COHOMOLOGY OF HOPF ALGEBRAS

SARAH WITHERSPOON

1. Introduction

In these expository notes, we collect some homological techniques for Hopf algebras, and prove in particular that the cohomology of a Hopf algebra embeds, as a subalgebra, into its Hochschild cohomology. Important special cases are group algebras and universal enveloping algebras of Lie algebras. The embedding was first noted in the general case by Ginzburg and Kumar [7]. Our proof is based on an explicit isomorphism of modules (Lemma 3.2), and we prove directly that the embedding is multiplicative (Corollary 3.8). This proof appeared first in the appendix of [13]; here we include more details.

These notes were developed in part for a short course given at the University of Buenos Aires in May 2010, and we thank the University of Buenos Aires for its hospitality. We thank Andrea Solotar and the students in the course, as well as Dave Benson and Paul Sobaje, for their comments which greatly improved the writing.

2. Some homological properties

Let A be a Hopf algebra over a field k with counit $\varepsilon : A \to k$ and antipode $S : A \to A$. Tensor products will be over k unless otherwise specified. We will use Sweedler notation for the coproduct $\Delta : A \to A \otimes A$, that is $\Delta(a) = \sum_{(a)} a_1 \otimes a_2$ for $a \in A$, although we will often leave out the subscript (a) on the summation symbol. Tensor products of A-modules will be considered to be A-modules via the coproduct Δ, unless otherwise stated.

We establish relations among A-modules given by Hom, $\otimes$, and dual. If V, W are left A-modules, then $\mathrm{Hom}_k(V, W)$ is a left A-module via

$$(a \cdot f)(v) = \sum a_1 f(S(a_2)v),$$

and similarly, if V, W are right A-modules, then $\mathrm{Hom}_k(V, W)$ is a right A-module via

$$(f \cdot a)(v) = \sum f(vS(a_1))a_2,$$

for all $a \in A$, $f \in \mathrm{Hom}_k(V, W)$, and $v \in V$.

Date: 10 January 2017.

Lemma 2.1. *(i) Let U, V, and W be left A-modules. Then there is a natural isomorphism of left A-modules*

$$\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)).$$

If S is bijective, there is a natural isomorphism of vector spaces

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(U, \mathrm{Hom}_k(V, W)).$$

(ii) Let U, V, and W be right A-modules. Then there is a natural isomorphism of right A-modules,

$$\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W)),$$

and a natural isomorphism of vector spaces

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(V, \mathrm{Hom}_k(U, W)).$$

Proof. (i) Define functions $\phi : \mathrm{Hom}_k(U \otimes V, W) \to \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$ by

$$(\phi(f)(u))(v) = f(u \otimes v),$$

and $\psi : \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \to \mathrm{Hom}_k(U \otimes V, W)$ by

$$(\psi(g))(u \otimes v) = (g(u))(v).$$

We check that ϕ is an A-module homomorphism. Similar calculations show that ψ is an A-module homomorphism and is inverse to ϕ. Let $a \in A$. Then, as S is an anti-coalgebra homomorphism,

$$\begin{aligned}(\phi(a \cdot f)(u))(v) &= (a \cdot f)(u \otimes v)\\ &= \sum a_1(f(S(a_2) \cdot (u \otimes v)))\\ &= \sum a_1(f(S(a_3)u \otimes S(a_2)v)).\end{aligned}$$

On the other hand,

$$\begin{aligned}(a \cdot \phi(f))(u)(v) &= \sum (a_1(\phi(f)(S(a_2)u)))(v)\\ &= \sum a_1((\phi(f))(S(a_3)u)(S(a_2)v)\\ &= \sum a_1(f(S(a_3)u \otimes S(a_2)v)).\end{aligned}$$

Therefore $\phi(a \cdot f) = a \cdot \phi(f)$.

For the second statement in part (i), we note that in general $\mathrm{Hom}_A(U, V)$ is precisely the subspace of A-invariant elements of $\mathrm{Hom}_k(U, V)$, that is

$$(\mathrm{Hom}_k(U, V))^A := \{f \in \mathrm{Hom}_k(U, V) \mid a \cdot f = \varepsilon(a) f \text{ for all } a \in A\}.$$

(This is a straightforward computation.)

(ii) The proof of the statements for right modules is similar. $\square$

If V is a left (respectively, right) A-module, its dual vector space $V^* = \mathrm{Hom}_k(V, k)$ has a left (respectively, right) A-module structure given by

$$(a \cdot f)(v) = f(S(a)v)$$

(respectively, $(f \cdot a)(v) = f(vS(a))$ for all $a \in A$, $v \in V$, and $f \in V^*$.

Lemma 2.2. *Let V, W be left (respectively, right) A-modules. If V is finite dimensional as a vector space over k, then $\mathrm{Hom}_k(V, W) \cong W \otimes V^*$ as left A-modules (respectively, $\mathrm{Hom}_k(V, W) \cong V^* \otimes W$ as right A-modules).*

Proof. We will prove the statement for left A-modules; that for right A-modules is similar. Let $\phi : W \otimes V^* \to \mathrm{Hom}_k(V, W)$ and $\psi : \mathrm{Hom}_k(V, W) \to W \otimes V^*$ be defined by $(\phi(w \otimes f))(v) = f(v)w$ and $\psi(f) = \sum_i f(v_i) \otimes v_i^*$ where $\{v_i\}, \{v_i^*\}$ are dual bases for V, V^*. Let $a \in A$. Then

$$\begin{aligned}
\phi(a \cdot (w \otimes f))(v) &= \sum \phi(a_1 w \otimes (a_2 \cdot f))(v) \\
&= \sum ((a_2 \cdot f)(v))(a_1 w) \\
&= \sum f(S(a_2)v) a_1 w.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(a \cdot (\phi(w \otimes f)))(v) &= \sum a_1 (\phi(w \otimes f)(S(a_2)v)) \\
&= \sum a_1 (f(S(a_2)v)w) \\
&= \sum f(S(a_2)v) a_1 w.
\end{aligned}$$

Therefore $\phi(a \cdot (w \otimes f)) = a \cdot (\phi(w \otimes f))$. Clearly ϕ is inverse to ψ. □

Remark 2.3. Alternatively, we may define a dual module as $V^\# = \mathrm{Hom}_k(V, k)$ with action $(a \cdot f)(v) = f(\overline{S}(a)v)$ where $\overline{S}$ is the composition inverse of S (under the assumption that S is bijective). Give $\mathrm{Hom}_k(V, W)$ the alternative A-module structure $(a \cdot f)(v) = \sum a_2 f(\overline{S}(a_1)v)$, under which it is still the case that the subspace of A-homomorphisms in $\mathrm{Hom}_k(V, W)$ is equal to the A-invariant subspace of $\mathrm{Hom}_k(V, W)$. It may be shown that under this A-module structure, $\mathrm{Hom}_k(V, W) \cong V^\# \otimes W$, and $\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W))$.

Lemma 2.4. *If P is a projective left A-module, and V any left A-module, then both $P \otimes V$ and $V \otimes P$ are projective left A-modules. If P is a projective right A-module, and V any right A-module, then both $P \otimes V$ and $V \otimes P$ are projective right A-modules. Similar statements apply with "projective" replaced by "flat".*

Proof. We give two proofs of the first statement. The first proof is essentially that given in Benson [2, Proposition 3.1.5]: The projective module P is a direct summand of a free module, so it suffices to prove that $A \otimes V$ and $V \otimes A$ are both free as left A-modules. There is an isomorphism $A \otimes V \cong A \otimes V_{tr}$, where V_{tr} is the underlying vector space of V, but with the trivial A-module structure (via ε). This isomorphism is similar to one in Montgomery [11, Theorem 1.9.4], and is given by $a \otimes v \mapsto \sum a_1 \otimes S(a_2)v$, the inverse function by $a \otimes v \mapsto \sum a_1 \otimes a_2 v$, for all $v \in V$, $a \in A$. Now V_{tr} is a direct sum of copies of the trivial module k, and so $A \otimes V_{tr}$ is a free left A-module. Similarly, there is an isomorphism of left A-modules, $V \otimes A \cong V_{tr} \otimes A$, via the A-module homomorphism $v \otimes a \mapsto \sum \overline{S}(a_1)v \otimes a_2$ whose inverse is $v \otimes a \mapsto \sum a_1 v \otimes a_2$, and so $V \otimes A$ is a free left A-module.

The second proof of the first statement uses properties of functors: As V is projective over the field k and P is projective over A, $\mathrm{Hom}_A(P, \mathrm{Hom}_k(V, -))$ is an exact functor. By Lemma 2.1, this is the same as $\mathrm{Hom}_A(P \otimes V, -)$. Therefore $P \otimes V$ is projective. A similar argument applies to $V \otimes P$, using the alternative A-action on Hom given in Remark 2.3.

The second statement, for right modules, may be proven similarly. The last statement follows since flat modules are direct limits of finitely generated free modules. $\square$

Lemma 2.5. *Let U, V, W be left A-modules. The isomorphisms of Lemmas 2.1 and 2.2 induce the following isomorphism:*

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(U, W \otimes V^*).$$

A similar statement holds for right modules.

Proof. Let $P_\bullet$ be a projective resolution of U, so that $P_\bullet \otimes V$ is a projective resolution of $U \otimes V$ (see Lemma 2.4). The natural isomorphisms of Lemmas 2.1 and 2.2 yield a chain homotopy equivalence

$$\mathrm{Hom}_A(P_\bullet, W \otimes V^*) \to \mathrm{Hom}_A(P_\bullet \otimes V, W),$$

and thus an isomorphism on Ext as claimed. $\square$

Remark 2.6. Using the alternative A-action on Hom described in Remark 2.3, we similarly find that

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(V, U^{\#} \otimes W).$$

Cup products. We use the following notation for (co)homology of the Hopf algebra A over the field k, and Hochschild (co)homology, to distinguish the two:

$$\begin{array}{rclrcl} \mathrm{H}^*(A, M) & := & \mathrm{Ext}^*_A(k, M), & \mathrm{H}_*(A, M) & := & \mathrm{Tor}^A_*(k, M), \\ \mathrm{HH}^*(A, M) & := & \mathrm{Ext}^*_{A^e}(A, M), & \mathrm{HH}_*(A, M) & := & \mathrm{Tor}^{A^e}_*(A, M), \end{array}$$

where M denotes a left A-module in the first line, and an A-bimodule in the second line.

We will use the following version of the Künneth Theorem to define products on cohomology. For a proof under the flat hypothesis on C, see e.g. Weibel [19, Theorem 3.6.3]; a symmetric argument gives a proof under the flat hypothesis on D. For more details on cup products, see e.g. Benson [2, §3.2].

Theorem 2.7 (Künneth)**.** *Let R be a ring. Let C be a chain complex of right R-modules, and let D be a chain complex of left R-modules. Assume either that C_n and $d(C_n)$ are both flat R-modules for each n, or that D_n and $d(D_n)$ are both flat R-modules for each n. Then for each n, there is a natural short exact sequence:*

$$0 \to \bigoplus_{i+j=n} \mathrm{H}_i(C) \otimes_R \mathrm{H}_j(D) \to \mathrm{H}_n(C \otimes_R D) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(\mathrm{H}_i(C), \mathrm{H}_j(D)) \to 0.$$

Some of our applications of the Künneth Theorem will be in the case that $R = k$ is a field, in which case the Tor term always vanishes, so that the remaining two terms are isomorphic for each n.

Again let A be a Hopf algebra over a field k, and let M, M', N, N' be left A-modules. We will define a cup product for each $i, j \geq 0$ as in Benson [2, §3.2],

$$\smile : \mathrm{Ext}_A^i(M, M') \times \mathrm{Ext}_A^j(N, N') \to \mathrm{Ext}_A^{i+j}(M \otimes N, M' \otimes N').$$

Let $P_{\bullet}$ be a projective resolution of M, and $Q_{\bullet}$ be a projective resolution of N. Consider the total complex of the tensor product complex $P_{\bullet} \otimes Q_{\bullet}$. By Lemma 2.4, each module in this complex is projective. By the Künneth Theorem, since the tensor product is over the field k and Tor_1^k is always 0, $P_{\bullet} \otimes Q_{\bullet}$ is a projective resolution of the A-module $M \otimes N$.

Let $f \in \mathrm{Hom}_A(P_i, M')$, $g \in \mathrm{Hom}_A(Q_j, N')$ represent elements of $\mathrm{Ext}_A^i(M, M')$, $\mathrm{Ext}_A^j(N, N')$, respectively. Then

$$f \otimes g \in \mathrm{Hom}_A(P_i \otimes Q_j, M' \otimes N')$$

and this function may be extended to an element of

$$\mathrm{Hom}_A(\bigoplus_{r+s=i+j} (P_r \otimes Q_s), M' \otimes N')$$

by defining it to be the 0 map on all components other than $P_i \otimes Q_j$. By definition of the differential on the total complex,

$$d(f \otimes g) = d(f) \otimes g + (-1)^{\deg f} f \otimes d(g),$$

so such a product of two cocycles is again a cocycle. It also follows from this formula that the product of a cocycle with a coboundary is a coboundary. Therefore this induces a well-defined product on cohomology.

We will need the following result from Benson [2].

Lemma 2.8. [2, Prop. 3.2.1] *If M, M', N, N' are left A-modules and $\zeta \in \mathrm{Ext}_A^m(M, M')$, $\eta \in \mathrm{Ext}_A^n(N, N')$, then the cup product*

$$\zeta \smile \eta \in \mathrm{Ext}_A^{m+n}(M \otimes N, M' \otimes N')$$

is equal to the Yoneda composite of

$$\zeta \otimes \mathrm{id}_{N'} \in \mathrm{Ext}_A^m(M \otimes N', M' \otimes N')$$

and

$$\mathrm{id}_M \otimes \eta \in \mathrm{Ext}^n_A(M \otimes N, M \otimes N').$$

Remark 2.9. Letting $M = M' = N = N' = k$, since $k \otimes k \cong k$, the cup product (equivalently, Yoneda composition) gives $\mathrm{H}^*(A, k)$ the structure of a graded ring. Another proof that the two products are the same also shows that the product is graded-commutative, that is $\alpha \smile \beta = (-1)^{\deg\alpha \deg\beta}\beta \smile \alpha$; this is the Eckmann-Hilton argument. See Suarez-Alvarez [17] for a general context for this type of argument, which does not require cocommutativity of A (cf. [2, Cor. 3.2.2]). More generally, when $M = N = k$ and $M' = N' = B$ is an A-module algebra (that is, $a \cdot (bb') = \sum (a_1 \cdot b)(a_2 \cdot b')$ for all $a \in A$, $b, b' \in B$), we may compose the cup product with the map induced by multiplication $B \otimes B \to B$ to obtain a ring structure on $\mathrm{H}^*(A, B)$. In the next section, we will let B be the algebra A itself, under the adjoint action of A, as defined there.

Let $M = M' = k$ and $N' = N$. By composing with the isomorphism $k \otimes N \cong N$, we thus obtain an action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*_A(N, N)$, via $- \otimes N$ followed by Yoneda composition. On the other hand, we have an action by Yoneda composition of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*_A(k, N \otimes N^*)$.

In the following statement, we apply Lemma 2.5 with $U = k$, $V = N$:

Theorem 2.10. *Let N be a left A-module. The action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*_A(N, N)$, given by $-\otimes N$ followed by Yoneda composition, corresponds to that on $\mathrm{Ext}^*_A(k, N \otimes N^*)$, given by Yoneda composition, under the isomorphism*

$$\mathrm{Ext}^*_A(N, N) \cong \mathrm{Ext}^*_A(k, N \otimes N^*).$$

Proof. Let $P_\bullet$ be a projective resolution of k, so that $P_\bullet \otimes N$ is a projective resolution of $k \otimes N \cong N$. We must check that the following diagram commutes for each m, n, where ϕ_m, ϕ_{m+n} are the isomorphisms given by Lemma 2.1(i) with $V = N$ and $U = P_m, P_{m+n}$, respectively, and the horizontal maps are the chain level maps corresponding to cup product (see Lemma 2.8).

$$\begin{tikzcd}
\mathrm{Hom}_A(P_m, k) \otimes \mathrm{Hom}_A(P_n \otimes N, N) \arrow[r] \arrow[d, "\mathrm{id} \otimes \phi_n"] & \mathrm{Hom}_A(P_{m+n} \otimes N, N) \arrow[d, "\phi_{m+n}"] \\
\mathrm{Hom}_A(P_m, k) \otimes \mathrm{Hom}_A(P_n, \mathrm{Hom}_k(N, N)) \arrow[r] & \mathrm{Hom}_A(P_{m+n}, \mathrm{Hom}_k(N, N))
\end{tikzcd}$$

Let $\zeta \in \mathrm{Ext}^m_A(k, k)$ and $\eta \in \mathrm{Ext}^n_A(N, N)$, represented by $f \in \mathrm{Hom}_A(P_m, k)$ and $g \in \mathrm{Hom}_A(P_n \otimes N, N)$, respectively. Identify f with the corresponding function from $\Omega^m(k)$ to k, and extend to a chain map $f_\bullet$ with $f_i \in \mathrm{Hom}_A(P_{m+i}, P_i)$. The top horizontal map takes $f \otimes g$ to $g \circ (f_n \otimes \mathrm{id}_N)$, and applying ϕ_{m+n} we have

$$\phi_{m+n}(g \circ (f_n \otimes \mathrm{id}_N))(x)(v) = g(f_n(x) \otimes v)$$

for all $x \in P_{m+n}$, $v \in N$. On the other hand, $(\mathrm{id} \otimes \phi_n)(f \otimes g) = f \otimes \phi_n(g)$, and applying the bottom horizontal map we find

$$(\phi_n(g) \circ f_n)(x)(v) = g(f_n(x) \otimes v).$$

Therefore the diagram commutes. □

There is another action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$ given by $M \otimes -$ followed by Yoneda composition. In case A is cocommutative (or more generally quasi-triangular), this action is the same as that given by $- \otimes M$ followed by Yoneda composition. In general it will not be the same; see, for example, [3]. We state next the counterpart of Theorem 2.10 for this action. Let $\mathrm{Hom}'_k(V,W)$ denote the A-module that is $\mathrm{Hom}_k(V,W)$ as a vector space, but with action as described in Remark 2.3. Let $V^{\#} = \mathrm{Hom}_k(V,k)$, with A-module structure as described in Remark 2.3. Then, as stated there, there are isomorphisms of A-modules:

$$\mathrm{Hom}'_k(U \otimes V, W) \cong \mathrm{Hom}'_k(V, \mathrm{Hom}'_k(U,W)) \cong \mathrm{Hom}'_k(V, U^{\#} \otimes W).$$

It follows that

$$\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A(V, U^{\#} \otimes W), \tag{2.1}$$

and consequently

$$\mathrm{Ext}^*_A(U \otimes V, W) \cong \mathrm{Ext}^*_A(V, U^{\#} \otimes W).$$

Theorem 2.11. *Let M be a left A-module. The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$, given by $M \otimes -$ followed by Yoneda composition, corresponds to that on $\mathrm{Ext}^*_A(k, M^{\#} \otimes M)$, given by Yoneda composition, under the isomorphism*

$$\mathrm{Ext}^*_A(M,M) \cong \mathrm{Ext}^*_A(k, M^{\#} \otimes M).$$

Proof. Let $P_{\bullet}$ be a projective resolution of k, so that $M \otimes P_{\bullet}$ is a projective resolution of $M \otimes k \cong M$. We must check that the following diagram commutes for each m, n, where ϕ_m, ϕ_{m+n} are the isomorphisms given in (2.1), with $U = M$ and $V = P_n, P_{m+n}$, respectively, and the horizontal maps are the chain level maps corresponding to cup product (see Lemma 2.8).

$$\begin{tikzcd}
\mathrm{Hom}_A(P_m,k) \otimes \mathrm{Hom}_A(M \otimes P_n, M) \arrow[r] \arrow[d, "\mathrm{id} \otimes \phi_n"] & \mathrm{Hom}_A(M \otimes P_{m+n}, M) \arrow[d, "\phi_{m+n}"] \\
\mathrm{Hom}_A(P_m,k) \otimes \mathrm{Hom}_A(P_n, \mathrm{Hom}'_k(M,M)) \arrow[r] & \mathrm{Hom}_A(P_{m+n}, \mathrm{Hom}'_k(M,M))
\end{tikzcd}$$

Let $\zeta \in \mathrm{Ext}^m_A(k,k)$ and $\eta \in \mathrm{Ext}^n_A(M,M)$, represented by $f \in \mathrm{Hom}_A(P_m,k)$ and $g \in \mathrm{Hom}_A(M \otimes P_n, M)$, respectively. Identify f with the corresponding function from $\Omega^m(k)$ to k, and extend to a chain map $f_{\bullet}$ with $f_i \in \mathrm{Hom}_A(P_{m+i}, P_i)$. The top horizontal map takes $f \otimes g$ to $g \circ (\mathrm{id}_M \otimes f_n)$, and applying ϕ_{m+n} we have

$$\phi_{m+n}(g \circ (\mathrm{id}_M \otimes f_n))(x)(v) = g(v \otimes f_n(x))$$

for all $x \in P_{m+n}$, $v \in M$. On the other hand, $(\mathrm{id} \otimes \phi_n)(f \otimes g) = f \otimes \phi_n(g)$, and applying the bottom horizontal map we find

$$(\phi_n(g) \circ f_n)(x)(v) = g(v \otimes f_n(x)).$$

Therefore the diagram commutes. $\square$

It can be shown that $(N^*)^\# \cong N$ and $(M^\#)^* \cong M$ as A-modules. As a consequence, we have the following theorem.

Theorem 2.12. *Let M, N be left A-modules.*

(i) *The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(N,N)$, given by $-\otimes N$ followed by Yoneda composition, corresponds to the action given by $N^* \otimes -$ followed by Yoneda composition.*

(ii) *The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$, given by $M \otimes -$ followed by Yoneda composition, corresponds to the action given by $-\otimes M^\#$ followed by Yoneda composition.*

Proof. (i) Let $M = N^*$. Then $M^\# \cong N$, as noted above. Apply Theorems 2.10 and 2.11.

(ii) Let $N = M^\#$. Then $N^* \cong M$, as noted above. Apply Theorems 2.10 and 2.11. $\square$

3. Bimodules and Hochschild cohomology

Let $A^e = A \otimes A^{op}$, the enveloping algebra of A. We next give some relations among the algebras A, A^e, and their modules.

Lemma 3.1. *Let $\delta : A \to A^e$ be the function defined by*

$$\delta(a) = \sum a_1 \otimes S(a_2)$$

for all $a \in A$. Then δ is an injective algebra homomorphism.

Proof. First note that $\delta(1) = 1 \otimes 1$, the identity in A^e. Let $a, b \in A$. Then

$$\begin{aligned} \delta(ab) &= \sum a_1 b_1 \otimes S(a_2 b_2) \\ &= \sum a_1 b_1 \otimes S(b_2) S(a_2) \\ &= (\sum a_1 \otimes S(a_2))(\sum b_1 \otimes S(b_2)) = \delta(a)\delta(b), \end{aligned}$$

as multiplication in the second factor is opposite that in A.

To see that δ is injective, compose with the function $\pi : A^e \to A$ defined by $\pi(a \otimes b) = a\varepsilon(b)$. We have, for all $a \in A$,

$$\begin{aligned} \pi \circ \delta(a) = \pi(\sum a_1 \otimes S(a_2)) &= \sum a_1 \varepsilon(S(a_2)) \\ &= \sum a_1 \varepsilon(a_2) = a, \end{aligned}$$

that is $\pi \circ \delta$ is the identity map on A. This implies that δ is injective. □

We will identify A with the subalgebra $\delta(A)$ of A^e. This will allow us to induce modules from A to A^e, using tensor products: Let M be a left A-module, and consider A^e to be a right A-module via right multiplication by elements of $\delta(A)$. Then the vector space $A^e \otimes_A M$ is a left A^e-module, the action given by left multiplication in the first factor.

Consider A to be a left A^e-module by left and right multiplication. Let k be the trivial module for A, that is k is the field on which $a \in A$ acts by multiplication by $\varepsilon(a)$.

Lemma 3.2. *There is an isomorphism of left A^e-modules*

$$A \cong A^e \otimes_A k,$$

where $A^e \otimes_A k$ is the induced A^e-module.

Proof. Let $f : A \to A^e \otimes_A k$ be the function defined by

$$f(a) = a \otimes 1 \otimes 1,$$

and $g : A^e \otimes_A k \to A$ be the function defined by

$$g(a \otimes b \otimes 1) = ab$$

for all $a, b \in A$. We will check that f and g are both A^e-module homomorphisms, and that they are inverses.

Let $a, b, c \in A$. Then, since $c = \sum c_1 \varepsilon(c_2)$, we have

$$\begin{aligned} f((b \otimes c)(a)) &= f(bac) \\ &= bac \otimes 1 \otimes 1 \\ &= \sum bac_1 \otimes \varepsilon(c_2) \otimes 1 \\ &= \sum bac_1 \otimes S(c_2)c_3 \otimes 1. \end{aligned}$$

Now identifying A with $\delta(A) \subset A^e$, since the rightmost factor is in k with action of A given by ε, we may rewrite this as

$$\begin{aligned} \sum ba \otimes c_2 \otimes \varepsilon(c_1) &= \sum ba \otimes \varepsilon(c_1)c_2 \otimes 1 \\ &= ba \otimes c \otimes 1 \\ &= (b \otimes c)(a \otimes 1 \otimes 1) \\ &= (b \otimes c)f(a). \end{aligned}$$

Therefore f is an A^e-module homomorphism.

Let $a, b, c, d \in A$. Then

$$\begin{aligned} g((c \otimes d)(a \otimes b \otimes 1)) &= g(ca \otimes bd \otimes 1) \\ &= cabd \\ &= (c \otimes d)(ab) \\ &= (c \otimes d) g(a \otimes b \otimes 1). \end{aligned}$$

Therefore g is an A^e-module homomorphism.

Now let $a, b \in A$. We have

$$\begin{aligned} g \circ f(a) &= g(a \otimes 1 \otimes 1) = a, \\ \text{and } f \circ g(a \otimes b \otimes 1) &= f(ab) \\ &= ab \otimes 1 \otimes 1 \\ &= \sum ab_1 \varepsilon(b_2) \otimes 1 \otimes 1 \\ &= \sum ab_1 \otimes \varepsilon(b_2) \otimes 1 \\ &= \sum ab_1 \otimes S(b_2) b_3 \otimes 1 \\ &= \sum a \otimes b_2 \otimes \varepsilon(b_1) \\ &= \sum a \otimes \varepsilon(b_1) b_2 \otimes 1 \\ &= a \otimes b \otimes 1. \end{aligned}$$

Therefore f and g are inverse functions. □

We will use the following general lemma, due to Eckmann and Shapiro, in the case $B = A^e$. There is a similar statement for coinduced modules (using Hom) that we do not give here; see e.g. Benson [2, Corollary 2.8.4].

Lemma 3.3 (Eckmann-Shapiro)**.** *Let B be a ring, and let A be a subring of B for which B is flat as a right A-module. Let M be a left A-module, and let N be a left B-module. Consider N to be an A-module via restriction of the action, and let $B \otimes_A M$ denote the induced B-module where B acts on the leftmost factor by multiplication. Then for all $i \geq 0$, there is an isomorphism of abelian groups,*

$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_B(B \otimes_A M, N).$$

Proof. Let $P_{\bullet} \to M$ be an A-projective resolution of M. Since $B \otimes_A A \cong B$, the induced modules $B \otimes_A P_i$ are projective as B-modules. The induced complex $B \otimes_A P_{\bullet} \to B \otimes_A M$ is exact, as B is flat over A, and so it is a projective resolution of $B \otimes_A M$ as a B-module. We will prove that for each i, $\mathrm{Hom}_A(P_i, N) \cong \mathrm{Hom}_B(B \otimes_A P_i, N)$ as abelian groups. By their definitions, these isomorphisms will comprise a chain map that induces an isomorphism on homology.

Let

$$\phi : \mathrm{Hom}_A(P_i, N) \to \mathrm{Hom}_B(B \otimes_A P_i, N)$$

be defined by $\phi(f)(b\otimes p)=bf(p)$ for all $b\in B$, $p\in P_i$, and $f\in \mathrm{Hom}_A(P_i,N)$, and let

$$\psi:\mathrm{Hom}_B(B\otimes_A P_i,N)\to \mathrm{Hom}_A(P_i,N)$$

be defined by $\psi(g)(p)=g(1\otimes p)$, for all $p\in P_i$ and $g\in \mathrm{Hom}_B(B\otimes_A P_i,N)$. Since each such function g is a homomorphism of B-modules, ϕ and ψ are inverse maps, and by their definitions they are homomorphisms of abelian groups. □

Remark 3.4. There is a Tor version of the lemma which is easier: Let N be a left B-module, let M be a right A-module, and let $M\otimes_A B$ denote the induced module. If B is flat as a left A-module, then for all i there is an isomorphism of abelian groups, $\mathrm{Tor}_i^A(M,N)\cong \mathrm{Tor}_i^B(M\otimes_A B,N)$.

Lemma 3.5. *Assume the antipode S is bijective. Then the right A-module A^e, where A acts by right multiplication by $\delta(A)$, is a projective A-module.*

Proof. We claim that $S:A\to A^{op}$ is an isomorphism of right A-modules, where A acts on the right by multiplication on A and by multiplication by $S(A)$ on A^{op}. We need only check S is an A-module map: $S(a\cdot b)=S(b)S(a)=S(a)\cdot S(b)$, in A^{op}, for all $a,b\in A$. This yields an isomorphism of right A-modules $A\otimes A\to A\otimes A^{op}=A^e$. Now $A\otimes A$ is projective as a right A-module by Lemma 2.4. Thus A^e is a projective right A-module, the action of A being precisely multiplication by $\delta(A)$. □

Remark 3.6. The hypothesis that S is bijective is not very restrictive: All finite dimensional Hopf algebras satisfy this, as well as many known infinite dimensional Hopf algebras.

There is a cup product on Hochschild cohomology for any algebra A, which may be expressed in terms of the bar resolution

$$\cdots\to A^{\otimes 4}\to A^{\otimes 3}\to A\otimes A\to A\to 0$$

as follows. Identify $\mathrm{Hom}_{A^e}(A^{\otimes(i+2)},A)\cong \mathrm{Hom}_k(A^{\otimes i},A)$. Let $f\in \mathrm{Hom}_k(A^{\otimes i},A)$, $g\in \mathrm{Hom}_k(A^{\otimes j},A)$ represent elements in $\mathrm{HH}^i(A,A)$, $\mathrm{HH}^j(A,A)$, respectively. Then

$$(f\smile g)(a_1\otimes\cdots\otimes a_{i+j}):=f(a_1\otimes\cdots\otimes a_i)g(a_{i+1}\otimes\cdots a_{i+j})$$

for all $a_1,\dots,a_{i+j}\in A$, where the images of f and g are multiplied as elements in A. This represents an element in $\mathrm{HH}^{i+j}(A,A)$. More generally, if B is an A-bimodule that is also an algebra for which $a(bb')=(ab)b'$ and $(bb')a=b(b'a)$ for all $a\in A$, $b,b'\in B$, this formula gives a cup product on $\mathrm{HH}^*(A,B)$.

There is an equivalent definition of the cup product on Hochschild cohomology, using a tensor product of complexes, as follows. Let $X_\bullet$ be any A^e-projective resolution of A. We claim that the total complex of $X_\bullet\otimes_A X_\bullet$ is also an A^e-projective resolution of A. First we will show that for each i,j, $X_i\otimes_A X_j$ is projective as an A^e-module. It suffices to show that $A^e\otimes_A A^e$ is projective. But

$A^e \otimes_A A^e \cong A^e \otimes_k A$ since A^e acts only on the outermost two factors of A. Since A is free as a k-module, we see that $A^e \otimes_k A$ is free as an A^e-module. Next, to see that $X_{\bullet} \otimes_A X_{\bullet}$ is a resolution of A, we apply the Künneth Theorem: First note that the module $A \otimes A^{\mathrm{op}}$, under left multiplication by elements of A in the left factor, is a free left A-module since A^{op} is a free k-module. It follows that $X_{\bullet}$ is, by restriction, a projective resolution of the free left A-module A. So the boundaries are also all projective A-modules, that is the hypotheses of the Künneth Theorem hold. The Tor terms in the Künneth sequence vanish: The only term in which not both arguments are 0 is $\mathrm{Tor}_1^A(\mathrm{H}_0(X), \mathrm{H}_0(X)) = \mathrm{Tor}_1^A(A, A) = 0$ since A is free. This implies that $X_{\bullet} \otimes_A X_{\bullet}$ is indeed a resolution of $A \otimes_A A \cong A$ by A^e-projective modules.

By the Comparison Theorem there is a chain map $D : X_{\bullet} \to X_{\bullet} \otimes_A X_{\bullet}$, and the cup product on Hochschild cohomology may be defined by this map: Let $f \in \mathrm{Hom}_{A^e}(X_i, A)$, $g \in \mathrm{Hom}_{A^e}(X_j, A)$ represent elements of $\mathrm{HH}^i(A, A)$, $\mathrm{HH}^j(A, A)$. Then $f \smile g = (f \otimes g) \circ D$. Such a map D is unique up to chain homotopy. For any two resolutions, there is also a chain map between them, and so this definition of cup product does not depend on choices of $X_{\bullet}$ and D. If $X_{\bullet}$ is the bar resolution, one choice of chain map D induces precisely the chain level cup product as given above; see Sanada [14].

The embedding. We will consider A to be an A-module by the left adjoint action, that is if $a, b \in A$,

$$a \cdot b = \sum a_1 b S(a_2).$$

Denote this A-module by A^{ad}. More generally, if M is an A-bimodule, denote by M^{ad} the left A-module with action given by $a \cdot m = \sum a_1 m S(a_2)$ for all $a \in A$, $m \in M$.

Theorem 3.7. *Assume the antipode S is bijective. Then there is an isomorphism of algebras*

$$\mathrm{HH}^*(A, A) \cong \mathrm{H}^*(A, A^{ad}).$$

Proof. By Lemma 3.5, A^e is a flat right A-module, and so we may apply Lemma 3.2 and Lemma 3.3 with $B = A^e$, $M = k$, and $N = A$ to obtain $\mathrm{Ext}_A^*(k, A^{ad}) \cong \mathrm{Ext}_{A^e}^*(A, A)$ as k-modules. It remains to prove that the cup products are preserved by this isomorphism. This follows from the proof of [15, Proposition 3.1], valid more generally in this context. We provide some details for the sake of completeness.

Let $P_{\bullet}$ denote an A-projective resolution of k, so $X_{\bullet} = A^e \otimes_A P_{\bullet}$ is an A^e-projective resolution of $A^e \otimes_A k \cong A$.

There is an A-chain map $\iota : P_{\bullet} \to X_{\bullet}$ defined by $\iota(p) = (1 \otimes 1) \otimes p$ for all $p \in P_i$. Let $f \in \mathrm{Hom}_{A^e}(X_i, A)$ be a cocycle representing a cohomology class in

$\mathrm{Ext}^*_{A^e}(A, A)$. The corresponding cohomology class in $\mathrm{Ext}^*_A(k, A^{ad})$ is represented by $f \circ \iota$.

Let $D : P_\bullet \to P_\bullet \otimes P_\bullet$ be a chain map. Such a map exists and is unique up to homotopy by the Comparison Theorem. Since $k \otimes k \cong k$, the Künneth Theorem implies that $P_\bullet \otimes P_\bullet$ is also a projective resolution of k as an A-module, via similar arguments to those given before. Therefore D induces an isomorphism on cohomology. The map D also induces a chain map $D' : X_\bullet \to X_\bullet \otimes_A X_\bullet$ as follows. There is a map of A^e-chain complexes $\theta : A^e \otimes_A (P_\bullet \otimes P_\bullet) \to X_\bullet \otimes_A X_\bullet$, given by

$$\theta((a \otimes b) \otimes (p \otimes q)) = ((a \otimes 1) \otimes p) \otimes ((1 \otimes b) \otimes q).$$

Now D induces a map from $A^e \otimes_A P_\bullet$ to $A^e \otimes_A (P_\bullet \otimes P_\bullet)$. Let D' be the composition of this map with θ. Again D' is unique up to homotopy.

Now let $f \in \mathrm{Hom}_{A^e}(X_i, A)$, $g \in \mathrm{Hom}_{A^e}(X_j, A)$ be cocycles. The above observations imply the following diagram commutes:

$$\begin{array}{ccccccc}
X_\bullet & \xrightarrow{D'} & X_\bullet \otimes_A X_\bullet & \xrightarrow{f \otimes g} & A \otimes_A A & \xrightarrow{\sim} & A \\
\iota \uparrow & & & & & & || \\
P_\bullet & \xrightarrow{D} & P_\bullet \otimes P_\bullet & \xrightarrow{f\iota \otimes g\iota} & A \otimes A & \xrightarrow{m} & A
\end{array}$$

where m denotes multiplication. The top row yields the product in $\mathrm{Ext}^*_{A^e}(A, A)$ and the bottom row yields the product in $\mathrm{Ext}^*_A(k, A^{ad})$. □

Corollary 3.8. *Assume the antipode S is bijective. Then $\mathrm{H}^*(A, k)$ is an algebra direct summand of $\mathrm{HH}^*(A, A)$.*

By "algebra direct summand," we mean that $\mathrm{H}^*(A, k)$ is isomorphic to a direct sum of a subalgebra of $\mathrm{HH}^*(A, A)$ and an ideal, so that we may also view $\mathrm{H}^*(A, k)$ as a quotient of $\mathrm{HH}^*(A, A)$.

Proof. Under the left adjoint action of A on itself, the trivial module k is isomorphic to the submodule of A^{ad} given by all scalar multiples of the identity 1. In fact k is a direct summand of A^{ad}, its complement being the kernel of ε. As $\mathrm{Ext}^*_A(k, -)$ is additive, the result follows. □

Remark 3.9. Again there is a Tor version that is easier: Apply Remark 3.4 to obtain $\mathrm{HH}_i(A, A) \cong \mathrm{H}_i(A, A^{ad})$ as abelian groups. It follows that $\mathrm{H}_i(A, k)$ is a direct summand of $\mathrm{HH}_i(A, A)$.

Corollary 3.10. *Assume the antipode S is bijective. If $\mathrm{HH}^*(A, A)$ is finitely generated, then $\mathrm{H}^*(A, k)$ is finitely generated.*

Proof. By Corollary 3.8, $\mathrm{H}^*(A, k)$ may be expressed as a quotient of $\mathrm{HH}^*(A, A)$, so it is generated by the images of the generators of $\mathrm{HH}^*(A, A)$. □

Corollary 3.8 was the idea used by Ginzburg and Kumar in [7] to prove graded-commutativity of $\mathrm{H}^*(A,k)$ via graded-commutativity of $\mathrm{HH}^*(A,A)$. This is a different proof from the Eckmann-Hilton argument. Gerstenhaber [8, §7, Corollary 1] first proved graded-commutativity of the cup product on Hochschild cohomology; see also Sanada [14, Proposition 1.2].

There are many finite dimensional Hopf algebras A for which $\mathrm{H}^*(A,k)$ is known to be finitely generated: Golod [9], Venkov [18], and Evens [5] proved this for group algebras, Friedlander and Suslin [6] for cocommutative Hopf algebras, Ginzburg and Kumar [7] for small quantum groups, and there are some other known classes of examples such as in [12]. Etingof and Ostrik [4] have conjectured that $\mathrm{H}^*(A,k)$ is finitely generated for all finite dimensional Hopf algebras A.

Question. Is the converse of Corollary 3.10 true in general? It is known to be true in many cases, including that of a finite group algebra.

References

[1] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, **21**, Amer. Math. Soc., Providence, RI, 2001.

[2] D. J. Benson, Representations and Cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge University Press, 1991.

[3] D. J. Benson and S. Witherspoon, *Examples of support varieties for Hopf algebras with noncommutative tensor products*, in preparation.

[4] P. Etingof and V. Ostrik, *Finite tensor categories*, Mosc. Math. J. **4** (2004), no. 3, 627–654, 782–783

[5] L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239.

[6] E. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270.

[7] V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), 179–198.

[8] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267–288.

[9] E. Golod, *The cohomology ring of a finite p-group*, (Russian) Dokl. Akad. Nauk SSSR **235** (1959), 703–706.

[10] P. J. Hilton and U. Stammbach, A Course in Homological Algebra, Springer-Verlag, 1971.

[11] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Conf. Math. Publ. **82**, Amer. Math. Soc., 1993.

[12] M. Mastnak, J. Pevtsova, P. Schauenburg, and S. Witherspoon, *Cohomology of finite dimensional pointed Hopf algebras*, Proc. London Math. Soc. (3) **100** (2010), no. 2, 377–404.

[13] J. Pevtsova and S. Witherspoon, *Varieties for modules of quantum elementary abelian groups*, Algebras and Rep. Th. **12** (2009), no. 6, 567–595.

[14] K. Sanada, *On the Hochschild cohomology of crossed products*, Comm. Algebra **21** (1993), 2727–2748.

[15] S. F. Siegel and S. J. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. **79** (1999), 131-157.

[16] N. Snashall and Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705–732.
[17] M. Suarez-Alvarez, *The Hilton-Eckmann argument for the anti-commutativity of cup-products*, Proc. Amer. Math. Soc. **132** (8) (2004), 2241–2246.
[18] B. B. Venkov, *Cohomology algebras for some classifying spaces*, Dokl. Akad. Nauk. SSR **127** (1959), 943–944.
[19] C. Weibel, An Introduction to Homological Algebra, Cambridge University Press, 1994.
[20] Y. Zhu, *Hopf algebras of prime dimension*, Int. Math. Research Notes **1** (1994), 53–59.

Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: `sjw@math.tamu.edu`