TWISTED GRADED HECKE ALGEBRAS

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ABSTRACT. We generalize graded Hecke algebras to include a twisting two-cocycle for the associated finite group. We give examples where the parameter spaces of the resulting *twisted* graded Hecke algebras are larger than that of the graded Hecke algebras. We prove that twisted graded Hecke algebras are particular types of deformations of a crossed product.

1. INTRODUCTION

Drinfel'd defined graded Hecke algebras for any finite subgroup G of $GL(V)$ [8]. This definition was shown by Ram and Shepler [20] to generalize the definition of graded versions of affine Hecke algebras for real reflection groups given by Lusztig, who was motivated by questions in representation theory [16, 17]. In the same paper Ram and Shepler classified all graded Hecke algebras for complex reflection groups. They found many nontrivial graded Hecke algebras, but also showed that some groups (such as $G(r, 1, n) = \mathbb{Z}/r\mathbb{Z} \wr S_n$ when $r > 2$, $n > 3$) have no nontrivial graded Hecke algebras. Etingof and Ginzburg showed that in the case of a finite symplectic group, the symplectic form itself arises naturally in the structure of any associated graded Hecke algebra, always yielding nontrivial examples, which they called symplectic reflection algebras [9]. Many authors have studied representations of graded Hecke algebras, their subalgebras generated by certain idempotents, and connections to the geometry of the corresponding orbifolds V/G. (See for example [13, 15, 18, 19].)

In this note we generalize these constructions by incorporating a two-cocycle α that represents an element of the cohomology group $\mathrm{H}^2(G, \mathbb{C}^\times)$. Chmutova introduced such a two-cocycle for symplectic groups, showing that it appears naturally in symplectic reflection algebras arising from nonfaithful group representations to $Sp(V)$ [7]. Here our motivation comes from numerous papers on orbifolds in which such a cocycle, called *discrete torsion*, appears ([1, 6, 22] are just a few), as well as our finding that for some groups there is a nontrivial *twisted* graded Hecke algebra even if there is no nontrivial graded Hecke algebra (such as Example 2.16 below). We adapt the direct linear-algebraic approach of Ram and

Shepler to derive criteria for existence of such nontrivial twisted graded Hecke algebras (Theorem 2.10, Corollary 2.13). We give examples for which the parameter spaces for twisted graded Hecke algebras are larger than that for graded Hecke algebras: $S_n \subset GL(\mathbb{C}^n)$ via the permutation representation (Example 2.17), as well as $(\mathbb{Z}/\ell Z)^{n-1} \subset SL(\mathbb{C}^n)$ (Example 2.16). On the other hand, if G is symplectic, this is not the case (Example 2.15). It would be interesting to do a more thorough analysis of the parameter spaces of twisted graded Hecke algebras for various types of groups.

In Section 3 we show that twisted graded Hecke algebras are precisely particular types of deformations of crossed product algebras $S(V)\#_\alpha G$ (Theorem 3.2). We use this idea to show that Ram and Shepler's isomorphism between the different definitions of (untwisted) graded Hecke algebra given by Drinfel'd and Lusztig arises as an equivalence of deformations whose infinitesimals are thus cohomologous. This also puts previous results into a larger context: There are in general many more infinitesimal deformations (that is, Hochschild two-cocycles) of $S(V)\#_\alpha G$ than those that lift to deformations that are (twisted) graded Hecke algebras. In some cases these other infinitesimals also lift to deformations of $S(V)\#_\alpha G$: See Remark 2.14 herein and [6, 24]. We are not aware of a general result regarding such deformations.

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Throughout, we will work over the complex numbers $\mathbb{C}$, so that $\otimes = \otimes_{\mathbb{C}}$ unless otherwise indicated.

2. Twisted graded Hecke algebras

Our approach to graded Hecke algebras will be through crossed products (generalizations of skew group algebras), so we begin by summarizing this construction. Our definition below of (twisted) graded Hecke algebras involves a parameter t, which may be taken to be an indeterminate or any complex number. In Section 3 we will assume t is an indeterminate, as we will discuss deformations over the polynomial ring $\mathbb{C}[t]$. Specializing t to any nonzero complex number results in a definition of graded Hecke algebra equivalent to those in the literature.

Let $V = \mathbb{C}^n$, G a finite subgroup of $GL(V)$, and $\alpha : G \times G \to \mathbb{C}^\times$ a two-cocycle, that is

$$(2.1) \qquad\qquad \alpha(g,h)\alpha(gh,k) = \alpha(h,k)\alpha(g,hk)$$

for all $g, h, k \in G$. The action of G on V induces actions by algebra automorphisms on the tensor algebra $T(V)$ and the symmetric algebra $S(V)$.

Suppose S is any associative $\mathbb{C}$-algebra with an action of G by automorphisms, for example $S = T(V)$ or $S = S(V)$. Then we may form the *crossed product*

algebra $S\#_\alpha G$, which is $S \otimes \mathbb{C}G$ as a vector space, and has multiplication

$$(r \otimes g)(s \otimes h) = \alpha(g,h)r(g \cdot s) \otimes gh$$

for all $r, s \in S$ and $g, h \in G$. This makes $S\#_\alpha G$ an associative algebra as α is a two-cocycle. Note that S is a subalgebra of $S\#_\alpha G$, via $S \xrightarrow{\sim} S \otimes 1$, and that the subalgebra $1 \otimes \mathbb{C}G$ of $S\#_\alpha G$ is also known as the *twisted group algebra* $\mathbb{C}^\alpha G$. We accordingly abbreviate $s \otimes g$ by $s\overline{g}$ ($s \in S$, $g \in G$). The action of G on S becomes an inner action on $S\#_\alpha G$, namely $s \mapsto \overline{g}s(\overline{g})^{-1}$ for all $s \in S$, $g \in G$.

The two-cocycle α is a *coboundary* if there is some function $\beta : G \to \mathbb{C}^\times$ such that

$$(2.2) \qquad \alpha(g,h) = \beta(g)\beta(h)\beta(gh)^{-1}$$

for all $g, h \in G$. The set of two-cocycles modulo coboundaries forms an abelian group under pointwise multiplication, called the *Schur multiplier* of G, and denoted $\mathrm{H}^2(G, \mathbb{C}^\times)$. If two cocycles differ by a coboundary, that is if they are *cohomologous*, the corresponding crossed products are isomorphic.

Replacing α by a cohomologous cocycle if necessary, we may assume that α is normalized so that $\alpha(1,g) = \alpha(g,1) = 1$ for all $g \in G$. It follows that $\overline{1}$ is the multiplicative identity for $\mathbb{C}^\alpha G$. Thus $(\overline{g})^{-1} = \alpha^{-1}(g,g^{-1})\overline{g^{-1}} = \alpha^{-1}(g^{-1},g)\overline{g^{-1}}$ for all $g \in G$.

For each $g \in G$, choose a skew-symmetric bilinear form $a_g : V \times V \to \mathbb{C}$ (arbitrary for now, and possibly 0). Let t be an indeterminate and extend scalars, $S\#_\alpha G[t] := \mathbb{C}[t] \otimes (S\#_\alpha G)$. Let A be the quotient of $T(V)\#_\alpha G[t]$ by the (two-sided) ideal $I[t]$ generated by all

$$(2.3) \qquad [v,w] - \sum_{g \in G} a_g(v,w)t\overline{g}$$

where $v, w \in V$ and $[v,w] = vw - wv$. (We have omitted tensor symbols in elements of $T(V)$ for brevity.) Note that A is *additively* isomorphic to a quotient of $S(V)\#_\alpha G[t]$ via any choice of linear section $S(V) \hookrightarrow T(V)$ of the canonical projection of $T(V)$ onto $S(V)$. This quotient is proper if and only if there is more than one way to rearrange factors in a word by applying the relations (2.3). We say that A is a *twisted graded Hecke algebra* if the above map yields $A \cong S(V) \otimes \mathbb{C}G[t]$ as vector spaces over $\mathbb{C}[t]$. Equivalently, if we assign degree 1 to elements of V and degree 0 to elements of G, then A is a filtered algebra over $\mathbb{C}[t]$ and is a twisted graded Hecke algebra in case the associated graded algebra $\mathrm{gr}A$ is isomorphic to $S(V)\#_\alpha \mathbb{C}G[t]$.

Remark 2.4. Let α be the trivial two-cocycle $\alpha(g,h) = 1$ for all $g, h \in G$, and let $t = 1$. Then A becomes the *graded Hecke algebra* of [8, 20].

We will use the techniques of Ram and Shepler [20] to determine the conditions on the bilinear forms a_g under which A is a twisted graded Hecke algebra. Let

$h \in G$, and use (2.3) to obtain two expressions involving $[h^{-1} \cdot v, h^{-1} \cdot w]$. By direct substitution: $[h^{-1} \cdot v, h^{-1} \cdot w] = \sum_{g \in G} a_g(h^{-1} \cdot v, h^{-1} \cdot w) t\overline{g}$. By conjugating (2.3) by $(\overline{h})^{-1}$:

$$\begin{aligned} [h^{-1} \cdot v, h^{-1} \cdot w] &= \sum_{g \in G} a_g(v, w) t(\overline{h})^{-1} \overline{g} \overline{h} \\ &= \sum_{g \in G} a_g(v, w) \alpha^{-1}(h, h^{-1}) \alpha(g, h) \alpha(h^{-1}, gh) t\overline{h^{-1}gh}. \end{aligned}$$

Both must be in the ideal $I[t]$, and the assumed additive isomorphism $A \cong S(V) \otimes \mathbb{C}G[t]$ allows us to equate coefficients of each $\overline{g}$ ($g \in G$) in these two expressions. Thus we have $a_{h^{-1}gh}(h^{-1} \cdot v, h^{-1} \cdot w) = \alpha^{-1}(h, h^{-1}) \alpha(g, h) \alpha(h^{-1}, gh) a_g(v, w)$, and replacing v, w by $h \cdot v, h \cdot w$, we obtain the first equation below. Again the isomorphism $A \cong S(V) \otimes \mathbb{C}G[t]$ implies that in any expression $wvu \in V^{\otimes 3}$, three applications of (2.3) in any order to obtain uvw plus an element in tA must yield the same element of A. Comparison results in the second equation below. Thus necessary conditions for A to be a twisted graded Hecke algebra are:

$$(2.5) \qquad a_{h^{-1}gh}(v, w) = \alpha^{-1}(h, h^{-1}) \alpha(g, h) \alpha(h^{-1}, gh) a_g(h \cdot v, h \cdot w)$$

$$(2.6) \qquad a_g(u, v)(g \cdot w - w) + a_g(v, w)(g \cdot u - u) + a_g(w, u)(g \cdot v - v) = 0$$

for all $g, h \in G$ and $u, v, w \in V$. These equations will be used in the proofs of the following lemma and theorem.

Not only are (2.5) and (2.6) necessary for A to be a twisted graded Hecke algebra, but they are also sufficient: The relations (2.3) allow for rearrangement of any expression in A to a particular form identified with an element of $S(V) \otimes \mathbb{C}G[t]$, and the relations (2.5) and (2.6) imply that such a form is unique. That is, (2.5) is equivalent to uniqueness of the canonical form of $\overline{h}vu$, and (2.6) is equivalent to uniqueness of the canonical form of wvu ($u, v, w \in V$, $h \in G$), as demonstrated in [20] for the case $\alpha = 1$ (in the text above Lemma 1.5). The uniqueness of the form of wvu is equivalent to the Jacobi identity in A,

$$(2.7) \qquad\qquad [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0,$$

which is another way to express the condition (2.6). For the uniqueness of the canonical form of a monomial having more than three factors, it is helpful to note that replacing any of u, v, w in the left side of (2.7) by an element of degree larger than 1 results in an element in the ideal generated by the left side of (2.7). (For more details, see [20].)

Let $(,)$ be any G-invariant nondegenerate Hermitian form on V. Define orthogonal complements of subspaces of V via this form. Let $g \in G$ and $V^g = \{v \in V \mid g \cdot v = v\}$. We will need the observation that $(V^g)^{\perp} = \operatorname{im}(g - 1)$, which follows from the standard facts $V^g = \ker(g - 1)$, $\operatorname{im}(g - 1) \subset (V^g)^{\perp}$, and $\dim(\operatorname{im}(g - 1)) = \dim(\ker(g - 1)^{\perp})$. For any $h \in G$, denote by $h^{\perp} : (V^g)^{\perp} \to (V^g)^{\perp}$

the composition of the linear maps $h : (V^g)^\perp \hookrightarrow V$, $V \to V/V^g$, and a choice of isomorphism $V/V^g \xrightarrow{\sim} (V^g)^\perp$.

Lemma 2.8. *Suppose that A is a twisted graded Hecke algebra for G defined by skew-symmetric bilinear forms a_g ($g \in G$) on V. For each $g \neq 1$, if $a_g \neq 0$ then $\ker a_g = V^g$, $\mathrm{codim}(V^g) = 2$, and*

$$a_g(h \cdot v, h \cdot w) = \det(h^\perp) a_g(v, w)$$

for all $h \in G$ and $v, w \in (V^g)^\perp$.

This proof is essentially the same as in Ram and Shepler [20, Lemma 1.8], with appropriate adjustments made for the possibly nontrivial two-cocycle α.

Proof. Let $g \neq 1$ with $a_g \neq 0$. Let $v \in V$ and $w \in V^g$. If v is also in V^g, then $a_g(v, w)(g \cdot u - u) = 0$ for all $u \in V$ by (2.6). As $g \neq 1$, there is some $u \in V$ with $g \cdot u \neq u$, so $a_g(v, w) = 0$. If $v \notin V^g$, let $u = \sum_{k=1}^r g^k \cdot v$, where r is the order of g, so that $u \in V^g$ (possibly $u = 0$). As before, this implies $a_g(u, w) = 0$, and we may rewrite this equation using the definition of u and (2.5) as

$$
\begin{aligned}
0 = a_g(u, w) &= \sum_{k=1}^r a_g(g^k \cdot v, w) \\
&= \sum_{k=1}^r \alpha^{-1}(g^{-k}, g^k) \alpha(g, g^{-k}) \alpha(g^k, g^{1-k}) a_g(v, w).
\end{aligned}
$$

The restriction of α to the cyclic subgroup $\langle g \rangle$ generated by g is a coboundary since the Schur multiplier of a cyclic group is trivial [14, Prop. 1.1]. Therefore there is a function $\beta : \langle g \rangle \times \langle g \rangle \to \mathbb{C}^\times$ such that $\alpha(h, l) = \beta(h)\beta(l)\beta^{-1}(hl)$ for all $h, l \in \langle g \rangle$. It follows that the above coefficients of $a_g(v, w)$ are

$$
\begin{aligned}
&\alpha^{-1}(g^{-k}, g^k) \alpha(g, g^{-k}) \alpha(g^k, g^{1-k}) \\
&= \beta^{-1}(g^{-k})\beta^{-1}(g^k)\beta(1)\beta(g)\beta(g^{-k})\beta^{-1}(g^{1-k})\beta(g^k)\beta(g^{1-k})\beta^{-1}(g) \\
&= \beta(1),
\end{aligned}
$$

that is they are independent of k. The above calculation reduces to $0 = a_g(u, w) = r\beta(1)a_g(v, w)$. As $r\beta(1) \neq 0$, this forces $a_g(v, w) = 0$. Consequently $V^g \subset \ker a_g$, and so $(\ker a_g)^\perp \subset (V^g)^\perp$.

As a_g is nonzero and skew-symmetric, we have $\dim((\ker a_g)^\perp) = \mathrm{codim}(\ker a_g) \geq 2$. Let u, v be two linearly independent elements of $(\ker a_g)^\perp \subset (V^g)^\perp$ with $a_g(u, v) \neq 0$, so that in particular $g \cdot u - u \neq 0$ and $g \cdot v - v \neq 0$. Let w be any element of $(V^g)^\perp = \mathrm{im}(g - 1)$, and write $w = g \cdot w' - w'$ for some $w' \in V$. By (2.6), w is a linear combination of $g \cdot u - u$ and $g \cdot v - v$, two elements of $(V^g)^\perp$. This implies $\dim(\ker a_g)^\perp = \dim(V^g)^\perp = 2$, so that $V^g = \ker a_g$ and $\mathrm{codim}(V^g) = 2$.

Finally, let $v, w \in (V^g)^\perp$, $h \in G$, and

$$
\begin{aligned}
h \cdot v &= a_{11}v + a_{21}w + x \\
h \cdot w &= a_{12}v + a_{22}w + x'
\end{aligned}
$$

where $x, x' \in V^g$ and a_{ij} are scalars. Applying (2.5), we have $a_{h^{-1}gh}(v, w) = \alpha^{-1}(h, h^{-1})\alpha(g, h)\alpha(h^{-1}, gh)a_g(h \cdot v, h \cdot w)$ on the one hand, while evaluating via the above equations yields

(2.9)
$$
\begin{aligned}
a_{h^{-1}gh}(v, w) &= \alpha^{-1}(h, h^{-1})\alpha(g, h)\alpha(h^{-1}, gh)a_g(a_{11}v + a_{21}w + x, a_{12}v + a_{22}w + x') \\
&= \alpha^{-1}(h, h^{-1})\alpha(g, h)\alpha(h^{-1}, gh)(a_{11}a_{22} - a_{12}a_{21})a_g(v, w) \\
&= \alpha^{-1}(h, h^{-1})\alpha(g, h)\alpha(h^{-1}, gh)\det(h^\perp)a_g(v, w).
\end{aligned}
$$

Equating the two expressions for $a_{h^{-1}gh}(v, w)$ gives $a_g(h \cdot v, h \cdot w) = \det(h^\perp)a_g(v, w)$, as desired. $\qquad\square$

Now suppose $g \neq 1$, $a_g \neq 0$, and $h \in C(g) = \{h \in G \mid hg = gh\}$. Then (2.9) is equivalent to $\det(h^\perp) = \alpha(h, h^{-1})\alpha^{-1}(g, h)\alpha^{-1}(h^{-1}, gh)$. Applying (2.1) to the triple h, h^{-1}, gh, we find that $\alpha(h, h^{-1})\alpha^{-1}(h^{-1}, gh) = \alpha(h, h^{-1}gh) = \alpha(h, g)$ when $h \in C(g)$, so that $\det(h^\perp) = \alpha(h, g)\alpha^{-1}(g, h)$ for all $h \in C(g)$. This condition is independent of the choice of α in a given coset modulo coboundaries since coboundaries are symmetric on commuting pairs as is evident from (2.2). This is as expected since the determinant function is independent of such choices. Note that as $g \in C(g)$, (2.9) implies $\det(g^\perp) = 1$, that is $g \in SL(V)$. Further, in case g is α-regular, that is $\alpha(g, h) = \alpha(h, g)$ for all $h \in C(g)$, this determinant condition is simply $\det(h^\perp) = 1$ for all $h \in C(g)$. For nonregular elements g, this condition is different from that in the case of the trivial cocycle, leading to new examples such as Examples 2.16 and 2.17 below.

Theorem 2.10. *Let G be a finite subgroup of $GL(V)$, $\alpha : G \times G \to \mathbb{C}^\times$ a normalized two-cocycle, and $g \in G - \{1\}$. There is a twisted graded Hecke algebra A with $a_g \neq 0$ if, and only if, $\ker a_g = V^g$, $\mathrm{codim}(V^g) = 2$, and*

(2.11)
$$
\det(h^\perp) = \alpha(h, g)\alpha^{-1}(g, h)
$$

for all $h \in C(g)$.

Again the proof is similar to that of Ram and Shepler [20, Thm. 1.9] in the untwisted case. See also the paper of Etingof and Ginzburg [9], who used a criterion of Braverman and Gaitsgory adapted to Koszul algebras over $\mathbb{C}G$ [3, 5].

Proof. If A is a twisted graded Hecke algebra, Lemma 2.8 and subsequent comments show the given conditions hold.

Conversely, suppose the stated conditions hold for g. Up to a scalar multiple, there is a unique skew-symmetric form on V that is nondegenerate on $(V^g)^\perp$ and

has kernel V^g. Fix such a form a_g, and let

$$(2.12) \quad a_k(v, w) = \begin{cases} \alpha^{-1}(h, h^{-1})\alpha(g, h)\alpha(h^{-1}, gh)a_g(h \cdot v, h \cdot w), & \text{if } k = h^{-1}gh \\ 0, & \text{otherwise} \end{cases}$$

for all $v, w \in V$. In order for (2.5) to hold for all pairs of group elements, we must check that the definition of a_k is independent of the choice of representative from the conjugacy class of g. Suppose that $k = l^{-1}h^{-1}ghl$, so that there are two ways to define a_k. One way yields

$$a_k(v, w) = \alpha^{-1}(hl, l^{-1}h^{-1})\alpha(g, hl)\alpha(l^{-1}h^{-1}, ghl)a_g(hl \cdot v, hl \cdot w),$$

and the other yields

$$a_k(v, w)$$

$$= \alpha^{-1}(l, l^{-1})\alpha(h^{-1}gh, l)\alpha(l^{-1}, h^{-1}ghl)\alpha^{-1}(h, h^{-1})\alpha(g, h)\alpha(h^{-1}, gh)a_g(hl \cdot v, hl \cdot w).$$

These two expressions for $a_k(v, w)$ are indeed equal, as follows from five applications of (2.1), to the triple h^{-1}, gh, l, to l^{-1}, h^{-1}, ghl, to g, h, l, to hl, l^{-1}, h^{-1}, and to h, l, l^{-1}. By the discussion before Theorem 2.10, if $h^{-1}gh = g$, then (2.12) coincides with $a_g(v, w)$. Therefore the definition of a_k is independent of the choice of representative from the conjugacy class of g, and (2.5) holds. The argument of [20, Lemma 1.8(b)] applies without change to show that (2.6) holds as well: Their key observation is that (2.6) holds trivially if any one of u, v, w is in V^g, while $\dim(V^g)^\perp = 2$ implies that any three elements u, v, w of $(V^g)^\perp$ must be linearly dependent. Substituting a linear dependence relation into the left side of (2.6) yields 0 after some manipulation. Therefore A, defined by the $\{a_k\}$ in (2.12), is a twisted graded Hecke algebra. □

In summary, under the conditions in the theorem, a_g is determined by its value $a_g(v, w)$ on a basis v, w of $(V^g)^\perp$, and for each $h \in G$, $a_{h^{-1}gh}(v, w)$ is given by (2.12).

Corollary 2.13. *Let d be the number of conjugacy classes of $g \in G$ such that* $\operatorname{codim} V^g = 2$ *and* $\det(h^\perp) = \alpha(h, g)\alpha^{-1}(g, h)$ *for all $h \in C(g)$. The sets $\{a_g\}_{g \in G}$ corresponding to twisted graded Hecke algebras A form a vector space of dimension* $d + \dim(\bigwedge^2 V)^G$.

Proof. By (2.5) and (2.6), the only condition on a_1 is $a_1(v, w) = a_1(h \cdot v, h \cdot w)$ for all $h \in G$ since $\alpha(h, h^{-1}) = \alpha(h^{-1}, h)$, that is a_1 is a G-invariant element of $(\bigwedge^2 V)^*$. For each conjugacy class of elements $g \in G$ with $\operatorname{codim}(V^g) = 2$ and $\det(h^\perp) = \alpha(h, g)\alpha^{-1}(g, h)$ for all $h \in C(g)$, there are skew-symmetric forms a_g with kernel V^g, determined by a single form a_g (unique up to scalar multiple) for a representative g, and given by (2.12). □

Remark 2.14. The results of this section apply if we relax one of the conditions on the forms a_g a little: Allow a_g to take values in $\mathbb{C} \oplus V^g$, that is g-invariant polynomials of degree at most 1. Again let A be the quotient of $T(V)\#_\alpha G[t]$ by the ideal $I[t]$ generated by all expressions of the form (2.3). As the image of each form a_g has degree at most 1, the techniques of Ram and Shepler [20] still apply to yield

$$a_{h^{-1}gh}(v,w) = \alpha^{-1}(h,h^{-1})\alpha(g,h)\alpha(h^{-1},gh)\left(h^{-1} \cdot a_g(h \cdot v, h \cdot w)\right)$$

in place of (2.5), and (2.6) is unchanged (but now considered as a relation in $S(V)$). That is, these are necessary and sufficient conditions for A to be additively isomorphic to $S(V) \otimes \mathbb{C}G[t]$, leading to a generalization of Theorem 2.10.

In the remainder of this section, we give several examples.

Example 2.15. (Symplectic groups.) Let V be a finite dimensional symplectic vector space over $\mathbb{C}$, that is $\dim V$ is even and there is a nondegenerate skew-symmetric form $\omega : V \times V \to \mathbb{C}$. Let G be a finite subgroup of the symplectic group $Sp(V)$ of all invertible linear transformations preserving ω. Let $\alpha : G \times G \to \mathbb{C}^\times$ be the trivial two-cocycle $\alpha(g,h) = 1$ for all $g, h \in G$. For each element $g \in G$ such that $\operatorname{codim} V^g = 2$ (the *symplectic reflections*), let $a_g = c_g\omega_g$, where ω_g is defined to be $\omega|_{(V^g)^\perp}$ on $(V^g)^\perp = \operatorname{im}(1-g)$ and 0 on $V^g = \ker(1-g)$, and where $c_g \in \mathbb{C}$ are scalars such that $c_{hgh^{-1}} = c_g$ for all $h \in G$. Let $a_1 = c\omega$ for a scalar c and $a_g = 0$ for all other $g \in G$. By [9, Thm. 1.3], the collection $\{a_g \mid g \in G\}$ determines a graded Hecke algebra, called a *symplectic reflection algebra*. By Corollary 2.13, the parameter space for the possible *twisted* graded Hecke algebras cannot be larger than that for graded Hecke algebras in the symplectic case (cf. [7]).

The next two examples, by contrast, involve groups for which the parameter space for twisted graded Hecke algebras is larger than that for graded Hecke algebras.

Example 2.16. (Elementary abelian groups.) Let $n \geq 3$, $\ell \geq 2$, $V = \mathbb{C}^n$ and $G \cong (\mathbb{Z}/\ell\mathbb{Z})^{n-1}$ the multiplicative subgroup of $M_{n \times n}(\mathbb{C})$ generated by the diagonal matrices (where q is a primitive ℓth root of 1):

$$\begin{aligned}
g_1 &= \operatorname{diag}(q, q^{-1}, 1, \ldots, 1) \\
g_2 &= \operatorname{diag}(1, q, q^{-1}, 1, \ldots, 1) \\
&\vdots \\
g_{n-1} &= \operatorname{diag}(1, \ldots, 1, q, q^{-1})
\end{aligned}$$

Define a function $\alpha : G \times G \to \mathbb{C}^\times$ by

$$\alpha(g_1^{i_1} \cdots g_{n-1}^{i_{n-1}}, \ g_1^{j_1} \cdots g_{n-1}^{j_{n-1}}) = q^{-\sum_{1 \leq k \leq n-2} i_k j_{k+1}}.$$

The relation (2.1) may be checked directly. For each generator g_i $(i = 1, \ldots, n-1)$, as well as for $g_n = g_1^{-1} \cdots g_{n-1}^{-1}$, we have $\det(h^\perp) = \alpha(h, g_i)\alpha^{-1}(g_i, h)$ for all $h \in G$,

as follows from a straightforward computation. Thus we have a twisted graded Hecke algebra in which the forms $a_{g_1}, \ldots, a_{g_n}$ are all nonzero (and other $a_g = 0$). (Note that some of the relevant determinants are not 1. There is no nontrivial (untwisted) graded Hecke algebra in this case.) If $\ell \neq 2$, it may be checked that this gives the full parameter space of twisted graded Hecke algebras corresponding to this cocycle α. If $\ell = 2$, this determinant condition is in fact satisfied for all g with $\mathrm{codim}(V^g) = 2$, and so there is a larger parameter space of twisted graded Hecke algebras in this case. This example is discussed in a different context in [24, Example 4.1], which generalizes [6, Example 4.7].

Example 2.17. (Symmetric groups.) Let $n \geq 4$ and $G = S_n$, acting on $V = \mathbb{C}^n$ by the permutation representation. Let α be the unique nontrivial two-cocycle (up to coboundary). The Schur representation group for S_n is

$$\Gamma_n = \langle t_1, \ldots, t_{n-1}, \tau \mid \tau^2 = 1, t_i^2 = 1, \tau t_i = t_i \tau, (t_i t_{i+1})^3 = 1, (t_i t_j)^2 = \tau (i \leq j - 2) \rangle.$$

(See for example [14, p. 179].) A two-cocycle $S_n \times S_n \to \mathbb{C}^\times$ is determined up to coboundary by an irreducible representation of the Schur representation group Γ_n for which the central element τ necessarily acts as multiplication by a scalar. As $\tau^2 = 1$, this scalar must be -1 in case of the nontrivial cocycle α. Let $g = (12)(34)$. The presentation of Γ_n may be used to check that

$$\det(h^\perp) = \alpha(h, g)\alpha^{-1}(g, h)$$

for all $h \in C(g)$. For example, letting $h = (12)$, we have $\det(h^\perp) = -1$. Choose a section $T : S_n \to \Gamma_n$ of the projection from Γ_n to S_n that sends τ to 1 and t_i to the transposition $(i, i+1)$ as follows: Let $T(12) = t_1$, $T(34) = t_3$, $T((12)(34)) = t_1 t_3$, and choose other images arbitrarily. Then $\alpha(h, g)\alpha^{-1}(g, h)$ is the scalar by which the following element acts on an irreducible representation of Γ_n:

$$T(12)T((12)(34))T((12)(12)(34))^{-1}T((12)(34)(12))T(12)^{-1}T((12)(34))^{-1}$$

$$\begin{aligned} &= T(12)T(12)T(34)T(34)^{-1}T(34)T(12)^{-1}T((12)(34))^{-1} \\ &= T(34)T(12)T(34)T(12) \\ &= \tau, \end{aligned}$$

thus the scalar is -1 as desired. Other elements h of $C(g)$ may be checked similarly; as $h \mapsto \det(h^\perp)$ is a group homomorphism, it suffices to check the condition (2.11) on generators of $C(g)$. Therefore there is a twisted graded Hecke algebra corresponding to α with $a_g \neq 0$. (Compare with trivial α, where a_g is necessarily 0 for this choice of g [20, Table 1].) With the choice $g = (123)$, we find as in the case of the trivial cocycle α, that $1 = \det(h^\perp) = \alpha(h, g)\alpha^{-1}(g, h)$ for all $h \in C(g)$. Therefore the parameter space of twisted graded Hecke algebras is larger than that of graded Hecke algebras, and involves the conjugacy classes of all $g \in G$ for which $\mathrm{codim}\, V^g = 2$.

3. Deformations of crossed products

In this section, we prove that the twisted graded Hecke algebras of the previous section are precisely the deformations of $S(V)\#_\alpha G$ of a particular type. We use this connection to deformations to put our results and examples in a larger context.

Let t be an indeterminate. Given any associative algebra R over $\mathbb{C}$ (for example $R = S(V)\#_\alpha G$), a *deformation of R over $\mathbb{C}[t]$* is an associative $\mathbb{C}[t]$-algebra with underlying vector space $R[t] = \mathbb{C}[t] \otimes R$ and multiplication of the form

$$r * s = rs + \mu_1(r,s)t + \mu_2(r,s)t^2 + \cdots + \mu_p(r,s)t^p$$

for all $r, s \in R$, where rs denotes the product of r and s in R, the $\mu_i : R \times R \to R$ are $\mathbb{C}$-bilinear maps extended to be $\mathbb{C}[t]$-bilinear, and p depends on r, s. Associativity of $*$ implies that μ_1 is a *Hochschild two-cocycle*, that is

$$(3.1) \qquad \mu_1(w,r)s + \mu_1(wr,s) = \mu_1(w,rs) + w\mu_1(r,s)$$

for all $w, r, s \in R$, as well as further conditions on the μ_i, $i \geq 1$. The cocycle μ_1 is called the *infinitesimal* of the deformation. (See [10] or [11] for the details from algebraic deformation theory.)

Recall that $S(V)\#_\alpha G$ is a graded algebra where we assign degree 1 to elements of V and degree 0 to elements of G.

Theorem 3.2. *Up to isomorphism, the twisted graded Hecke algebras are precisely the deformations of $S(V)\#_\alpha G$ over $\mathbb{C}[t]$ for which $\deg \mu_i = -2i$ ($i \geq 1$).*

Proof. Let A be a twisted graded Hecke algebra, as defined in Section 2. Let $v_1, \ldots, v_n$ be a basis of V and choose the section of the projection from $T(V)$ to $S(V)$ in which a word in $T(V)$ is written in the order $v_1^{i_1} \cdots v_n^{i_n}$. Express all elements of A in terms of this section, writing group elements on the right. Such expressions in A exist and are unique due to the additive isomorphism $A \cong S(V)\#_\alpha G[t]$. Now let $r = v_1^{i_1} \cdots v_n^{i_n}\overline{g}$ and $s = v_1^{j_1} \cdots v_n^{j_n}\overline{h}$ be elements of A. Denoting the product in A by $*$, we have

$$r * s = \alpha(g,h)v_1^{i_1} \cdots v_n^{i_n} * (g \cdot v_1^{j_1} \cdots v_n^{j_n})\overline{gh}.$$

The factor $v_1^{i_1} \cdots v_n^{i_n}(g \cdot v_1^{j_1} \cdots v_n^{j_n})$ may now be rearranged using (2.3) repeatedly until it is in the standard form for elements of A discussed above. The result will be of the form

$$r * s = rs + \mu_1(r,s)t + \mu_2(r,s)t^2 + \cdots + \mu_p(r,s)t^p,$$

where rs is identified with the product of r and s in $S(V)\#_\alpha G$, via the additive isomorphism $A \cong S(V)\#_\alpha G[t]$, and $\mu_1(r,s), \ldots, \mu_p(r,s)$ are elements of A also identified with elements of $S(V)\#_\alpha G$. This is a finite process as each time (2.3) is applied, the degree drops. As the multiplication $*$ in the twisted graded Hecke algebra A is bilinear and associative, the maps μ_i are bilinear and thus A is a deformation of $S(V)\#_\alpha G$ over $\mathbb{C}[t]$. (Alternatively, the existence of the

obvious algebra isomorphism $A/tA \cong S(V)\#_\alpha G$ implies that A is a deformation of $S(V)\#_\alpha G$ over $\mathbb{C}[t]$.) The conditions on the μ_i stated in the theorem are consequences of the relations (2.3), by induction on the degree $\sum_{k=1}^{n}(i_k + j_k)$ of a product $v_1^{i_1} \cdots v_n^{i_n} * v_1^{j_1} \cdots v_n^{j_n}$.

Conversely, suppose that A is a deformation of $S(V)\#_\alpha G$ over $\mathbb{C}[t]$ satisfying the given conditions. By definition, $A \cong S(V)\#_\alpha G[t]$ as a vector space over $\mathbb{C}[t]$. Define a $\mathbb{C}[t]$-linear map $\phi : T(V)\#_\alpha G[t] \to A$ by

$$\phi(v_{i_1} \cdots v_{i_m}\overline{g}) = v_{i_1} * \cdots * v_{i_m} * \overline{g}$$

for all words $v_{i_1} \cdots v_{i_m}$ in $T(V)$ and $g \in G$. It may be checked that ϕ is an algebra homomorphism since $T(V)$ is free on $v_1, \ldots, v_n$, and since $\mu_i(\mathbb{C}^\alpha G, \mathbb{C}^\alpha G) = \mu_i(\mathbb{C}^\alpha G, V) = \mu_i(V, \mathbb{C}^\alpha G) = 0$ for all $i \geq 1$ by the degree condition on μ_i. We will show, by induction on degree, that ϕ is surjective: First note that $\phi(\overline{g}) = \overline{g}$ and $\phi(v\overline{g}) = v\overline{g}$ for all $v \in V$, $g \in G$. Now we would like to show that an arbitrary basis monomial $v_{i_1} \cdots v_{i_m}\overline{g}$ $(i_1 \leq \cdots \leq i_m)$ of A is in $\mathrm{im}(\phi)$. Assume $v_{i_2} \cdots v_{i_m}\overline{g} = \phi(X)$ for some element X of $T(V)\#_\alpha G[t]$. Then

$$
\begin{aligned}
\phi(v_{i_1}X) &= \phi(v_{i_1}) * \phi(X) \\
&= v_{i_1} * v_{i_2} \cdots v_{i_m}\overline{g} \\
&= v_{i_1} \cdots v_{i_m}\overline{g} + \mu_1(v_{i_1}, v_{i_2} \cdots v_{i_m}\overline{g})t + \mu_2(v_{i_1}, v_{i_2} \cdots v_{i_m}\overline{g})t^2 + \cdots
\end{aligned}
$$

By induction on m, as μ_i is a map of degree $-2i$, each $\mu_j(v_{i_1}, v_{i_2} \cdots v_{i_m}\overline{g})$ is in $\mathrm{im}(\phi)$. This implies $v_{i_1} \cdots v_{i_m}\overline{g} \in \mathrm{im}(\phi)$, and thus ϕ is surjective.

It remains to determine the kernel of ϕ. Letting $v, w \in V$, we find that

$$
\begin{aligned}
\phi(vw) &= v * w = vw + \mu_1(v, w)t \\
\phi(wv) &= w * v = wv + \mu_1(w, v)t,
\end{aligned}
$$

since $\deg \mu_i = -2i$. As $vw = wv$ in $S(V)$, we may subtract to obtain $\phi(vw - wv) = (\mu_1(v, w) - \mu_1(w, v))t$. Since $\deg \mu_1 = -2$ and $\phi(\overline{g}) = \overline{g}$ for all $g \in G$, this implies that

(3.3) $$vw - wv - (\mu_1(v, w) - \mu_1(w, v))t$$

is in the kernel of ϕ for all $v, w \in V$. It also follows, as $\deg \mu_1 = -2$, that

$$\mu_1(v, w) - \mu_1(w, v) = \sum_{g \in G} a_g(v, w)\overline{g}$$

for some functions $a_g : V \times V \to \mathbb{C}$. By definition, a_g is bilinear and skew-symmetric for each $g \in G$. Let $I[t]$ be the ideal of $T(V)\#_\alpha G[t]$ generated by all such expressions (3.3), so that $I[t] \subset \ker \phi$. We will use a dimension count to show that $I[t] = \ker \phi$: By the arguments of the previous section, $T(V)\#_\alpha G[t]/I[t]$ is a quotient of $S(V)\#_\alpha G[t]$, and so has dimension in each degree no greater than that of $S(V)\#_\alpha G[t]$. Since ϕ induces a map from $T(V)\#_\alpha G[t]/I[t]$ onto $S(V)\#_\alpha G[t]$, this forces $I[t] = \ker \phi$ and thus A is a twisted graded Hecke algebra. $\square$

The proof of Theorem 3.2 shows that the skew-symmetric forms a_g appearing in the relations (2.3) of a twisted graded Hecke algebra arise as coefficients in the skew-symmetrization of a Hochschild two-cocycle μ_1 on $S(V)\#_\alpha G$. Hochschild cohomology thus provides an alternate approach to the study of twisted graded Hecke algebras. An advantage of this approach is that it puts the computation of the possible skew-symmetric forms a_g into a larger context. A disadvantage is that, given a Hochschild two-cocycle μ_1 on an algebra R, there is no general method for determining whether it lifts to a deformation of R, nor for finding the μ_i $(i \geq 2)$ in case it does lift. In case $\deg(\mu_1) = -1$, Remark 2.14 points to existence of deformations of $S(V)\#_\alpha G$ defined by quotients of $T(V)\#_\alpha G[t]$ analogous to twisted graded Hecke algebras. There are a few special cases where deformations of $S(V)\#_\alpha G$ are known whose infinitesimals μ_1 have arbitrarily high degree [6, 24]. These examples were discovered after examining the relevant Hochschild cohomology:

$$\mathrm{HH}^\bullet(S(V)\#_\alpha G) \cong \left(\bigoplus_{g \in G} \textstyle\bigwedge^{\bullet - \operatorname{codim} V^g}((V^g)^*) \otimes \det(((V^g)^\perp)^*) \otimes S(V^g)\overline{g} \right)^G$$

additively, where the superscript G denotes elements invariant under the action induced by conjugation in $\mathbb{C}^\alpha G$ and the action of G on V, and det denotes the top exterior power. This follows from [12], see (6.2) and the formula before (6.4) where $S(V)$ is replaced by $\mathbb{C}[M]$, or more directly from the techniques of [2], where $S(V)$ is replaced by a Weyl algebra.

We conclude by giving a broader context for the isomorphism found by Ram and Shepler between Lusztig's and Drinfeld's definitions of (untwisted) graded Hecke algebras for real reflection groups [8, 17, 20]. Let $W \subset GL(V)$ be a finite real reflection group. Specifically, W is generated by simple reflections $s_1, \ldots, s_n$ corresponding to simple roots $\alpha_1, \ldots, \alpha_n \in V$ for a root system R in V. If $\alpha \in R$ is any root, we write s_α for the reflection corresponding to α, so that

$$s_\alpha \cdot v = v - \langle v, \alpha^\smallsmile \rangle \alpha$$

where $\alpha^\smallsmile = 2\alpha/\langle \alpha, \alpha \rangle$, and $\langle\ ,\ \rangle$ is the inner product. For each $\alpha \in R$, choose $k_\alpha \in \mathbb{C}$ in such a way that $k_{g \cdot \alpha} = k_\alpha$ for all $g \in W$, so that the number of independent parameters k_α is simply the number of different lengths of roots. Lusztig's graded version of an affine Hecke algebra, as defined over $\mathbb{C}[t]$, is the quotient A' of $T(V)\#W[t]$ by the ideal generated by all

$$[v, w] \quad \text{and} \quad \overline{s}_i v - (s_i \cdot v)\overline{s}_i + k_{\alpha_i}\langle v, \alpha_i^\smallsmile \rangle t$$

where $v, w \in V$ and $i = 1, \ldots, n$.

Ram and Shepler defined the following graded Hecke algebra and showed that it is isomorphic to Lusztig's algebra A' given above (and their isomorphism extends

to one over $\mathbb{C}[t]$): For each $g \in W$, let

$$a_g(v, w) = \frac{1}{4} \sum_{\substack{\alpha, \beta > 0 \\ g = s_\alpha s_\beta}} k_\alpha k_\beta \left(\langle v, \beta^\vee \rangle \langle w, \alpha^\vee \rangle - \langle v, \alpha^\vee \rangle \langle w, \beta^\vee \rangle \right).$$

In order to make the connection to Lusztig's version, replace t by t^2 in (2.3) and take the quotient A of $T(V)\#W[t]$ by the ideal generated by all

$$[v, w] - \sum_{g \in W} a_g(v, w) t^2 \overline{g}.$$

The resulting quotient algebra A is in fact a graded Hecke algebra defined over $\mathbb{C}[t^2]$, however the isomorphism with Lusztig's algebra A' is not defined over $\mathbb{C}[t^2]$. This isomorphism $\Phi_t : A' \to A$ is given by

$$\Phi_t(v) = v - t\langle v, h \rangle \quad \text{and} \quad \Phi_t(\overline{g}) = \overline{g}$$

for all $v \in V, g \in W$, where $h = \frac{1}{2} \sum_{\alpha > 0} k_\alpha \alpha^\vee \overline{s}_\alpha$, and $\langle v, h \rangle = \frac{1}{2} \sum_{\alpha > 0} k_\alpha \langle v, \alpha^\vee \rangle \overline{s}_\alpha$.
This may be extended uniquely to an algebra isomorphism by the calculations of Ram and Shepler. The reason for replacing t by t^2 in the definition of A is that it is necessary in order to extend their isomorphism to one defined over $\mathbb{C}[t]$. The only calculation that changes significantly in this context occurs in the proof that

$$[\Phi_t(v), \Phi_t(w)] = [v, w] + t^2[\langle v, h \rangle, \langle w, h \rangle] - t[v, \langle w, h \rangle] + t[w, \langle v, h \rangle] = 0.$$

This is true as $[v, \langle w, h \rangle] = [w, \langle v, h \rangle]$ and $[\langle v, h \rangle, \langle w, h \rangle] = -\sum_{g \in W} a_g(v, w)\overline{g}$ by computations in the proof of [20, Thm. 3.5].

In particular, the isomorphism Φ_t is an equivalence of deformations over $\mathbb{C}[t]$. This implies that the Hochschild two-cocycle corresponding to the coefficients of t in the deformation A' of $S(V)\#W$ is a coboundary, however there is a Hochschild two-cocycle arising from the coefficients of t^2 in products in A' that will be cohomologous to that for A. (Again see [10] or [11] for details from the general theory of algebraic deformations.)

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