $|G|$. If $D(G)$ were semisimple, then the one dimensional trivial $D(G)$-module k would appear once in a decomposition of $D(G)$ into a direct sum of irreducible $D(G)$-modules. In particular, any composition series of $D(G)$ would contain exactly one composition factor isomorphic with k.

As the counit $\epsilon : D(G) \to k$ is an algebra homomorphism, $\mathrm{Ker}(\epsilon)$ is an ideal of $D(G)$. Thus it is a right submodule of $D(G)$. We claim that the module $D(G)/\mathrm{Ker}(\epsilon)$ is isomorphic to the trivial module: Note that $1_{D(G)} = \sum_{g\in G}\phi_g 1 \in D(G) - \mathrm{Ker}(\epsilon)$, and

$$
1_{D(G)} \cdot \phi_g h = 1_{D(G)} + (\phi_g h - 1_{D(G)}) \in \begin{cases} 1_{D(G)} + \mathrm{Ker}(\epsilon), & \text{if } g = 1 \\ \mathrm{Ker}(\epsilon), & \text{if } g \neq 1. \end{cases}
$$

On the other hand, consider the element $t = \sum_{g \in G} \phi_1 g$. As p divides $|G|$, we have $t \in \mathrm{Ker}(\epsilon)$. Also if $g', h' \in G$, then

$$t \cdot \phi_{g'} h' = \delta_{1,g'} t = \epsilon(\phi_{g'} h') t,$$

so kt is a submodule of $D(G)$ which is also isomorphic to the trivial module. Thus we have a series of submodules,

$$D(G) \supset \mathrm{Ker}(\epsilon) \supset kt \supset 0,$$

whose refinement to a composition series contains the trivial module more than once as a composition factor. □

There are other proofs of Theorem 2.3.3. For example, see [12]. Alternatively, the theorem follows directly from Maschke's Theorem for finite dimensional Hopf algebras, which states that a finite dimensional Hopf algebra A is semisimple if and only if $\epsilon(t) \neq 0$ for some $t \in \int_A^r$ [13, 15]. The set of *right integrals* $\int_A^r$ is the set of all $t \in A$ such that $t \cdot a = \epsilon(a)t$ for all $a \in A$. In the case $A = D(G)$, we may take $t = \sum_{g \in G} \phi_1 g$, so that $\epsilon(t) = |G|$.

We finally make a simple observation which results in a proof that the cohomology groups of the quantum double $D(G)$ and of the group algebra kG are isomorphic. This result will not be needed in the sequel.

We may embed kG as an ideal direct summand of $D(G)$ in the following way. Denote by $D(G)_1$ the subspace $\sum_{g \in G} k\phi_1 g$ of $D(G)$. Then $D(G)_1$ is an ideal direct summand of $D(G)$, with complement the ideal $\sum_{g,h \in G, g \neq 1} k\phi_g h$. Further, $D(G)_1$ is isomorphic to kG as an algebra. In this way, kG-modules have $D(G)$-module structures in which projective kG-modules become projective $D(G)$-modules. We write $\mathrm{incl}_{G,1}$ for this map, which takes a kG-module U to the $D(G)$-module on the vector space U with $\phi_g h$ acting as h if $g = 1$ and as 0 otherwise. The map $\mathrm{incl}_{G,1}$ may be extended to a functor embedding the category of kG-modules as a full subcategory of the category of $D(G)$-modules. Note also that the trivial $D(G)$-module is the image of the trivial kG-module under $\mathrm{incl}_{G,1}$.

Lemma 2.3.4 *For each natural number n, the cohomology groups $H^n(D(G), k)$ and $H^n(G, k)$ are isomorphic.*

Proof: Consider k to be the trivial kG-module, and let

$$\dots \to P_1 \to P_0 \to k \to 0$$

be a projective resolution of k. By the above observation, we may apply the functor $\mathrm{incl}_{G,1}$ to obtain a projective resolution of the trivial $D(G)$-module $\mathrm{incl}_{G,1}(k)$ (which we also denote by k). Further, the resulting Hom sequences

$$0 \to \mathrm{Hom}_{kG}(k, k) \to \mathrm{Hom}_{kG}(P_0, k) \to \mathrm{Hom}_{kG}(P_1, k) \to \dots,$$

and

$$0 \to \mathrm{Hom}_{D(G)}(k, k) \to \mathrm{Hom}_{D(G)}(\mathrm{incl}_{G,1}(P_0), k) \to \mathrm{Hom}_{D(G)}(\mathrm{incl}_{G,1}(P_1), k) \to \dots,$$

are isomorphic as complexes. Thus the cohomology groups $H^n(G, k)$ and $H^n(D(G), k)$ computed from these sequences are isomorphic. □

We may extend the idea of embedding kG as an ideal direct summand of $D(G)$ to obtain a decomposition of $D(G)$ into a direct sum of ideals indexed by the conjugacy classes of G, an approach taken in [12] to characterize $D(G)$-modules. We shall take a slightly different approach in the next chapter.

CHAPTER 3

G-EQUIVARIANT VECTOR BUNDLES

In this chapter, we consider G-equivariant k-vector bundles on finite G-sets. These are analogous to the G-equivariant complex vector bundles on finite sets discussed by Lusztig in [11]. The utility of vector bundles in the study of the quantum double $D(G)$ is indicated by the result that $D(G)$-modules are essentially G-vector bundles on the G-set G under the conjugation action (Theorem 3.2.2).

3.1. Vector bundles and the algebra $D_X(G)$

Fix a finite right G-set X. A *G-equivariant k-vector bundle U* on X is a collection of finite dimensional vector spaces $\{U_x\}_{x\in X}$, together with a representation of G on their direct sum $\sum_{x\in X} U_x$ such that $U_x \cdot g = U_{x^g}$. We call U_x the *x-component* or *fiber* of U. If u is an element of the kG-module $\sum_{x\in X} U_x$, we write $u = \sum_{x\in X} u_x$, where $u_x \in U_x$ for each $x \in X$.

If U and V are G-vector bundles on X, a *morphism* $f : U \to V$ is a kG-module map $f : \sum_{x\in X} U_x \to \sum_{x\in X} V_x$ which preserves fibers, that is $f(U_x) \subseteq V_x$ for all $x \in X$. Note that a morphism is determined by its action on the x-components, where x ranges over a set of representatives of orbits of G on X. An *isomorphism* is an invertible morphism. The *direct sum* of two G-vector bundles on X is the G-vector bundle whose x-component is the direct sum of the x-components of the original modules for each $x \in X$. This is the product in the category of G-vector bundles on X.

The underlying kG-module $\sum_{x\in X} U_x$ of a G-vector bundle U on X may be decomposed into a direct sum of kG-modules

$$\sum_{\mathcal{O}} U_{\mathcal{O}},$$

taken over all G-orbits $\mathcal{O}$ in X, with $U_{\mathcal{O}} = \sum_{x\in\mathcal{O}} U_x$. For any $x \in \mathcal{O}$, $U_{\mathcal{O}}$ is determined by U_x, since $U_{x^g} = U_x \cdot g$ for all $g \in G$. By restriction of action, the subspace U_x may be considered to be a kG_x-module, where $G_x = \{g \in G \mid x^g = x\}$ is the isotropy subgroup of x. As a kG-module then, $U_{\mathcal{O}}$ is isomorphic to the induced module $i^G_{G_x}(U_x) = U_x \otimes_{kG_x} kG$, also written $U_x\uparrow^G$; this follows from standard results about induced modules [1]. Thus the indecomposable (respectively, irreducible) G-vector bundles on X are indexed by pairs (V, x) where x is a representative of an orbit of X and V is an indecomposable (respectively, irreducible) kG_x-module.

We shall consider another way to view G-vector bundles on X. Let $k[X]$ be the algebra of all functions on X taking values in k, with pointwise addition and multiplication. Note that $k[X]$ is isomorphic to a diagonal matrix algebra with standard basis all dual functions ϕ_x for $x \in X$. Define an action of G as automorphisms of $k[X]$ by

$$f^g(x) = f(x^{g^{-1}})$$

for all $g \in G$, $x \in X$, and $f \in k[X]$. The action of $g \in G$ on the basis of dual functions is given by $\phi_x^g = \phi_{x^g}$, for all $x \in X$.

We next consider the skew group algebra that is the smash product of $k[X]$ with kG, as in [12]. Explicitly, we build an algebra structure on $k[X] \otimes kG$, similar to that for the quantum double, as follows. For $x \in X$ and $h \in H$, we denote the basis element $\phi_x \otimes h$ of $k[X] \otimes kG$ by $\phi_x h$. We define a product by requiring that $\phi_x h \phi_y \ell = \phi_x \phi_{y^{h^{-1}}} h\ell$, which is nonzero if and only if $x = y^{h^{-1}}$. Denote the resulting algebra by $D_X(G)$. Its identity is $1_{D_X(G)} = \sum_{x\in X} \phi_x 1$. In case X is the G-set G under conjugation, then $D_X(G) = D(G)$, the quantum double of G, has a Hopf algebra structure as well.

3.2. A category equivalence

Let **mod**-$D_X(G)$ denote the category of (finite dimensional right) $D_X(G)$-modules. Let **vect**(X, G) denote the category of G-vector bundles on X. Note that both are abelian categories. Theorem 3.2.2 below states that these categories are equivalent. The lemma and theorem are analogous to results in [8].

Lemma 3.2.1 *(i) Let U be a $D_X(G)$-module. Then $U_{\text{vect}} = \{U \cdot \phi_x\}_{x \in X}$ is a G-vector bundle on X.*

(ii) Let $U = \{U_x\}_{x \in X}$ be a G-vector bundle on X. Then $U_{D_X} = \sum_{x \in X} U_x$ is a $D_X(G)$-module where $u \cdot \phi_x g = u_x \cdot g$ for all $u \in U_{D_X}$, $x \in X$, and $g \in G$.

Proof: (i) Clearly $U = \sum_{x \in X} U \cdot \phi_x$ as a direct sum of vector spaces, since the ϕ_x, for all $x \in X$, are orthogonal idempotents whose sum is 1. Further, $U\phi_x g = Ug\phi_{x^g} = U\phi_{x^g}$. Thus U_{vect} is a G-vector bundle on X.

(ii) For all $g, g' \in G$, $x, x' \in X$, and $u \in \sum_{x \in X} U_x$, we have

$$\begin{aligned} u \cdot (\phi_x g \phi_{x'} g') &= u(\phi_x \phi_{x'^{g^{-1}}} g g') \\ &= \delta_{x, x'^{g^{-1}}} u_x g g' \\ &= \delta_{x, x'^{g^{-1}}} (u_x g) g' \\ &= \delta_{x, x'^{g^{-1}}} (u \phi_x g) g' \\ &= (u \cdot \phi_x g) \cdot \phi_{x'} g'. \end{aligned}$$

And $u \cdot 1_{D(G)} = u \cdot \sum_{x \in X} \phi_x 1 = \sum_{x \in X} u_x = u$. Thus U_{D_X} is a $D_X(G)$-module. □

We now define the functors $(\cdot)_{\text{vect}}$ and $(\cdot)_{D_X}$ on morphisms of **mod**-$D_X(G)$ and of **vect**(X, G). For any morphism $f : U \to V$ in **mod**-$D_X(G)$, define the map $f_{\text{vect}} : U_{\text{vect}} \to V_{\text{vect}}$ by setting $f_{\text{vect}} = f$. We see that f_{vect} is a morphism of G-vector bundles as follows. Clearly f_{vect} is a kG-module map on the underlying kG-modules U and V of U_{vect} and V_{vect}. Further,

$$f_{\text{vect}}(U_x) = f(U \cdot \phi_x) = f(U) \cdot \phi_x \subseteq V \cdot \phi_x = V_x.$$

Similarly, for any morphism $f : U \to V$ in **vect**(X, G), define $f_{D_X} : U_{D_X} \to V_{D_X}$ by setting $f_{D_X} = f$. If $u \in U, x \in X$ and $g \in G$, then

$$f(u \cdot \phi_x g) = f(u_x g) = f(u_x) g = f(u)_x g = f(u) \cdot \phi_x g.$$

Thus f_{D_X} is a $D_X(G)$-morphism.

Theorem 3.2.2 *There is an equivalence between the category of $D_X(G)$-modules and the category of G-vector bundles on X given by the functors*

$$\begin{aligned}(\cdot)_{\mathsf{vect}} &: \textbf{mod-}D_X(G) \rightarrow \textbf{vect}(X,G),\\ (\cdot)_{D_X} &: \textbf{vect}(X,G) \rightarrow \textbf{mod-}D_X(G).\end{aligned}$$

Proof: We have seen that morphisms in $\textbf{mod-}D_X(G)$ correspond one-to-one to morphisms in $\textbf{vect}(X,G)$ under the functors $(\cdot)_{\mathsf{vect}}$ and $(\cdot)_{D_X}$. Thus it suffices to prove that $((\cdot)_{\mathsf{vect}})_{D_X}$ and $((\cdot)_{D_X})_{\mathsf{vect}}$ are isomorphisms on modules.

To see that $((\cdot)_{\mathsf{vect}})_{D_X}$ is an isomorphism, let U be a $D_X(G)$-module, $u \in (U_{\mathsf{vect}})_{D_X}$, $x \in X$, and $g \in G$. Then $u\phi_x g = u_x g$ in U_{vect}, and $u_x g = u\phi_x g$ in U. Thus mapping u to itself in U is a $D_X(G)$-module isomorphism.

To see that $((\cdot)_{D_X})_{\mathsf{vect}}$ is an isomorphism, let U be a G-vector bundle on X. Let u be an element of the underlying kG-module $\sum_{x\in X} U_x$, $g \in G$, and $x \in X$. Then $u_x g = u\phi_x g = u_x g$. Thus mapping u to itself in $\sum_{x\in X} U_x$ is an isomorphism of G-vector bundles. □

3.3. A characterization of $D(G)$-modules

We now turn to the situation relevant to the quantum double of G. Let H be a subgroup of G, and consider G to be an H-set under the conjugation action.

Corollary 3.3.1 *Up to isomorphism, the indecomposable (respectively, irreducible) $D_G(H)$-modules are indexed by pairs (V,g) where g is a representative of an H-orbit on G, and V is an indecomposable (respectively, irreducible) $kC_H(g)$-module.*

Proof: By Theorem 3.2.2, the category of $D_G(H)$-modules is equivalent to the category of H-vector bundles on G. By the discussion in Section 3.1, the indecomposable (respectively, irreducible) H-vector bundles on G are indexed by the pairs (V,g), and the H-vector bundle corresponding to (V,g) is given by

$$V\uparrow^H = \sum_{x\in C_H(g)\backslash H} V_{g^x},$$

with g^x-component $V_{g^x} = V^x$ the conjugate module for $kC_H(g^x) = kC_H(g)^x$. We consider this H-vector bundle to be a $D_G(H)$-module by defining

$$v \cdot \phi_{g'} h' = v_{g'} \cdot h'$$

for all $v \in V\uparrow^H$, $g' \in G$, and $h' \in H$, as in Lemma 3.2.1. $\square$

In particular, when $H = G$, we have the well-known characterization of modules for the quantum double $D(G)$ of G [7, 9, 12]: The indecomposable (respectively, irreducible) $D(G)$-modules are indexed by pairs (V, g) where g is a representative of a conjugacy class in G, and V is an indecomposable (respectively, irreducible) $kC(g)$-module.

We now describe an equivalent construction of indecomposable $D_G(H)$-modules, similar to that given in [12], which will be used in Chapter 5. Again, let g be a representative of an H-orbit on G, let $J = C_H(g)$, and V be an indecomposable kJ-module. Let $\mathrm{incl}_{J,g}(V)$ be the $D_G(J)$-module corresponding to the J-vector bundle on G that is V in the g-component and 0 in all other components. Then the corresponding indecomposable $D_G(H)$-module is the induced module

$$\mathrm{incl}_{J,g}(V) \otimes_{D_G(J)} D_G(H).$$

As $D_G(H)$ is a free left $D_G(J)$-module, we see that this $D_G(H)$-module is a sum of the subspaces $\mathrm{incl}_{J,g}(V) \otimes_{D_G(J)} D_G(J)h$, for $h \in J\backslash H$, which are the g^h-components of the corresponding H-vector bundle on G.

Next we obtain a ring of H-vector bundles on G that is isomorphic to the representation ring $R(D_G(H))$. This will give us an alternative description of these representation rings which will be used in Chapters 4 and 5. Let $r_{\mathrm{vect}}(G, H)$ be the additive group generated by isomorphism classes of H-vector bundles on G with direct sum for addition. We define a product on elements of $r_{\mathrm{vect}}(G, H)$, as in [11]. (In case $k = \mathbb{C}$, our $r_{\mathrm{vect}}(G, H)$ is Lusztig's $K_H(G)$.) Let U and V be H-vector bundles on

G. The product $U \otimes V$ of U and V is defined to be the H-vector bundle having components

$$(U \otimes V)_g = \sum_{x \in G} U_x \otimes V_{x^{-1}g}$$

for $g \in G$. The sum of these components has kH-module structure given by the action of H on the tensor product of the underlying kH-modules of U and V. We check the action of H on fibers. If $h \in H$, then

$$\begin{aligned} (U \otimes V)_g \cdot h &= \sum_{x \in G} U_x h \otimes V_{x^{-1}g} h \\ &= \sum_{x \in G} U_{x^h} \otimes V_{(x^{-1}g)^h} \\ &= (U \otimes V)_{g^h}. \end{aligned}$$

Thus $U \otimes V$ is an H-vector bundle on G.

With this product, $r_{\text{vect}}(G, H)$ becomes an associative ring. The identity is the H-vector bundle which is the trivial kH-module k in the 1-component, and 0 in all other components.

Theorem 3.3.2 *Let H be a subgroup of G, and consider G to be an H-set under conjugation by elements of H. Then there is a ring isomorphism*

$$r(D_G(H)) \simeq r_{\text{vect}}(G, H).$$

Proof: Theorem 3.2.2 provides a one-to-one correspondence between the set of isomorphism classes of indecomposable $D_G(H)$-modules and the set of isomorphism classes of indecomposable H-vector bundles on G. This correspondence may be extended to an isomorphism of the additive groups of $r(D_G(H))$ and of $r_{\text{vect}}(G, H)$. Thus it suffices to prove that the functor $(\cdot)_{\text{vect}} : \textbf{mod-}DH \to \textbf{vect}(G, H)$ preserves tensor products.

Let U and V be $D_G(H)$-modules. Then $(U \otimes V)_{\text{vect}}$ is an H-vector bundle on G with underlying kH-module $U \otimes V$ and components, for each $g \in G$,

$$((U \otimes V)_{\text{vect}})_g = (U \otimes V)\phi_g = \sum_{x \in G} U\phi_x \otimes V\phi_{x^{-1}g}.$$

On the other hand, $U_{\text{vect}} \otimes V_{\text{vect}}$ is an H-vector bundle on G with underlying kH-module $U \otimes V$ and components, for each $g \in G$,

$$\begin{aligned}(U_{\text{vect}} \otimes V_{\text{vect}})_g &= \sum_{x \in G} (U_{\text{vect}})_x \otimes (V_{\text{vect}})_{x^{-1}g} \\ &= \sum_{x \in G} U\phi_x \otimes V\phi_{x^{-1}g}. \ \square\end{aligned}$$

We shall use the language of $D_G(H)$-modules and of H-vector bundles on G interchangeably in the sequel. If U is a $D_G(H)$-module, we shall write $U_g = U \cdot \phi_g$, and consider this space to be a $kC_H(g)$-module, or a kL-module for any subgroup L of $C_H(g)$. For an arbitrary element of $R(D_G(H))$, we take its g-component to be the element of $R(C_H(g))$ which is the corresponding linear combination of g-components of the vector bundles involved.

CHAPTER 4

CHARACTERS OF THE REPRESENTATION RING $R(D(G))$

In this chapter we construct characters of the representation ring $R(D(G))$ of the quantum double from characters of the Green ring $R(G)$, that is of the representation ring of the group algebra kG. In Chapter 5 it will be shown that the characters constructed here are all of the characters of $R(D(G))$. When k has prime characteristic, we show that those characters of $R(D(G))$ arising from the Brauer characters of the group provide a quantum double analog of the Brauer characters, in that they correspond one-to-one to the characters of the semisimple Grothendieck ring $\mathcal{R}(D(G))$.

We shall use the notation $Z(A)$ for the center of an algebra A.

4.1. Characters in the complex case

In case $k = \mathbb{C}$, an observation of Lusztig [11, p. 242] and Theorem 3.3.2 imply that

$$R(D(G)) \simeq \prod_g Z(\mathbb{C}C(g))$$

as algebras, where the product is taken over a set of representatives g of conjugacy classes in G. This isomorphism is given by mapping a $D(G)$-module U to

$$\sum_{h\in C(g)} \mathrm{Tr}(\phi_h g, U)h = \sum_{h\in C(g)} \mathrm{Tr}(g, U_h)h$$

for each g, where Tr denotes taking the trace of the linear transformation indicated, and $U_h = U\phi_h$ is the h-component of U as discussed in Chapter 3. Lemma 4.2.3 below gives a generalization of these maps for fields of arbitrary characteristic, and Theorem 4.3.2 gives an analog of the above isomorphism for fields of prime characteristic.

We note that if H is any finite group, and ρ is an irreducible character of H, then the map

$$\frac{1}{\deg \rho}\rho : Z(\mathbb{C}H) \to \mathbb{C}$$

is an algebra homomorphism, that is a character of $Z(\mathbb{C}H)$. To see this, consider the decomposition of the semisimple algebra $\mathbb{C}H$ into a direct sum of matrix algebras. The elements of $Z(\mathbb{C}H)$ are sums of scalar multiples of the identities for these matrix algebras. The map $\frac{1}{\deg \rho}\rho$ singles out one of these matrix algebras, and maps elements of $Z(\mathbb{C}H)$ to their corresponding scalar coordinates. These maps, one for each irreducible character ρ of H, provide an algebra isomorphism of $Z(\mathbb{C}H)$ onto a direct sum of copies of $\mathbb{C}$.

In this way, for the case $k = \mathbb{C}$, we obtain characters of $R(D(G))$ via the decomposition $R(D(G)) \simeq \prod_g Z(\mathbb{C}C(g))$ given above. Such a character maps a $D(G)$-module U to

$$\chi_{g,\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in C(g)} \mathrm{Tr}(g, U_h)\rho(h),$$

in accordance with [11]. These characters completely separate elements of $R(D(G))$, so that $R(D(G))$ is semisimple in this case. For other fields, there is a similar decomposition, given in Theorem 4.3.2, of the Grothendieck ring $\mathcal{R}(D(G))$ of $D(G)$-modules, with the result that $D(G)$-modules may be separated at least up to composition factors by characters of $R(D(G))$. The proof of Theorem 4.3.2 involves certain homomorphisms mapping $R(D(G))$ to the algebras $Z(\mathbb{C}C(g))$ whose definition depends on Lemma 4.2.3 below.

4.2. Characters in the general case

A *species* of the Green ring $R(G)$ is an algebra homomorphism from $R(G)$ to $\mathbb{C}$ [5]. In other words, a species is a character of the Green ring. Distinct species are linearly independent [5].

Let L be a subgroup of G, and let $r^G_L : R(G) \to R(L)$ denote the linear map defined by restriction of modules. Note that r^G_L is an algebra homomorphism. If U

is a kG-module, we also denote the restriction $r_L^G(U)$ by $U\downarrow_L$ when the group G is understood. An *origin* of a species $s : R(G) \to \mathbb{C}$ is a minimal subgroup L of G such that $s = t \circ r_L^G$ for some species t of $R(L)$. In [5] it is shown that the origins of a species form a single conjugacy class of subgroups.

We shall need the following lemma, which is Proposition 6.9 of [5]. Let $i_H^G : R(H) \to R(G)$ denote the induction map, that is $i_H^G(U) = U \otimes_{kH} kG$ for a kH-module U.

Lemma 4.2.1 (Benson-Parker) *The following are equivalent for a species s of $R(G)$ and a subgroup H of G:*

(i) There is a species t of $R(H)$ such that $s = t \circ r_H^G$.

(ii) $\mathrm{Ker}(s) \supseteq \mathrm{Ker}(r_H^G)$,

(iii) $\mathrm{Ker}(s) \not\supseteq \mathrm{Im}(i_H^G)$. □

One consequence of the lemma is that if L is an origin of s and $s = t \circ r_L^G$, then the kernel of t contains any module induced to L from a proper subgroup of L. To see this, note that here t cannot factor through r_J^L for a proper subgroup J of L, as L is the origin of s, and restriction of modules is transitive. Now apply the lemma to the new situation with $G = L$ and H a proper subgroup of L.

The next lemma is a key part of the proof of Lemma 4.2.3, which results in the description of characters of $R(DG)$ in Theorem 4.2.4. It is also used in the next chapter.

Lemma 4.2.2 *Let U and V be $D(G)$-modules, L a subgroup of G, $H = C_G(L)$, and $h \in H$. Then the kL-submodule $\sum_{x\in G-H} U_x \otimes V_{x^{-1}h}$ of $\mathrm{res}_L^{D(G)}(U \otimes V)$ is isomorphic to a direct sum of kL-modules induced from proper subgroups of L.*

Proof: First note that $\sum_{x\in G-H} U_x \otimes V_{x^{-1}h}$ is indeed a kL-submodule of the restriction $\mathrm{res}_L^{D(G)}(U\otimes V)$ of the $D(G)$-module $U\otimes V$ to a kL-module: As $H = C_G(L)$, conjugates of elements of $G - H$ by an element $\ell \in L$ are still in $G - H$, and ℓ-conjugates of two elements whose product is h will still have product h.

Without loss of generality, we may assume that U and V are indecomposable $D(G)$-modules. Considering U and V as G-vector bundles on G, their underlying kG-modules must satisfy $\mathrm{res}_G^{D(G)}(U) \simeq U_x{\uparrow^G}$ and $\mathrm{res}_G^{D(G)}(V) \simeq V_g{\uparrow^G}$ for some elements $x, g \in G$, where U_x is considered here as a $kC(x)$-module, and V_g a $kC(g)$-module. Then $\mathrm{res}_L^{D(G)}(U \otimes V) = r_L^G \circ \mathrm{res}_G^{D(G)}(U \otimes V) \simeq (U_x{\uparrow^G} \otimes V_g{\uparrow^G}){\downarrow_L}$ as a kL-module, since $\mathrm{res}_G^{D(G)}$ is an algebra homomorphism (Lemma 2.2.1). By the Mackey Product Theorem [1], this is isomorphic to

$$\sum_{\sigma \in C(x)\backslash G/C(g)} \Big((U_{x^\sigma} \otimes V_g){\downarrow_{C(x)^\sigma \cap C(g)}}\Big){\uparrow^G}{\downarrow_L},$$

since $(U_x)^\sigma \simeq U_{x^\sigma}$ as modules for $kC(x)^\sigma = kC(x^\sigma)$. By the Mackey Subgroup Theorem [1], we see finally that $\mathrm{res}_L^{D(G)}(U \otimes V)$ is isomorphic to

$$\sum_{\sigma \in C(x)\backslash G/C(g)} \; \sum_{\tau \in C(x)^\sigma \cap C(g)\backslash G/L} (U_{x^{\sigma\tau}} \otimes V_{g^\tau}){\downarrow_{C(x)^{\sigma\tau} \cap C(g)^\tau \cap L}}{\uparrow^L}.$$

The subspaces $U_x \otimes V_{x^{-1}h}$ of $U \otimes V$ appearing in $\sum_{x \in G-H} U_x \otimes V_{x^{-1}h}$ correspond to some of the $U_{x^{\sigma\tau}} \otimes V_{g^\tau}$ (or their L-conjugates) in the above sum for which neither $x^{\sigma\tau}$ nor g^τ is an element of H. Further, if $U_{x^{\sigma\tau}} \otimes V_{g^\tau}$ appears in $\sum_{x \in G-H} U_x \otimes V_{x^{-1}h}$, then so does the entire kL-module $(U_{x^{\sigma\tau}} \otimes V_{g^\tau}){\downarrow_{C(x)^{\sigma\tau} \cap C(g)^\tau \cap L}}{\uparrow^L}$, as L-conjugates of elements of $G - H$ are also elements of $G - H$. If neither $x^{\sigma\tau}$ nor g^τ is an element of $H = C_G(L)$, then L is contained in neither $C(x^{\sigma\tau})$ nor $C(g^\tau)$. Thus $C(x)^{\sigma\tau} \cap C(g)^\tau \cap L = C(x^{\sigma\tau}) \cap C(g^\tau) \cap L$ is properly contained in L. Therefore $\sum_{x \in G-H} U_x \otimes V_{x^{-1}h}$ is equal to a sum of certain summands in the direct sum decomposition of $\mathrm{res}_L^{D(G)}(U \otimes V)$ above that are modules induced from proper subgroups of L. $\square$

Lemma 4.2.3 *Let s be a species of $R(G)$ with origin L, $s = t \circ r_L^G$, and $H = C_G(L)$. Define a linear function $f_t : R(D(G)) \to Z(\mathbb{C}H)$ by*

$$f_t(U) = \sum_{h \in H} t(U_h)h$$

for all $D(G)$-modules U. Then f_t is an algebra homomorphism.

Proof: First note that the U_h are indeed kL-modules, since elements of L commute with elements of H. Also note that the image of f_t is in the center of $\mathbb{C}H$ as claimed, again since L commutes elementwise with H, any two conjugate elements of H will correspond to components which are isomorphic as kL-modules. Thus t takes the same value on these components. Clearly f_t takes the trivial $D(G)$-module to the identity of $Z(\mathbb{C}H)$. It remains to prove that f_t is multiplicative.

Let U and V be $D(G)$-modules. Then

$$\begin{aligned} f_t(U)f_t(V) &= \sum_{x,h\in H} t(U_x)t(V_h)xh \\ &= \sum_{x,h\in H} t(U_x \otimes V_{x^{-1}h})h, \end{aligned}$$

where in the second sum, h has been replaced by $x^{-1}h$, and we have used the multiplicativity of the species t. On the other hand,

$$\begin{aligned} f_t(U\otimes V) &= \sum_{h\in H} t((U\otimes V)_h)h \\ &= \sum_{h\in H} t\left(\sum_{x\in G} U_x \otimes V_{x^{-1}h}\right) h \\ &= \sum_{x,h\in H} t(U_x \otimes V_{x^{-1}h})h + \sum_{h\in H} t\left(\sum_{x\in G-H} U_x \otimes V_{x^{-1}h}\right) h. \end{aligned}$$

Comparing the two calculations, we need only see that the second sum above is equal to zero. But this follows directly from Lemmas 4.2.1 and 4.2.2. Thus this sum is zero and f_t is multiplicative. □

We point out that f_t may not be surjective. In particular, if we let

$$\mathrm{Stab}_G(t) = \{g \in N_G(L) \mid t(a) = t(a^g) \text{ for all } a \in R(L)\}$$

be the *stabilizer* of t in G [5], then clearly the image of f_t is contained in the set $(\mathbb{C}H)^{\mathrm{Stab}_G(t)}$ of points of $\mathbb{C}H$ fixed under conjugation by $\mathrm{Stab}_G(t)$.

We are now ready to describe certain characters of the representation ring $R(D(G))$ of the quantum double.

Theorem 4.2.4 *Let s be a species of $R(G)$ with origin L, $s = t \circ r_L^G$, $H = C_G(L)$, and ρ an irreducible character of H. Define a linear function $\chi_{t,\rho} : R(D(G)) \to \mathbb{C}$ by*

$$\chi_{t,\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in H} t(U_h)\rho(h)$$

for all $D(G)$-modules U. Then $\chi_{t,\rho}$ is a character of $R(D(G))$.

Proof: The function $\chi_{t,\rho}$ is just the composition of the homomorphism $f_t : R(D(G)) \to Z(\mathbb{C}H)$ given in Lemma 4.2.3 with the homomorphism $\frac{1}{\deg \rho}\rho$: $Z(\mathbb{C}H) \to \mathbb{C}$ discussed in Section 4.1. □

In case $k = \mathbb{C}$, the characters described in the theorem are precisely those given in Section 4.1. Indeed, the Green ring $R(G)$ in this case coincides with the character ring, and species are essentially the columns of the character table for G. Each is given by the trace of an element $g \in G$ on a module. The fact that g is a group-like element in the Hopf algebra kG implies that

$$\mathrm{Tr}(g, U \otimes V) = \mathrm{Tr}(g, U)\mathrm{Tr}(g, V)$$

for any two kG-modules U and V, and so this trace function is an algebra homomorphism. An origin of the species $\mathrm{Tr}(g, \cdot)$ is the cyclic subgroup $L = \langle g \rangle$ generated by g, with centralizer $H = C(g)$. The corresponding character of $R(D(G))$ described in Theorem 4.2.4 is thus the map sending a $D(G)$-module U to

$$\chi_{\mathrm{Tr}(g,\cdot),\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in C(g)} \mathrm{Tr}(g, U_h)\rho(h),$$

for g a representative of a conjugacy class in G, and ρ an irreducible character of $C(g)$. These are the characters $\chi_{g,\rho}$, given at the beginning of this chapter, that appear in [11]. We also note that these characters of $R(D(G))$ may be expressed as trace functions of the elements

$$\frac{1}{\deg \rho} \sum_{h \in C(g)} \rho(h)\phi_h g$$

of $D(G)$. However these are not group-like elements in general.

We point out that any character of $R(D(G))$ of the form $\chi_{t,\rho}$ given in the theorem may be thought of as an extension to $R(D(G))$ of the species $s = t \circ r_L^G$ of $R(G)$: The map $\mathrm{incl}_{G,1} : R(G) \to R(D(G))$, sending a kG-module U to the $D(G)$-module which is U in the 1-component and 0 elsewhere, embeds $R(G)$ as a subalgebra of $R(D(G))$. If $a \in R(G)$, then

$$\begin{aligned}
\chi_{t,\rho}(\mathrm{incl}_{G,1}(a)) &= \frac{1}{\deg \rho} \sum_{h \in H} t\left((\mathrm{incl}_{G,1}(a))_h\right) \rho(h) \\
&= \frac{1}{\deg \rho} t\left((\mathrm{incl}_{G,1}(a))_1\right) \rho(1) \\
&= t \circ r_L^G(a) \\
&= s(a).
\end{aligned}$$

Thus distinct species of $R(G)$ yield distinct characters of $R(D(G))$.

4.3. A quantum analog of Brauer characters

In the rest of this chapter, we develop an analog of Brauer characters for the quantum double $D(G)$. We assume now that the characteristic p of k is prime.

We consider certain species, called *Brauer species*, that are essentially the columns of the table of Brauer characters for the group G. The Brauer species corresponding to a given p'-element g of G, that is an element whose order is not divisible by p, is given by lifting eigenvalues of the action of g on a module to $\mathbb{C}$ and taking the trace there. Let s_g denote this species. Just as in the case $k = \mathbb{C}$, an origin of s_g is $L = \langle g \rangle$ with centralizer $H = C(g)$. Write $s_g = t_g \circ r^G_{\langle g \rangle}$ where t_g is the corresponding Brauer species of $R(\langle g \rangle)$.

Recall that $R_0(D(G))$ denotes the ideal of $R(D(G))$ generated by elements of the form $U - U' - U''$ where

$$0 \to U' \to U \to U'' \to 0$$

is a short exact sequence of $D(G)$-modules. By Theorem 2.3.3, $R_0(D(G)) = 0$ if and only if the characteristic p of k does not divide the order of G.

Lemma 4.3.1 *For each p'-element g of G, let $s_g = t_g \circ r^G_{<g>}$ denote the corresponding Brauer species of $R(G)$. The ideal $R_0(D(G))$ is the kernel of the homomorphism*

$$\pi : R(D(G)) \rightarrow \prod_g Z(\mathbb{C}C(g))$$

given by the product of the maps f_{t_g}, defined in Lemma 4.2.3, taken over a set of representatives g of p'-conjugacy classes of G.

Proof: First let $0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0$ be a short exact sequence of $D(G)$-modules. Considering $D(G)$-modules as G-vector bundles on G, we have short exact sequences

$$0 \rightarrow U'_x \rightarrow U_x \rightarrow U''_x \rightarrow 0$$

of $kC(x)$-modules for all $x \in G$. Let g be a p'-element of G. If $x \in C(g)$, then $t_g(U_x) = t_g(U'_x) + t_g(U''_x)$, as t_g is a Brauer species of the subgroup $<g>$ of $C(g)$. Thus $f_{t_g}(U) = f_{t_g}(U') + f_{t_g}(U'')$ for all p'-elements g of G, and $R_0(D(G)) \subseteq \mathrm{Ker}(\pi)$.

Let $a \in R(D(G))$ with $\pi(a) = 0$. Fix $x \in G$ and let g be a p'-element of $C(x)$. Then $f_{t_g}(a) = 0$ implies that $t_g(a_h) = 0$ for all $h \in C(g)$; in particular, $t_g(a_x) = 0$ where a_x here is considered as an element of $R(<g>)$. Let $b_g = t_g \circ r^{C(x)}_{<g>}$ be the Brauer species of $C(x)$ corresponding to the p'-element g of $C(x)$. Then $b_g(a_x) = 0$ for all such g, where a_x is now considered as an element of $R(C(x))$. But this implies that $a_x \in R_0(C(x))$, the ideal of short exact sequences of $kC(x)$-modules [4].

Now, by Theorem 3.2.2, $D(G)$-modules and $D(G)$-module maps are determined by their x-components, where x ranges over a set of representatives of conjugacy classes of G. Further, a short exact sequence of $kC(x)$-modules induces to a short exact sequence of kG-modules [1] which may also be considered to be a short exact sequence of G-vector bundles on G. The latter corresponds to a short exact sequence of $D(G)$-modules under the equivalence of categories of Theorem 3.2.2. Thus $a \in R_0(D(G))$ and $\mathrm{Ker}(\pi) \subseteq R_0(D(G))$. $\square$

The lemma implies that there is an induced map from the Grothendieck ring

$$\mathcal{R}(D(G)) = R(D(G))/R_0(D(G))$$

to $\prod_g Z(\mathbb{C}C(g))$ provided by the product of the maps f_{t_g} for Brauer species $s_g = t_g \circ r^G_{<g>}$. Further, this map is injective. It turns out to be surjective as well, which we demonstrate next. First we recall that the number of irreducible kH-modules, for any finite group H, is equal to the number of conjugacy classes of p'-elements in H [1].

Theorem 4.3.2 *For each representative g of a p'-conjugacy class in G, let $s_g = t_g \circ r^G_{<g>}$ denote the corresponding Brauer species. The product π of the maps f_{t_g}, defined in Lemma 4.2.3, induces an algebra isomorphism*

$$\mathcal{R}(D(G)) \simeq \prod_g Z(\mathbb{C}C(g)).$$

In particular, the Grothendieck ring $\mathcal{R}(D(G))$ is semisimple.

Proof: By Lemma 4.2.3, π is an algebra homomorphism. By Lemma 4.3.1, π induces an injection from $\mathcal{R}(D(G)) = R(D(G))/R_0(D(G))$ to $\prod_g Z(\mathbb{C}C(g))$. It remains to prove that π is a surjection, which will follow once we see that the dimensions of these two finite dimensional algebras are the same.

The dimension of the Grothendieck ring $\mathcal{R}(D(G))$ is equal to the number of irreducible $D(G)$-modules. By Corollary 3.3.1 this is

$$\sum_x (\text{number of } p'\text{-conjugacy classes in } C(x)),$$

where the sum is taken over a set of representatives x of conjugacy classes in G. The dimension of $\prod_g Z(\mathbb{C}C(g))$ is

$$\sum_g (\text{number of conjugacy classes in } C(g)),$$

where the sum is taken over a set of representatives g of p'-conjugacy classes in G. The two sums are equal, as they represent two ways of counting the number of orbits in the G-set, under G-conjugation, consisting of all pairs (x, g) where x is an element of G, g is a p'-element of G, and x and g commute.

That $\mathcal{R}(D(G))$ is semisimple now follows from the fact that the algebras $Z(\mathbb{C}C(g))$ are semisimple, as discussed in Section 4.1. □

Theorem 4.2.4 now allows us to write down a table consisting of the complete set of characters of the Grothendieck ring $\mathcal{R}(D(G))$. Such a character maps the image of a $D(G)$-module U in $\mathcal{R}(D(G))$ to

$$\chi_{t_g,\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in H} t_g(U_h)\rho(h),$$

where $s_g = t_g \circ r^G_{<g>}$ is a Brauer species of $R(G)$, $H = C(g)$, and ρ is an irreducible character of H. By Theorem 4.3.2 the corresponding characters $\chi_{t_g,\rho}$ of the representation ring $R(D(G))$ separate $D(G)$-modules up to composition factors.

In case the characteristic p of k does not divide the order of G, we have by Theorems 2.3.3 and 4.3.2 that $R(D(G)) \simeq \mathcal{R}(D(G))$ is semisimple, and the above discussion is the full story. Otherwise, there are in general more species than the Brauer species [5], so that there will in general be characters of $R(D(G))$, defined by Theorem 4.2.4, which separate some modules having the same composition factors. When this happens, the ideal $R_0(D(G))$ of short exact sequences properly contains the radical of $R(D(G))$.

CHAPTER 5

THE GREEN FUNCTOR $R(D_G(\cdot))$

In this chapter we prove that the collection of all the representation rings $R(D_G(H))$, for subgroups H of G, constitutes a Green functor (Theorem 5.1.5). We then obtain a decomposition of $R(D(G))$ into a direct sum of ideals (Theorem 5.2.3) based on a result of Thévenaz about Green functors. This decomposition leads to an induction theorem (Corollary 5.2.4), a result regarding questions of semisimplicity of $R(D(G))$ (Corollary 5.2.5), and the result that the characters of $R(D(G))$ described in the previous chapter are all of its characters (Theorem 5.3.2).

5.1. $R(D_G(\cdot))$ is a Green functor

We first define a Green functor for G over an arbitrary commutative ring Λ, although here we shall always take Λ to be the complex numbers. See [16] for standard definitions and results regarding Green functors.

Suppose that for each subgroup H of G there is a Λ-algebra $A(H)$, and for every pair of subgroups $L \leq H$ and every element $g \in G$, there are linear maps:

- The *restriction* maps $r_L^H : A(H) \to A(L)$;
- The *induction* (or *transfer*) maps $i_L^H : A(L) \to A(H)$;
- The *conjugation* maps $c_{H,g} : A(H) \to A(H^g)$.

We say that $A(\cdot)$ is a *Green functor* (or algebra G-functor) if these maps satisfy the following properties:

(i) The maps r_L^H are algebra homomorphisms and the maps $c_{H,g}$ algebra isomorphisms for all subgroups $L \leq H$ and all $g \in G$.

(ii) $r_H^H = \mathrm{id}_{A(H)}$, the identity map on $A(H)$, and $r_L^H \circ r_H^J = r_L^J$ for all subgroups $L \leq H \leq J$.

(iii) $i_H^H = \mathrm{id}_{A(H)}$ and $i_H^J \circ i_L^H = i_L^J$ for all subgroups $L \leq H \leq J$.

(iv) $c_{H,h} = \mathrm{id}_{A(H)}$ and $c_{H^g,g'} \circ c_{H,g} = c_{H,gg'}$ for all subgroups H, all elements $h \in H$, and $g, g' \in G$.

(v) $c_{L,g} \circ r_L^H = r_{L^g}^{H^g} \circ c_{H,g}$ and $c_{H,g} \circ i_L^H = i_{L^g}^{H^g} \circ c_{L,g}$ for all subgroups $L \leq H$ and $g \in G$.

(vi) (Frobenius axiom) For all subgroups $L \leq H$ and all elements $a \in A(L)$ and $b \in A(H)$, we have

$$b \cdot i_L^H(a) = i_L^H(r_L^H(b) \cdot a) \qquad \text{and} \qquad i_L^H(a) \cdot b = i_L^H(a \cdot r_L^H(b)).$$

(vii) (Mackey axiom) For all subgroups $H, L \leq J$, we have

$$r_H^J \circ i_L^J = \sum_{g \in L \backslash J / H} i_{L^g \cap H}^H \circ r_{L^g \cap H}^{L^g} \circ c_{L,g}.$$

We point out that in our definition, if instead the $A(H)$ are merely Λ-modules, and properties (i) and (vi) are deleted, then $A(\cdot)$ is a *Mackey functor* (or module G-functor) [16]. There are important examples of such functors. Group cohomology becomes a Green functor given appropriate maps. We shall make use of the Green functor assigning the Green ring $R(H)$ to each subgroup H with the usual restriction, induction, and conjugation maps [16].

Now we define such linear maps for the representation ring of the quantum double. These correspond to the standard definitions of maps on the modules of fully group-graded algebras [6], and we choose notation consistent with this situation. However, in general a group-graded algebra may not be a Hopf algebra, and so it may not have a representation ring.

Let $L \leq H$ be subgroups of G, and $g \in G$.

- The restriction map $r_{D_G(L)}^{D_G(H)} : R(D_G(H)) \to R(D_G(L))$ sends a $D_G(H)$-module U to a $D_G(L)$-module by restriction of the action from $D_G(H)$ to $D_G(L)$. The resulting module will be denoted $r_{D_G(L)}^{D_G(H)}(U)$ or $U\!\downarrow_{D_G(L)}$.

- The induction map $i_{D_G(L)}^{D_G(H)} : R(D_G(L)) \to R(D_G(H))$ sends a $D_G(L)$-module V to the $D_G(H)$-module $V \otimes_{D_G(L)} D_G(H)$. This module will be denoted $i_{D_G(L)}^{D_G(H)}(V)$ or $V{\uparrow}^{D_G(H)}$.

- The conjugation map $c_{H,g} : R(D_G(H)) \to R(D_G(H^g))$ sends a $D_G(H)$-module U to the $D_G(H^g)$-module $U^g = U \otimes_{D_G(H)} D_G(H)g = U \otimes_{D_G(H)} g$.

We first show that the Frobenius axiom holds in this situation. By linearity of the maps involved, it suffices to prove this for modules.

Lemma 5.1.1 (Frobenius axiom) *Let $L \leq H$ be subgroups of G. Let U be a $D_G(L)$-module and V a $D_G(H)$-module. Then*

(i) $V \otimes U{\uparrow}^{D_G(H)} \simeq (V{\downarrow}_{D_G(L)} \otimes U){\uparrow}^{D_G(H)}$, *and*

(ii) $U{\uparrow}^{D_G(H)} \otimes V \simeq (U \otimes V{\downarrow}_{D_G(L)}){\uparrow}^{D_G(H)}$.

Proof: We prove (i); the proof of (ii) is similar. Define a linear map $f : (V{\downarrow}_{D_G(L)} \otimes U) \otimes_{D_G(L)} D_G(H) \to V \otimes (U \otimes_{D_G(L)} D_G(H))$ by

$$f((v \otimes u) \otimes h) = vh \otimes (u \otimes h)$$

for all $u \in U$, $v \in V$, and $h \in H$. Then f is well-defined. It is bijective with inverse $f^{-1} : V \otimes (U \otimes_{D_G(L)} D_G(H)) \to (V{\downarrow}_{D_G(L)} \otimes U) \otimes_{D_G(L)} D_G(H)$ defined by $f^{-1}(v \otimes (u \otimes h)) = (vh^{-1} \otimes u) \otimes h$. It remains to check that f is a $D_G(H)$-module map. Let $g \in G$ and $h' \in H$. Then

$$\begin{aligned} f((v \otimes u) \otimes h \cdot \phi_g h') &= f((v \otimes u)\phi_{hgh^{-1}} \otimes hh') \\ &= f\left(\sum_{x \in G} (v\phi_x \otimes u\phi_{x^{-1}hgh^{-1}}) \otimes hh' \right) \\ &= \sum_{x \in G} v\phi_x hh' \otimes (u\phi_{x^{-1}hgh^{-1}} \otimes hh'). \end{aligned}$$

On the other hand,

$$f((v \otimes u) \otimes h) \cdot \phi_g h' \;=\; (vh \otimes (u \otimes h))\phi_g h'$$

$$= \sum_{x\in G} vh\phi_x h' \otimes (u\otimes h)\phi_{x^{-1}g}h'$$
$$= \sum_{x\in G} v\phi_{hxh^{-1}}hh' \otimes (u\phi_{hx^{-1}gh^{-1}} \otimes hh')$$
$$= \sum_{x\in G} v\phi_x hh' \otimes (u\phi_{x^{-1}hgh^{-1}} \otimes hh'). \ \square$$

We next show that the conjugation maps are algebra homomorphisms.

Lemma 5.1.2 *Let H be a subgroup of G, $g \in G$, and U and V be $D_G(H)$-modules. Then there is an isomorphism of $D_G(H^g)$-modules*

$$(U\otimes V)^g \simeq U^g \otimes V^g.$$

Proof: The map $f : (U\otimes V)^g \to U^g \otimes V^g$ defined by $f((u\otimes v)\otimes g) = (u\otimes g)\otimes(v\otimes g)$ is a $D_G(H^g)$-module isomorphism. This follows from calculations similar to those in the proof of the previous lemma. $\square$

We next state two results which are proved in [6] for any fully group-graded algebra, and finally prove that $R(D_G(\cdot))$ is a Green functor.

Lemma 5.1.3 *Let $L \leq H$ be subgroups of G, $g, g' \in G$, and U a $D_G(L)$-module. Then*

(i) $(U \otimes_{D_G(L)} D_G(L)g) \otimes_{D_G(L^g)} D_G(L^g)g' \simeq U \otimes_{D_G(L)} D_G(L)gg'$,

(ii) $(U \otimes_{D_G(L)} D_G(L)g) \otimes_{D_G(L^g)} D_G(H^g) \simeq (U \otimes_{D_G(L)} D_G(H)) \otimes_{D_G(H)} D_G(H)g$.

$\square$

Lemma 5.1.4 (Mackey Subgroup Theorem) *Let $L, H \leq J \leq G$, and U be a $D_G(L)$-module. Then*

$$U{\uparrow}^{D_G(J)}{\downarrow}_{D_G(H)} \simeq \sum_{g\in L\backslash J/H} U^g{\downarrow}_{D_G(L^g)\cap H}{\uparrow}^{D_G(H)}. \ \square$$

Theorem 5.1.5 *$R(D_G(\cdot))$ is a Green functor.*

Proof: Let $L \leq H \leq J$ be subgroups of G, and $g, g' \in G$. Clearly the maps $r^{D_G(H)}_{D_G(L)}$ are algebra homomorphisms. The maps $c_{H,g}$ are algebra isomorphisms by Lemma 5.1.2 and the fact that action by g is invertible. Note that $r^{D_G(H)}_{D_G(H)} = \mathrm{id}_{R(D_G(H))}$ trivially, and $r^{D_G(H)}_{D_G(L)} \circ r^{D_G(J)}_{D_G(H)} = r^{D_G(J)}_{D_G(L)}$ by the definition of the restriction maps. We have $i^{D_G(H)}_{D_G(H)} = \mathrm{id}_{R(D_G(H))}$ by definition, and $i^{D_G(J)}_{D_G(H)} \circ i^{D_G(H)}_{D_G(L)} = i^{D_G(J)}_{D_G(L)}$ follows from the definitions and freeness of these Hopf algebras over their Hopf subalgebras. Clearly $c_{H,h}$ is the identity map on $R(D_G(H))$ whenever $h \in H$. Lemma 5.1.3 yields $c_{H^g,g'} \circ c_{H,g} = c_{H,gg'}$ and $c_{H,g} \circ i^{D_G(H)}_{D_G(L)} = i^{D_G(H^g)}_{D_G(L^g)} \circ c_{L,g}$. By their definitions, restriction commutes with conjugation in the sense that $c_{L,g} \circ r^{D_G(H)}_{D_G(L)} = r^{D_G(H^g)}_{D_G(L^g)} \circ c_{H,g}$. Lemma 5.1.1 is the Frobenius axiom. The Mackey axiom,

$$r^{D_G(J)}_{D_G(H)} \circ i^{D_G(J)}_{D_G(L)} = \sum_{g \in L\backslash J/H} i^{D_G(H)}_{D_G(L^g)\cap H} \circ r^{D_G(L^g)}_{D_G(L^g)\cap H} \circ c_{L,g}$$

for $L, H \leq J \leq G$, is Lemma 5.1.4. □

Next we state the Mackey Product Theorem for $R(D(G))$, which holds more generally for all Green functors [16]. It follows directly from the axioms, particularly the Mackey and Frobenius axioms.

Proposition 5.1.6 (Mackey Product Theorem) *Let $L, H \leq J \leq G$. Let U be a $D_G(L)$-module and V a $D_G(H)$-module. Then*

$$U\!\uparrow^{D_G(J)} \otimes V\!\uparrow^{D_G(J)} \simeq \sum_{g \in L\backslash J/H} \left((U^g \otimes V)\downarrow_{D_G(L^g)\cap H} \right) \uparrow^{D_G(J)} . \; \square$$

5.2. A direct sum decomposition of $R(D(G))$

We introduce the Brauer morphisms for a Green functor $A(\cdot)$ [16]. If $J \leq L$ are subgroups of G, the Frobenius axiom implies that the image of $i^L_J : A(J) \to A(L)$ is an ideal of $A(L)$. The *residue algebra* of $A(L)$ is

$$\overline{A}(L) = A(L) / \sum_{J<L} \mathrm{Im}(i^L_J),$$

the sum being taken over all proper subgroups J of L. Let $\mathrm{br}_L^L : A(L) \to \overline{A}(L)$ be the canonical surjection, and let $\mathrm{br}_L^G = \mathrm{br}_L^L \circ r_L^G : A(G) \to \overline{A}(L)$, called the *Brauer morphism.* We next state a result of Thévenaz about Green functors over rings with the order of G invertible [16, Corollary 3.6]. In particular, it will apply to our situation, in which $R(D_G(\cdot))$ is a Green functor for G over $\mathbb{C}$.

Proposition 5.2.1 (Thévenaz) *Let $A(\cdot)$ be a Green functor for G over a ring Λ in which the order of G is invertible. Then the product of the Brauer morphisms br_L^G yields an algebra isomorphism*

$$A(G) \simeq \prod_L \overline{A}(L)^{N_G(L)},$$

where the product is taken over a set of representatives L of conjugacy classes of subgroups of G. $\square$

Applied to the situation of the quantum double, the proposition yields an algebra isomorphism

$$R(D(G)) \simeq \prod_L \overline{R}(D_G(L))^{N_G(L)},$$

given by the product of Brauer morphisms $\mathrm{br}_{D_G(L)}^{D(G)}$. We next modify this ideal direct sum decomposition by examining the quotients $\overline{R}(D_G(L))$ more closely.

We shall need the following lemma. The algebra homomorphism $\mathrm{res}_L^{D_G(L)} : R(D_G(L)) \to R(L)$ is given by the restriction of a $D_G(L)$-module to a kL-module. We shall use the maps $\mathrm{incl}_{J,h} : R(J) \to R(D_G(J))$, where J is a subgroup of G and $h \in C_G(J)$, introduced in Section 3.3: Given a kJ-module U, $\mathrm{incl}_{J,h}(U)$ is the $D_G(J)$-module which is U in the h-component and 0 elsewhere. Note that $\mathrm{res}_J^{D_G(J)} \circ \mathrm{incl}_{J,h}$ is the identity map on $R(J)$.

Lemma 5.2.2 *Let $J \leq L$ be subgroups of G. If V is a $D_G(J)$-module then*

$$\mathrm{res}_L^{D_G(L)}(V \otimes_{D_G(J)} D_G(L)) \simeq \mathrm{res}_J^{D_G(J)}(V) \otimes_{kJ} kL$$

as kL-modules. If U is a kJ-module and $h \in C_G(L)$ then

$$\mathrm{incl}_{L,h}(U \otimes_{kJ} kL) \simeq \mathrm{incl}_{J,h}(U) \otimes_{D_G(J)} D_G(L)$$

as $D_G(L)$-modules. That is, $\mathrm{res}_L^{D_G(L)} \circ i_{D_G(J)}^{D_G(L)} = i_J^L \circ \mathrm{res}_J^{D_G(J)}$, *and* $\mathrm{incl}_{L,h} \circ i_J^L = i_{D_G(J)}^{D_G(L)} \circ \mathrm{incl}_{J,h}$.

Proof: As $D_G(L)$ is a free left $D_G(J)$-module, the elements $v \otimes \ell$, where v runs over a basis of V and ℓ runs over a set of representatives of the right cosets $J\backslash L$, are a basis of $V \otimes_{D_G(J)} D_G(L)$ and of $\mathrm{res}_J^{D_G(J)}(V) \otimes_{kJ} kL$. Clearly the actions of L on these basis elements correspond. Similarly, the elements $u \otimes \ell$, where u runs over a basis of U and ℓ runs over a set of representatives of the right cosets $J\backslash L$, are a basis of $U \otimes_{kJ} kL$ and of $\mathrm{incl}_{J,h}(U) \otimes_{D_G(J)} D_G(L)$. The actions of $D_G(L)$ on these basis elements correspond. □

A *p-hypoelementary* subgroup of G is a subgroup L for which $L/O_p(L)$ is cyclic [4, 5], where $O_p(L)$ denotes the unique maximal normal p-subgroup of L. If L is any subgroup of G, then the residue algebra of the Green ring, $\overline{R}(L) = R(L)/\sum_{J<L} \mathrm{Im}(i_J^L)$, is nonzero if and only if L is p-hypoelementary [16].

Theorem 5.2.3 *For each subgroup L of G, $\overline{R}(D_G(L)) \simeq \overline{R}(L) \otimes \mathbb{C}C_G(L)$. In particular, the product of the Brauer morphisms $\mathrm{br}_{D_G(L)}^{D(G)}$ induces an isomorphism of algebras*

$$R(D(G)) \simeq \prod_L \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)},$$

where the product is taken over a set of representatives L of conjugacy classes of p-hypoelementary subgroups of G.

Proof: The second statement follows from the first and Proposition 5.2.1, since $\overline{R}(L) = 0$ for all subgroups L of G which are not p-hypoelementary.

We consider $D_G(L)$-modules to be L-vector bundles on the L-set G as in Section 3.3. If U is a $D_G(L)$-module and $h \in C_G(L)$, then the h-component U_h is a $D_G(L)$-submodule of U. Define a linear map $\psi_L : R(D_G(L)) \to \overline{R}(L) \otimes \mathbb{C}C_G(L)$ by

$$\psi_L(U) = \sum_{h \in C_G(L)} \mathrm{br}_L^L \circ \ \mathrm{res}_L^{D_G(L)}(U_h) \otimes h$$

for all $D_G(L)$-modules U, where $\mathrm{br}_L^L : R(L) \to \overline{R}(L)$ is the canonical map. We claim that ψ_L is an algebra homomorphism, and that it is surjective with kernel equal to $\mathrm{Ker}\left(\mathrm{br}_{D_G(L)}^{D_G(L)}\right) = \sum_{J<L} \mathrm{Im}\left(i_{D_G(J)}^{D_G(L)}\right)$. This will imply that $\overline{R}(D_G(L))$ and $\overline{R}(L) \otimes \mathbb{C}C_G(L)$ are isomorphic.

That ψ_L is an algebra homomorphism results from a calculation very similar to that in Lemma 4.2.3 where it is shown that a certain function $f_t : R(D(G)) \to Z(\mathbb{C}C_G(L))$ is an algebra homomorphism. (In fact, such a function f_t factors through $\psi_L \circ \mathrm{res}_{D_G(L)}^{D(G)}$, as will be shown in Lemma 5.3.1). The key step in showing that ψ_L is an algebra homomorphism is applying Lemma 4.2.2, which implies that the kL-submodule $\sum_{x \in G - C_G(L)} U_x \otimes V_{x^{-1}h}$ of $\mathrm{res}_L^{D(G)}(U \otimes V)$, for $D(G)$-modules U and V, is in the kernel of br_L^L.

To see that ψ_L is surjective, choose $a \in R(L)$ and $h \in C_G(L)$. Then the element $\mathrm{incl}_{L,h}(a)$ of $R(D_G(L))$ which is a in the h-component, and 0 in all other components, will map to $\mathrm{br}_L^L(a) \otimes h$ under ψ_L.

Finally we show that $\mathrm{Ker}(\psi_L) = \mathrm{Ker}\left(\mathrm{br}_{D_G(L)}^{D_G(L)}\right)$. First let $a \in \mathrm{Im}\left(i_{D_G(J)}^{D_G(L)}\right)$ for a proper subgroup J of L, say $a = \sum_i \alpha_i V_i \otimes_{D_G(J)} D_G(L)$ where the V_i are $D_G(J)$-modules, $\alpha_i \in \mathbb{C}$. If $h \in C_G(L)$, then $a_h = \sum_i \alpha_i \left(V_i \otimes_{D_G(J)} D_G(L)\right)_h$ by definition, and

$$\begin{aligned}\left(V_i \otimes_{D_G(J)} D_G(L)\right)_h &= V_i \otimes_{D_G(J)} D_G(L)\phi_h \\ &= V_i\phi_h \otimes_{D_G(J)} D_G(L) \\ &= (V_i)_h \otimes_{D_G(J)} D_G(L).\end{aligned}$$

Thus $\mathrm{br}_L^L \circ \mathrm{res}_L^{D_G(L)}(a_h) = 0$ by Lemma 5.2.2 and the definition of the Brauer map br_L^L. Therefore $\psi_L(a) = 0$, and $\mathrm{Ker}\left(\mathrm{br}_{D_G(L)}^{D_G(L)}\right) \subseteq \mathrm{Ker}(\psi_L)$.

Now let $y \in \mathrm{Ker}(\psi_L)$, so that

$$\mathrm{br}_L^L \circ \mathrm{res}_L^{D_G(L)}(y_h) = 0$$

for each $h \in C_G(L)$. In other words, $\mathrm{res}_L^{D_G(L)}(y_h) \in \mathrm{Ker}(\mathrm{br}_L^L) = \sum_{J<L} \mathrm{Im}(i_J^L)$ for each $h \in C_G(L)$. Note that y_h is in the image of the map $\mathrm{incl}_{L,h} : R(L) \to R(D_G(L))$.

Applying $\mathrm{incl}_{L,h}$ to $\mathrm{res}_L^{D_G(L)}(y_h)$, we see that y_h is in $\sum_{J<L} \mathrm{Im}\left(i_{D_G(J)}^{D_G(L)}\right)$ by Lemma 5.2.2, and so $\mathrm{br}_{D_G(L)}^{D_G(L)}(y_h) = 0$. To see that the sum of the remaining components of y is also in $\mathrm{Ker}\left(\mathrm{br}_{D_G(L)}^{D_G(L)}\right)$, note that they correspond to L-orbits on G containing more than one element. By the construction of $D_G(L)$-modules following Corollary 3.3.1, this sum is a sum of $D_G(L)$-modules induced from subalgebras $D_G(J)$ for proper subgroups J of L, and so will be in $\mathrm{Ker}\left(\mathrm{br}_{D_G(L)}^{D_G(L)}\right)$. □

In fact, it is clear now that the isomorphism

$$R(D(G)) \xrightarrow{\sim} \prod_L \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$$

of the theorem is given by the product of the maps $\psi_L \circ \mathrm{res}_{D_G(L)}^{D(G)}$. We shall use this fact in the proof of Lemma 5.3.1.

We have the following induction theorem as an immediate corollary.

Corollary 5.2.4 *The representation ring $R(D(G))$ is generated by the images of the maps $i_{D_G(L)}^{D(G)}$ for all p-hypoelementary subgroups L of G.*

Proof: By Theorem 5.2.3 and the preceding discussion, $\overline{R}(D(G)) \neq 0$ if and only if G is p-hypoelementary. If G is p-hypoelementary, we are done. If not, then $\overline{R}(D(G)) = 0$, and thus $R(D(G)) = \sum_{L<G} \mathrm{Im}\left(i_{D_G(L)}^{D(G)}\right)$. By induction on the partially ordered set of subgroups of G, $R(D_G(L))$ is generated by the images $\mathrm{Im}\left(i_{D_G(J)}^{D_G(L)}\right)$ for all p-hypoelementary subgroups J of L. By transitivity of the induction maps, we now have the same result for $R(D(G))$. □

The direct sum decomposition given in Theorem 5.2.3 provides a connection between questions of semisimplicity of $R(D(G))$ and questions of semisimplicity of the Green ring $R(G)$. This connection is stated in the next corollary. An arbitrary commutative algebra over $\mathbb{C}$ is *semisimple* if the intersection of the kernels of its characters is zero, or equivalently, if characters separate elements of the algebra.

We summarize what is known about questions of semisimplicity of the Green ring $R(G)$ [5]. If the characteristic p of k is odd, then $R(G)$ is semisimple if and only if G has cyclic Sylow p-subgroups. If the characteristic of k is 2, then $R(G)$ is semisimple if a Sylow 2-subgroup of G is either a cyclic group or the Klein four-group. For some other cases in characteristic 2, $R(G)$ is not semisimple, and there are also some unresolved cases.

We note that Proposition 5.2.1 applied to the Green ring functor yields

$$R(G) \simeq \prod_L \overline{R}(L)^{N_G(L)},$$

where the product is taken over a set of representatives L of conjugacy classes of subgroups of G (or of p-hypoelementary subgroups). In case the quotient $\overline{R}(G)$ is nonzero (that is, in case G is p-hypoelementary), it is realized as an ideal direct summand of $R(G)$ in this way.

Corollary 5.2.5 *If $R(D(G))$ is semisimple, then $R(G)$ is semisimple. If the characteristic p of the field k is odd and $R(G)$ is semisimple, then $R(D(G))$ is semisimple.*

Proof: We may embed $R(G)$ as a subalgebra of $R(D(G))$ via the map $\mathrm{incl}_{G,1} : R(G) \to R(D(G))$. As both $R(G)$ and $R(D(G))$ are commutative algebras, the first statement is clear.

Suppose the characteristic p of k is odd and assume that $R(G)$ is semisimple. By the above discussion, G then has cyclic Sylow p-subgroups. Thus for any subgroup L of G, L also has cyclic Sylow p-subgroups, and so $R(L)$ is semisimple. As discussed above, $\overline{R}(L)$ may be identified with an ideal direct summand of $R(L)$ for each subgroup L of G. Thus $\overline{R}(L)$ is semisimple. Consider the decomposition of Theorem 5.2.3,

$$R(D(G)) \simeq \prod_L \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}.$$

Each summand satisfies

$$\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)} \subseteq \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{C_G(L)} = \overline{R}(L) \otimes Z(\mathbb{C}C_G(L)),$$

as $C_G(L)$ acts trivially on $R(L)$, and therefore on $\overline{R}(L)$. But $\overline{R}(L) \otimes Z(\mathbb{C}C_G(L))$ is the tensor product of two semisimple commutative algebras, and so is semisimple itself. Since $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$ is a subalgebra of the semisimple algebra $\overline{R}(L) \otimes Z(\mathbb{C}C_G(L))$, it is semisimple as well. By the direct sum decomposition of $R(D(G))$ above, we now have that $R(D(G))$ is semisimple. □

5.3. Characters revisited

Next we show how the characters of $R(D(G))$ defined in Section 4.2 behave with respect to the ideal direct sum decomposition

$$R(D(G)) \simeq \prod_L \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$$

of Theorem 5.2.3. We point out that any character must factor through exactly one of the summands.

Lemma 5.3.1 *Let s be a species of $R(G)$ with origin L, $s = t \circ r_L^G$, $H = C_G(L)$, and ρ an irreducible character of H. The character $\chi_{t,\rho}$ of $R(D(G))$, defined in Theorem 4.2.4, factors through the summand $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$ of the decomposition of Theorem 5.2.3.*

Proof: We recall that $\chi_{t,\rho} = \dfrac{1}{\deg \rho}\rho \circ f_t$, where $f_t : R(D(G)) \to Z(\mathbb{C}H)$ is defined in Lemma 4.2.3. First note that $t : R(L) \to \mathbb{C}$, having origin L, factors through the quotient $\overline{R}(L)$ of $R(L)$ by Lemma 4.2.1. We write $\overline{t}$ for the induced map from $\overline{R}(L)$ to $\mathbb{C}$. We claim that f_t is merely the composition of the maps

$$R(D(G)) \xrightarrow{\psi_L \circ r_{D_G(L)}^{D(G)}} \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)} \xrightarrow{\overline{t}\otimes \mathrm{id}} \mathbb{C} \otimes_{\mathbb{C}} Z(\mathbb{C}C_G(L)) \xrightarrow{\sim} Z(\mathbb{C}C_G(L)),$$

where ψ_L is defined in the proof of Theorem 5.2.3. Recall that the product of the maps $\psi_L \circ r_{D_G(L)}^{D(G)}$ yields the isomorphism $R(D(G)) \xrightarrow{\sim} \prod_L \left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$ given in Theorem 5.2.3. Let U be a $D(G)$-module. Then $\psi_L \circ r_{D_G(L)}^{D(G)}(U) = \sum_{h \in C_G(L)} \mathrm{br}_L^L \circ \mathrm{res}_L^{D_G(L)}(U_h) \otimes h$. Under $\overline{t} \otimes \mathrm{id}$ and the indicated identification, this is mapped to $\sum_{h\in C_G(L)} t(U_h)h = f_t(U)$, as t applied to a kL-module is the same as $\overline{t}$ applied to its equivalence class in $\overline{R}(L)$. Following this map by $\frac{1}{\deg \rho}\rho$, we see that $\chi_{t,\rho}$ factors through $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$. $\square$

The lemma yields a proof that the characters $\chi_{t,\rho}$ described in Theorem 4.2.4 are all of the characters of $R(D(G))$.

Theorem 5.3.2 *Any character of $R(D(G))$ is of the form $\chi_{t,\rho}$, defined in Theorem 4.2.4, for some species $s = t \circ r_L^G$ of $R(G)$ with origin L and some irreducible character ρ of $C_G(L)$.*

Proof: We need only determine the structure of characters of the individual ideal direct summands $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$ of $R(D(G))$. Note that $\overline{R}(L) \otimes Z(\mathbb{C}C_G(L))$ is integral over $(\overline{R}(L) \otimes \mathbb{C}C_G(L))^{N_G(L)}$: Any $\alpha \in \overline{R}(L) \otimes Z(\mathbb{C}C_G(L)) = (\overline{R}(L) \otimes \mathbb{C}C_G(L))^{C_G(L)}$ satisfies the monic polynomial

$$\prod_{n \in C_G(L) \backslash N_G(L)} (x - \alpha^n),$$

which has coefficients in $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$. Thus by the going-up theorem [2], all characters of $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$ may be extended to characters of $\overline{R}(L) \otimes Z(\mathbb{C}C_G(L))$. The latter are merely products of characters of $\overline{R}(L)$ and of $Z(\mathbb{C}C_G(L))$. By Lemma 4.2.1 and the discussion in Section 4.1, these correspond to products of species t of $R(L)$ having origin L with the functions $\frac{1}{\deg \rho}\rho$, where the ρ are irreducible characters of $C_G(L)$. We restrict such characters to $\left(\overline{R}(L) \otimes \mathbb{C}C_G(L)\right)^{N_G(L)}$. By the proof of Lemma 5.3.1, we obtain in this way the characters $\chi_{t,\rho} = \frac{1}{\deg \rho}\rho \circ f_t$ of $R(D(G))$ described in Theorem 4.2.4. $\square$

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