HOCHSCHILD COHOMOLOGY AND LINCKELMANN COHOMOLOGY FOR BLOCKS OF FINITE GROUPS

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Abstract. Let G be a finite group, $\mathbb{F}$ an algebraically closed field of finite characteristic p, and let B be a block of $\mathbb{F}G$.

We show that the Hochschild and Linckelmann cohomology rings of B are isomorphic, modulo their radicals, in the cases where
(1) B is cyclic and
(2) B is arbitrary and G either a nilpotent group or a Frobenius group (p odd).
(The second case is a consequence of a more general result).

We give some related results in the more general case that B has a Sylow p-subgroup P as a defect group, giving a precise local description of a quotient of the Hochschild cohomology ring. In case P is elementary abelian, this quotient is isomorphic to the Linckelmann cohomology ring of B, modulo radicals.

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1. Introduction

Let G be a finite group and $\mathbb{F}$ an algebraically closed field of positive characteristic p dividing the order of G. Let B be a block of the group algebra $\mathbb{F}G$, that is an indecomposable ideal direct summand of $\mathbb{F}G$. In [14, 15], Linckelmann defines the cohomology ring $\mathrm{LH}^*(B)$ (our notation) of the block B of $\mathbb{F}G$ to be a subring of certain stable elements in the group cohomology ring $\mathrm{H}^*(P, \mathbb{F})$, where P is a defect group of B. (See Definition 2.1.) Linckelmann then defines an injective ring homomorphism γ from the block cohomology ring $\mathrm{LH}^*(B)$ to the Hochschild cohomology ring $\mathrm{HH}^*(B)$ of B [14].

We are interested in a better understanding of the map γ connecting these two cohomology rings. As $\mathrm{HH}^0(B)$ generally has dimension over $\mathbb{F}$ larger than one, γ is not in general an isomorphism. However, if we take the quotient of each ring by its (Jacobson) radical, we still have an injective ring homomorphism, which is now an isomorphism *in degree* 0. One is now led to the following question:

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When does Linckelmann's injection

$$\gamma : \mathrm{LH}^*(B) \to \mathrm{HH}^*(B)$$

induce an isomorphism $\overline{\gamma}$ *modulo radicals?*

We point out that as these cohomology rings are finitely generated graded commutative rings, we need only check that such an isomorphism exists, and Linckelmann's injection γ will then automatically induce an isomorphism $\overline{\gamma}$. It is known that $\mathrm{LH}^*(B)$ and $\mathrm{HH}^*(B)$ have the same Krull dimension, that of $\mathrm{H}^*(P, \mathbb{F})$, the rank of P ([15, Corollary 4.3(ii)] or [11, Theorem 4.4]).

If $B = B_0$ is the principal block, $\overline{\gamma}$ is known to be an isomorphism in the cases where G is a p-group, G is abelian, and a few other specific cases [21, Sections 10 and 11], as well as the case where B_0 is *cyclic*, that is its defect groups are cyclic [22, Theorem 3].

In §3, we extend these results to prove:

Theorem 3.1. *Let G be a group with normal Sylow p-subgroup P such that for any $k \in PC_G(P) - 1$, $C_G(k) \leq PC_G(P)$.*
(For example, G a nilpotent group or G a Frobenius group where p divides the order of the Frobenius kernel.)

Then for any block B of $\mathbb{F}G$, we have that $\mathrm{LH}^*(B)$ *and* $\mathrm{HH}^*(B)$ *are isomorphic modulo their radicals.*

As a consequence, we give an affirmative answer to the question for all cyclic blocks:

Corollary 3.5. *Let G be any finite group, and B any block of $\mathbb{F}G$ having a cyclic defect group. Then the Linckelmann cohomology ring* $\mathrm{LH}^*(B)$ *is isomorphic to the Hochschild cohomology ring* $\mathrm{HH}^*(B)$*, modulo radicals.*

We then give further examples: An analogous result is true for the principal blocks of A_5 and $SL_2(8)$. These examples use Theorem 3.1 and Broué's abelian defect conjecture, which is known to hold for these groups by work of Rickard and Rouquier. They also suggest a strategy for handling a larger class of examples.

In §4, we give some related results in case P is a Sylow p-subgroup of G (now not necessarily normal), and B any block of G with defect group P (e.g. the principal block). We study the quotient ring $\mathrm{HH}^*_P(B)$ of the Hochschild cohomology ring of B modulo the ideal of proper transfers (see Definition 4.1) after proving some general module-theoretic results about this cohomology quotient. We give the structure of this quotient $\mathrm{HH}^*_P(B)$ in terms of local information:

Theorem 4.2. *Let B be a block of $\mathbb{F}G$ with defect group the Sylow p-subgroup P of G. Let $K = PC_G(P)$, and b a block of $\mathbb{F}K$ such that B is the unique block covering b. Then*

$$\mathrm{HH}^*_P(B) \cong (\mathrm{H}^*_P(P, \mathbb{F}) \otimes \mathbb{F}Z(P))^{N_G(b)}.$$

In particular, when P is elementary abelian, this quotient $\mathrm{HH}^*_P(B)$ is isomorphic, modulo radicals, to Linckelmann cohomology (Corollary 4.10). We give further examples of blocks of symmetric groups of defect 2 (p odd). In this case, Hochschild cohomology and Linckelmann cohomology are again isomorphic, modulo their radicals. These examples use Theorem 3.1 and Chuang's proof of Broué's abelian defect conjecture for these blocks.

A possible application of our work, particularly if it may be extended to include larger classes of groups and/or blocks, is to the study of varieties for blocks. In [15], Linckelmann develops such a theory, where the variety associated to a block B is the maximal ideal spectrum of the block cohomology ring $\mathrm{LH}^*(B)$. Some unpublished work of Siegel [20] also gives a theory of varieties for blocks, where this time the variety associated to a block is the maximal ideal spectrum of its *Hochschild* cohomology ring $\mathrm{HH}^*(B)$. In cases where Linckelmann's block cohomology and the Hochschild cohomology of the block are isomorphic modulo their radicals (or more generally F-isomorphic), these two varieties associated to the block will be the same, and so both theories may potentially be exploited to obtain further information.

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2. Preliminary remarks

We will use a number of results from [3] on subpairs and their partial order.

Let B be any block of $\mathbb{F}G$. Let (P, B_P) be a Sylow B-subpair of G, unique up to conjugacy. In particular, P is a defect group of B and B_P is a block of $\mathbb{F}C_G(P)$. If R is any subgroup of P, then there exists a unique block B_R of $\mathbb{F}C_G(R)$ such that $(R, B_R) \leq (P, B_P)$. Let $N_G(B_R)$ be the subgroup of $N_G(R)$ fixing B_R setwise, under conjugation.

Definition 2.1 (Linckelmann)**.** Let B be any block of $\mathbb{F}G$ with defect group P. The *cohomology ring* of the block B of G is the subring $\mathrm{LH}^*(B)$ of $\mathrm{H}^*(P,\mathbb{F})$ consisting of all $[\zeta] \in \mathrm{H}^*(P,\mathbb{F})$ satisfying

$$^g\operatorname{res}^P_R([\zeta]) = \operatorname{res}^P_R([\zeta])$$

for any subgroup R of P, and any $g \in N_G(B_R)$.

Our definition is equivalent to that of Linckelmann: By [5, Theorem 1.8], B_R is precisely the block of $\mathbb{F}C_G(R)$ with block idempotent e_R of [14, Definition 5.1]. (See also [15, p. 468].) We suppress Linckelmann's pointed group P_γ in our notation, as we do not explicitly use pointed groups in our definition, and in any case the definition (up to isomorphism) does not depend on the choice of P_γ.

The next two remarks are due to Linckelmann [14, 15].

Remark 2.2. If $B = B_0$ is the principal block, with defect group a Sylow p-subgroup P of G, it is easy to see that $N_G(B_R) = N_G(R)$ for all $R \leq P$. Thus by the Alperin Fusion Theorem [1] and the standard description of stable elements [9, Corollary 4.2.7], it follows that

$$\mathrm{LH}^*(B_0) = \mathrm{H}^*(G,\mathbb{F}).$$

In this case, Linckelmann's injection $\gamma : \mathrm{LH}^*(B_0) \to \mathrm{H}^*(B_0, B_0)$ is the composition of the canonical injection $\mathrm{H}^*(G,\mathbb{F}) \hookrightarrow \mathrm{H}^*(\mathbb{F}G,\mathbb{F}G)$ with the canonical projection $\mathrm{H}^*(\mathbb{F}G,\mathbb{F}G) \twoheadrightarrow \mathrm{HH}^*(B_0)$.

Remark 2.3. If the defect group P of B is abelian, then the inertial quotient $E = N_G(B_P)/C_G(P)$ controls fusion [3, Proposition 4.2], and is in general a p'-group [8, Theorem 61.15]. Therefore

$$\mathrm{LH}^*(B) \cong \mathrm{H}^*(P,\mathbb{F})^E \cong \mathrm{H}^*(P \rtimes E,\mathbb{F}).$$

(Here the superscript E denotes fixed points.) If a block b of $N_G(P)$ is the Brauer correspondent of B, it follows that $\mathrm{LH}^*(B) \cong \mathrm{LH}^*(b)$, that is their block cohomology rings are isomorphic. It is not known whether their Hochschild cohomology rings $\mathrm{HH}^*(B)$ and $\mathrm{HH}^*(b)$ are isomorphic in this case. This would be a consequence of Broué's abelian defect conjecture, that B and b are derived equivalent. Broué's conjecture is known to hold in case P is cyclic [13, 16], as well as in a number of other cases.

Let us look at an example so that the reader may see a sample of how Linckelmann's injection induces an isomorphism modulo radicals.

Example 2.4. Let $G = S_3$, the symmetric group on three letters, and $p = 2$. The principal block B_0 of $\mathbb{F}S_3$ is isomorphic to $\mathbb{F}C_2$ (where C_2

denotes a cyclic group of order 2). As C_2 is abelian, [7, Theorem 2.1] or [21, Proposition 3.2] implies that

$$\mathrm{HH}^*(B_0) \cong \mathbb{F}C_2 \otimes_{\mathbb{F}} \mathrm{H}^*(C_2, \mathbb{F}).$$

On the other hand, by Remark 2.2, the Linckelmann cohomology of B_0 is

$$\mathrm{LH}^*(B_0) \cong \mathrm{H}^*(S_3, \mathbb{F}) \cong \mathrm{H}^*(C_2, \mathbb{F}).$$

Linckelmann's injection $\gamma : \mathrm{LH}^*(B_0) \to \mathrm{HH}^*(B_0)$ sends $\mathrm{H}^*(C_2, \mathbb{F})$ to $1 \otimes \mathrm{H}^*(C_2, \mathbb{F}) \subset \mathbb{F}C_2 \otimes \mathrm{H}^*(C_2, \mathbb{F})$ (see [15, Theorem 4.2(ii)]). As $\mathbb{F}C_2$ is a local ring, γ is indeed an isomorphism, modulo radicals.

3. Frobenius groups

In this section, we will assume the following: *G is a group with normal Sylow p-subgroup P, such that for any $k \in PC_G(P) - 1$, the centralizer $C_G(k)$ is contained in $PC_G(P)$.* In particular, this is true of any group G that is equal to $PC_G(P)$ for a Sylow p-subgroup P, for example any nilpotent group. It is also true of any Frobenius group $G = K \rtimes H$ with Frobenius kernel K and complement H, in which p divides $|K|$, by [10, Theorem 2.7.6(ii),(iv) and Theorem 10.3.1(iii)]. In this case, $K = PC_G(P)$.

The main aim of this section is to show that for any block B of such a group G, the Linckelmann and Hochschild cohomology rings of the block are isomorphic modulo radicals.

Theorem 3.1. *Let G be a group with normal Sylow p-subgroup P such that for any $k \in PC_G(P) - 1$, $C_G(k) \le PC_G(P)$. Let B be a block of $\mathbb{F}G$. Then $\mathrm{LH}^*(B)$ and $\mathrm{HH}^*(B)$ are isomorphic modulo their radicals.*

Let $K = PC_G(P)$, and $H = G/K$. First we will prove the following result on the structure of the Hochschild cohomology ring $\mathrm{HH}^*(\mathbb{F}G)$.

Lemma 3.2. *There is an additive decomposition*

$$\mathrm{HH}^*(\mathbb{F}G) \cong \mathrm{HH}^*(\mathbb{F}K)^G \oplus (\mathbb{F}(G-K))^G.$$

The first summand is a subalgebra, and the second is an ideal consisting of nilpotent elements.

Proof. Let $\Delta G = \{(g, g) \mid g \in G\}$. Consider $\mathbb{F}K$ as a module for the subgroup $(K \times K)\Delta G$ of $G \times G$, where the element (x, y) acts as left multiplication by x and right multiplication by y^{-1}. There is an isomorphism of $\mathbb{F}(G \times G)$-modules:

$$\mathbb{F}G \cong \mathbb{F}K \uparrow_{(K\times K)\Delta G}^{G\times G},$$

where the arrow denotes induction from the subgroup $(K \times K)\Delta G$. Thus by the Eckmann-Shapiro Lemma,

$$\mathrm{HH}^*(\mathbb{F}G) \cong \mathrm{Ext}^*_{(K\times K)\Delta G}(\mathbb{F}K, \mathbb{F}G). \tag{1}$$

As $K \times K$ is normal in $(K \times K)\Delta G$ of index prime to p, the latter is isomorphic to $(\mathrm{Ext}^*_{K\times K}(\mathbb{F}K, \mathbb{F}G))^G$ (see e.g. [4, Proposition 3.8.2]). The $\mathbb{F}(K \times K)$-module $\mathbb{F}G$ is the direct sum $\mathbb{F}K \oplus \mathbb{F}(G - K)$, each summand of which is invariant under the G-action, so we obtain the additive decomposition

$$\mathrm{HH}^*(\mathbb{F}G) \cong (\mathrm{Ext}^*_{K\times K}(\mathbb{F}K, \mathbb{F}K))^G \oplus (\mathrm{Ext}^*_{K\times K}(\mathbb{F}K, \mathbb{F}(G - K))^G. \tag{2}$$

The first term is $(\mathrm{HH}^*(\mathbb{F}K))^G$. As an $\mathbb{F}(K\times K)$-module, $\mathbb{F}K \cong \mathbb{F} \uparrow_{\Delta K}^{K\times K}$, so we may apply the Eckmann-Shapiro Lemma to the second term to obtain $(\mathrm{Ext}^*_K(\mathbb{F}, \mathbb{F}(G-K))^G$. The hypotheses imply that $\mathbb{F}(G-K)$ is a free $\mathbb{F}K$-module, so in fact the second term is $(\mathrm{Ext}^0_K(\mathbb{F}, \mathbb{F}(G - K))^G \cong (\mathbb{F}(G - K))^G$. We have therefore proven the first statement of the lemma.

Next we will consider the ring structure of Hochschild cohomology, which is induced by the ring structure of $\mathbb{F}G$ in (1). If $g \in G - K$, note that the sum of the elements in its conjugacy class may be written as $c_g \cdot \kappa$ for a sum c_g of group elements, where $\kappa = \sum_{k\in K} k$. If $g, h \in G-K$, then

$$(c_g\kappa)(c_h\kappa) = c_g c_h \kappa^2 = |K| c_g c_h \kappa = 0,$$

as K is normal in G and p divides $|K|$. This shows in particular that $(\mathbb{F}(G - K))^G$ consists of nilpotent elements.

Finally, the image of the multiplication map on $\mathbb{F}K \times \mathbb{F}(G - K)$ is $\mathbb{F}(G - K)$, so the second term in (2) is an ideal. Clearly the first is a subalgebra. □

Next we look at the structure of the cohomology ring $\mathrm{LH}^*(B)$.

Lemma 3.3. *Let G be a group with normal Sylow p-subgroup P and set $K = PC_G(P)$. Suppose that for any $k \in K - 1$, $C_G(k) \le K$. Let B be a block of $\mathbb{F}G$ and let b be a block of $\mathbb{F}K$ that is covered by B. Then* $\mathrm{LH}^*(B) \cong H^*(P, \mathbb{F})^{N_G(b)}$.

Proof. Since $K = P \times Q$ where $Q = O_{p'}(K)$, we may write $C_G(P) = Z(P) \times Q$. The blocks of $\mathbb{F}C_G(P)$ correspond bijectively with blocks of $\mathbb{F}Q$, and may be written $\mathbb{F}Z(P) \otimes b'$, where b' is a block of $\mathbb{F}Q$.

By [3, (2.9)(3)], the block idempotent E of B is just the trace of the block idempotent e of $\mathbb{F}Z(P) \otimes b'$, from $N_G(e)$ to G, where B covers $\mathbb{F}Z(P)\otimes b'$. Therefore we have $(1, B) \le (P, \mathbb{F}Z(P)\otimes b')$ by [3, Definition 3.2 and Theorem 3.4]. That is, we have shown that $(P, \mathbb{F}Z(P) \otimes b')$ is a Sylow B-subpair. We will write $B_P = \mathbb{F}Z(P) \otimes b'$.

As in fact the block idempotent of $\mathbb{F}Z(P) \otimes b'$ lies in $\mathbb{F}Q$, and the same is true for the block idempotent of $\mathbb{F}Z(R) \otimes b'$ for any $R \leq P$, $R \neq 1$, we now have $B_R = \mathbb{F}Z(R) \otimes b'$, with

$$(R, B_R) \leq (P, B_P).$$

Next we show that $N_G(B_R) \leq N_G(B_P) = N_G(b)$, where $b = \mathbb{F}P \otimes b'$. To see this, note that as P is normal in G, we also have that $Z(P)$ is normal in G. Thus the normalizer in G of $B_P = \mathbb{F}Z(P) \otimes b'$ is just the normalizer of b', which is the same as $N_G(b)$. Further, any element of $N_G(B_R)$ must normalize b'.

Applying the definition of $\mathrm{LH}^*(B)$, we now have that

$$\mathrm{LH}^*(B) \cong \mathrm{H}^*(P, \mathbb{F})^{N_G(b)}.$$

□

We are now ready to finish the proof of Theorem 3.1.

Proof of Theorem 3.1. Let B be a block of G, and b a block of K that is covered by B. By Lemma 3.3,

$$\mathrm{LH}^*(B) \cong \mathrm{H}^*(P, \mathbb{F})^{N_G(b)} \cong \mathrm{LH}^*(b).$$

It remains to show that the Hochschild cohomology ring $\mathrm{HH}^*(B)$ is isomorphic to $\mathrm{H}^*(P, \mathbb{F})^{N_G(b)}$, modulo radicals.

Let E and e be the primitive central idempotents of $\mathbb{F}G$ and $\mathbb{F}K$ corresponding to B and b, respectively. We have $\mathrm{HH}^*(B) \cong \mathrm{HH}^*(\mathbb{F}G)E$, where we identify E with an element of $\mathrm{HH}^0(\mathbb{F}G) \cong Z(\mathbb{F}G)$. Therefore by Lemma 3.2,

$$\mathrm{HH}^*(B) \cong (\mathrm{HH}^*(\mathbb{F}K)E)^G + (\mathbb{F}(G - K)E)^G.$$

Here we have used the fact that the G-invariant elements are the image of the trace map tr_K^G as $|G : K|$ is prime to p, and E is itself G-invariant. By [3, (2.9)(3)], we also have $E = \mathrm{tr}_{N_G(b)}^G(e)$, and so modulo radicals, $\mathrm{HH}^*(B) \cong \mathrm{HH}^*(b)^{N_G(b)}$. Now $K = P \times Q$, and the idempotent e may be considered to be an element of $\mathbb{F}Q$, as a block idempotent involves only p'-elements. Therefore $b = \mathbb{F}Ke \cong \mathbb{F}P \otimes_{\mathbb{F}} \mathbb{F}Qe$. As Q is a p'-group, $\mathbb{F}Qe$ is a matrix algebra, and so $\mathrm{HH}^*(b) \cong \mathrm{HH}^*(\mathbb{F}P)$. We thus have

$$\mathrm{HH}^*(B) \cong \mathrm{HH}^*(\mathbb{F}P)^{N_G(b)}.$$

By [21, Theorem 10.1], $\mathrm{HH}^*(\mathbb{F}P)$ is isomorphic to $\mathrm{H}^*(P, \mathbb{F})$, modulo radicals. As $N_G(b)/K$ is a p'-group, $N_G(b)/K$-invariant elements of the quotient by an ideal are the same as the quotient of the $N_G(b)/K$-invariant elements, and so $\mathrm{HH}^*(\mathbb{F}P)^{N_G(b)}$ is isomorphic to $\mathrm{H}^*(P, \mathbb{F})^{N_G(b)}$, modulo their radicals. □

As stated at the beginning of the section, any nilpotent group, or Frobenius group with p dividing the order of the Frobenius kernel, satisfies the hypotheses of Theorem 3.1. Thus we have the following corollary.

Corollary 3.4. *Let G be a nilpotent group, or a Frobenius group in which p divides the order of the Frobenius kernel. Let B be a block of $\mathbb{F}G$. Then $\mathrm{LH}^*(B)$ and $\mathrm{HH}^*(B)$ are isomorphic, modulo their radicals.* □

We further obtain the following general result about cyclic blocks.

Corollary 3.5. *Let G be any finite group, and B any block of $\mathbb{F}G$ having a cyclic defect group. Then the Linckelmann cohomology ring $\mathrm{LH}^*(B)$ is isomorphic to the Hochschild cohomology ring $\mathrm{HH}^*(B)$, modulo radicals.*

Proof. Suppose P is a defect group of B, and P is cyclic. Let E denote the inertial quotient, $e = |E|$, and $m = (|P| - 1)/e$. By Remark 2.3,

$$\mathrm{LH}^*(B) \cong \mathrm{H}^*(P \rtimes E, \mathbb{F}).$$

Note that by [8, Lemma 60.9], $\mathbb{F}(P \rtimes E)$ has only one block. This block has cyclic defect group P, and so is a Brauer tree algebra with inertial index e and multiplicity m (see [2, p. 123]).

Since B is also a Brauer tree algebra with e edges and multiplicity m (see [2]), B and $\mathbb{F}(P \rtimes E)$ are derived equivalent by [16, Theorem 4.2]. By [18, Proposition 2.5], derived equivalent algebras have isomorphic Hochschild cohomology rings, and so $\mathrm{HH}^*(B) \cong \mathrm{HH}^*(\mathbb{F}(P \rtimes E))$.

Finally, as $P \rtimes E$ satisfies the hypotheses of Theorem 3.1, we have

$$\mathrm{HH}^*(\mathbb{F}(P \rtimes E)) \cong \mathrm{LH}^*(\mathbb{F}(P \rtimes E)),$$

modulo radicals. Since $\mathrm{LH}^*(\mathbb{F}(P \rtimes E)) \cong \mathrm{H}^*(P \rtimes E, \mathbb{F})$, this shows $\mathrm{HH}^*(B) \cong \mathrm{LH}^*(B)$, modulo radicals, as desired. □

The case in which G is a Frobenius group and p divides the order of the Frobenius complement H is covered by Corollary 3.5 in all cases except when $p = 2$ and a Sylow 2-subgroup of H is generalized quaternion (otherwise a Sylow p-subgroup of G is cyclic, see [10, Theorem 10.3.1(iv)]). We do not know whether $\mathrm{LH}^*(B)$ and $\mathrm{HH}^*(B)$ are isomorphic, modulo their radicals, in this case.

Finally, as an application of Corollary 3.4, we give two further examples in which Linckelmann's injection $\gamma : \mathrm{LH}^*(B) \to \mathrm{HH}^*(B)$ is an isomorphism modulo radicals. These groups do *not* satisfy the hypotheses of this section; instead the corollary is applied to the normalizers of Sylow p-subgroups.

Example 3.6. Let $G = A_5$, the alternating group on five letters, and $p = 2$. Let $P = \langle (12)(34), (13)(24) \rangle$, a Sylow 2-subgroup, and $H = N_G(P) \cong P \rtimes \langle (123) \rangle \cong A_4$. Let b_0 be the principal block of H. As H is a Frobenius group, Corollary 3.4 implies that $\mathrm{LH}^*(b_0)$ and $\mathrm{HH}^*(b_0)$ are isomorphic modulo their radicals. By Remark 2.2 or 2.3, $\mathrm{LH}^*(b_0) \cong \mathrm{H}^*(H, \mathbb{F}) \cong \mathrm{H}^*(P, \mathbb{F})^H$.

On the other hand, the principal block B_0 of $\mathbb{F}G$ is Rickard equivalent to b_0 by [17] or [19, Example 1]. Therefore $\mathrm{HH}^*(B_0) \cong \mathrm{HH}^*(b_0)$. By Remark 2.2 and [9, Theorem 4.2.8],

$$\mathrm{LH}^*(B_0) \cong \mathrm{H}^*(G, \mathbb{F}) \cong \mathrm{H}^*(P, \mathbb{F})^H.$$

Considering the previous statements about the cohomology rings of b_0, we now have $\mathrm{LH}^*(B_0) \cong \mathrm{HH}^*(B_0)$, modulo radicals.

Example 3.7. Let $G = SL_2(8)$ and

$$P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_8 \right\}$$

the Sylow 2-subgroup. Then

$$H = N_G(P) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{F}_8, a \neq 0 \right\}$$

is a semidirect product of P with a cyclic group of order 7. Let b_0 be the principal block of H, and note that H is a Frobenius group with Frobenius kernel P. Therefore $\mathrm{LH}^*(b_0)$ and $\mathrm{HH}^*(b_0)$ are isomorphic modulo their radicals. Further, $\mathrm{LH}^*(b_0) \cong \mathrm{H}^*(H, \mathbb{F}) \cong \mathrm{H}^*(P, \mathbb{F})^H$.

Rouquier gives a Rickard equivalence between the principal block B_0 of $\mathbb{F}G$ and b_0 [19, Example 2], and so $\mathrm{HH}^*(B_0) \cong \mathrm{HH}^*(b_0)$. As before,

$$\mathrm{LH}^*(B_0) \cong \mathrm{H}^*(G, \mathbb{F}) \cong \mathrm{H}^*(P, \mathbb{F})^H,$$

and we now have $\mathrm{LH}^*(B_0) \cong \mathrm{HH}^*(B_0)$, modulo radicals.

4. Sylow defect blocks

Let P be a Sylow p-subgroup of G, and B a block of G with defect group P (e.g. the principal block). To obtain some results in this general case, we will consider quotients of cohomology rings by the ideals generated by the images of transfers from all p-subgroups *properly* contained in a Sylow p-subgroup. We will write $Q <_G P$ to indicate that Q is properly contained in a subgroup conjugate to P by an element of G.

We take a module-theoretic view of blocks, as in [2]. Thus we will first make some general statements about modules, and later specialize to blocks. We refer the reader to [2] or [4] for the necessary facts about relatively projective modules, Green correspondence, and Brauer correspondence.

Definition 4.1. *If M is an $\mathbb{F}G$-module, we let*

$$\mathrm{H}^*_P(G,M) := \mathrm{H}^*(G,M)/\sum_{Q<_G P} \mathrm{cor}_Q^G(\mathrm{H}^*(Q,M)).$$

If B is a block of G, we let

$$\mathrm{HH}^*_P(B) := \mathrm{H}^*_P(G,B), \text{ and similarly } \mathrm{HH}^*(\mathbb{F}G) := \mathrm{H}^*_P(G,\mathbb{F}G),$$

where the G-action on $\mathbb{F}G$ and B is by conjugation.

We will sometimes use the notation

$$\mathcal{X}_G = \{Q \leq G \mid Q <_G P\},$$

so that the sum in the definition is over all elements of $\mathcal{X}_G$.

Remark 4.2. Let Ω denote the Heller operator, that is ΩV is the kernel of the projective cover of a module V. Then

(i) $$\mathrm{H}^n_P(G,M) = \mathrm{Hom}_{\mathbb{F}G}(\Omega^n\mathbb{F}, M)/\sum_{Q<_G P} \mathrm{tr}_Q^G(\mathrm{Hom}_{\mathbb{F}Q}(\Omega^n\mathbb{F}, M)),$$

where $\mathrm{tr}_Q^G(f)(v) = \sum_{g\in G/Q} gf(g^{-1}v)$.

(ii) In particular, if M is a relatively $\mathcal{X}_G$-projective $\mathbb{F}G$-module, then $\mathrm{H}^*_P(G,M) = 0$ by Higman's criterion.

The goal of this section is to prove the following theorem and derive consequences for Linckelmann cohomology in the case P is elementary abelian.

Theorem 4.3. *Let B be a block of $\mathbb{F}G$ with defect group the Sylow p-subgroup P of G. Let $K = PC_G(P)$, and b a block of $\mathbb{F}K$ such that B is the unique block covering b. Then*

$$\mathrm{HH}^*_P(B) \cong (\mathrm{H}^*_P(P,\mathbb{F}) \otimes \mathbb{F}Z(P))^{N_G(b)}.$$

Before we can prove this theorem, we must develop some preliminary properties of the functor $\mathrm{H}^*_P(G,-)$. The proof of the following lemma is a straightforward consequence of the proof of the Eckmann-Shapiro Lemma.

Lemma 4.4. *Let H be a subgroup of G containing P. Let M be an $\mathbb{F}H$-module. Then*

$$\mathrm{H}^*_P(G, M\uparrow_H^G) \cong H^*_P(H,M). \quad \square$$

We next observe that H^*_P is determined locally:

Lemma 4.5. *Let L be any subgroup of G such that $L \geq N_G(P)$, and M an $\mathbb{F}G$-module. Then res^G_L induces an isomorphism*

$$\mathrm{H}^*_P(G, M) \cong \mathrm{H}^*_P(L, M_L).$$

If M is indecomposable, Q is a vertex of M, and V its Green correspondent, then

$$\mathrm{H}^*_P(G, M) \cong \begin{cases} \mathrm{H}^*_P(L, V), & \text{if } Q =_G P \\ 0, & \text{if } Q <_G P. \end{cases}$$

Proof. This follows from Remark 4.2(i) and [4, Theorem 3.12.2(v)]. (For relevant facts about Ω, see [2, Theorem 20.5 and 20.7].) Alternatively, the lemma follows directly from Remark 4.2(ii) and Lemma 4.4. □

The next lemma allows us further to express the H^*_P-cohomology groups as fixed points of cohomology groups of $\mathbb{F}P$-modules.

Lemma 4.6. *Let K, L be any subgroups of G such that $P \leq K \leq L \leq N_G(P)$, K is normal in L, and M an $\mathbb{F}L$-module. Then res^L_K induces an isomorphism*

$$\mathrm{H}^*_P(L, M) \cong \mathrm{H}^*_P(K, M_K)^L.$$

Proof. As P is the Sylow p-subgroup of L, K is normal in L, and K contains P, standard arguments show that res^L_K induces an isomorphism on cohomology, $\mathrm{H}^*(L, M) \cong \mathrm{H}^*(K, M_K)^L$. It may be checked that this induces the isomorphism in the lemma. □

We will next turn our attention to blocks of group algebras, first deriving a result for the group algebras themselves. Let $K = PC_G(P)$. As $K \leq N_G(P)$ and P is the Sylow p-subgroup of $N_G(P)$, the arguments of Section 3 show that $K \cong P \times Q$ for a p'-group $Q \cong C_G(P)/Z(P)$.

Lemma 4.7. *(i)* $\mathrm{HH}^*_P(\mathbb{F}G) \cong (\mathrm{H}^*_P(P, \mathbb{F}) \otimes \mathbb{F}C_G(P))^{N_G(P)}$.

(ii) $\mathrm{HH}^*_P(\mathbb{F}G)$ *is isomorphic to* $(\mathrm{H}^*_P(P, \mathbb{F}) \otimes \mathbb{F}Q)^{N_G(P)}$, *modulo radicals.*

Proof. By Lemmas 4.5 and 4.6, we have

$$\mathrm{H}^*_P(G, \mathbb{F}G) \cong \mathrm{H}^*_P(P, \mathbb{F}G)^{N_G(P)}.$$

Now write $\mathbb{F}G \cong \mathbb{F}K \oplus \mathbb{F}(G-K)$ as $\mathbb{F}P$-modules. If $g \in G - K$, then $C_P(g) \neq P$, so $\mathbb{F}(G-K)$ is relatively $\mathcal{X}_P$-projective as an $\mathbb{F}P$-module. By Remark 4.2(ii),

$$\mathrm{H}^*_P(P, \mathbb{F}G) \cong \mathrm{H}^*_P(P, \mathbb{F}K).$$

As $K \cong P \times Q$, and P acts trivially on Q by conjugation,

$$\mathrm{H}^*(P, \mathbb{F}K) \cong \mathrm{H}^*(P, \mathbb{F}P) \otimes \mathbb{F}Q$$

(see [9, p. 18]). By analyzing this isomorphism, we find that it factors to yield

$$\mathrm{H}^*_P(P, \mathbb{F}K) \cong \mathrm{H}^*_P(P, \mathbb{F}P) \otimes \mathbb{F}Q.$$

Taking fixed points, we now have

$$\mathrm{H}^*_P(G, \mathbb{F}G) \cong (\mathrm{H}^*_P(P, \mathbb{F}P) \otimes \mathbb{F}Q)^{N_G(P)},$$

which by [21, Theorem 10.2] yields statement (i) of the theorem.

Statement (ii) follows as P acts trivially on $\mathrm{H}^*_P(P, \mathbb{F}) \otimes \mathbb{F}C_G(P)$, $N_G(P)/P$ is a p'-group, and $\mathbb{F}Z(P)$ is a local ring. □

Now we will show how the H^*_P-cohomology of B is determined locally.

Lemma 4.8. *Let B be a block of $\mathbb{F}G$. If B has a defect group properly contained in P, then $\mathrm{HH}^*_P(B) = 0$. If P is a defect group of B and b is the Brauer correspondent of B in $\mathbb{F}N_G(P)$, then*

$$\mathrm{HH}^*_P(B) \cong \mathrm{HH}^*_P(b).$$

Proof. By Lemma 4.5, $\mathrm{H}^*_P(G, B) = 0$ if B has a defect group Q properly contained in P, since the vertices of its components as an $\mathbb{F}\Delta(G)$-module will all be contained in Q. Otherwise, Lemma 4.5 implies that $\mathrm{H}^*_P(G, B) \cong \mathrm{H}^*_P(N_G(P), B)$.

As b is the Brauer correspondent of B, we have $B \cong b \oplus T$ as $\mathbb{F}(N_G(P) \times N_G(P))$-modules, where T is a module that is relatively projective for the set

$$\{s(\Delta(P))s^{-1} \cap (N_G(P) \times N_G(P)) \mid s \in G \times G, s \notin N_G(P) \times N_G(P)\}.$$

Restricting to $\Delta(N_G(P))$, we have

$$B_{\Delta(N_G(P))} \cong b_{\Delta(N_G(P))} \oplus T_{\Delta(N_G(P))},$$

where by the Mackey formula, $T_{\Delta(N_G(P))}$ is relatively projective for the set

$$\{s(\Delta(P))s^{-1} \cap \Delta(N_G(P)) \mid s \in G \times G, s \notin N_G(P) \times N_G(P)\}.$$

The elements in this set are all properly contained in $\Delta(P)$. Therefore by Remark 4.2(ii), $\mathrm{H}^*_P(N_G(P), B) \cong \mathrm{H}^*_P(N_G(P), b)$. □

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. By Lemma 4.8, we may assume that P is normal in G. As $\mathbb{F}(K \times K)$-modules, B divides

$$\mathbb{F}G \cong \mathbb{F}K \oplus \mathbb{F}(G - K),$$

and by [2, Lemma 13.7(3)], no indecomposable summand of $\mathbb{F}(G - K)$ has a vertex containing $\Delta(P)$. Therefore all summands of $B_{K \times K}$ with vertex containing $\Delta(P)$ are in fact blocks of K. By [2, Theorem 15.1], these form a single G-conjugacy class of blocks of K. Restricting to $\Delta(K)$ and applying Mackey's formula and Remark 4.2(ii), we obtain

$$\mathrm{H}^*_P(K,B) \cong \bigoplus_{g \in G/N_G(b)} \mathrm{H}^*_P(K, {}^g b).$$

By Lemma 4.6 with $L = G$, restricting from G to K yields

$$\begin{aligned} \mathrm{H}^*_P(G,B) &\cong \mathrm{H}^*_P(K,B)^G \\ &\cong \bigoplus_{g \in G/N_G(b)} (\mathrm{H}^*_P(K, {}^g b))^G \\ &\cong \mathrm{H}^*_P(K,b)^{N_G(b)} \\ &\cong \mathrm{H}^*_P(P,b)^{N_G(b)}, \end{aligned}$$

by Lemma 4.6 again (with K taking the place of L, and P taking the place of K). As in §3, the block b of $\mathbb{F}K$ must have the form $b = \mathbb{F}P \otimes b'$, where b' is a block of the semisimple algebra $\mathbb{F}Q$. Therefore

$$\mathrm{H}^*(P,b) \cong \mathrm{H}^*(P, \mathbb{F}P) \otimes b'.$$

By analyzing this isomorphism as in Lemma 4.7, we find that it factors to yield an isomorphism

$$\mathrm{H}^*_P(P,b) \cong \mathrm{H}^*_P(P, \mathbb{F}P) \otimes b'.$$

By [21, Theorem 10.2], $\mathrm{H}^*_P(P, \mathbb{F}P) \cong \mathrm{H}^*_P(P, \mathbb{F}) \otimes \mathbb{F}Z(P)$, and so we have

$$\mathrm{H}^*_P(G,B) \cong (\mathrm{H}^*_P(P,\mathbb{F}) \otimes \mathbb{F}Z(P) \otimes b')^{N_G(b)}.$$

Now $Q \leq N_G(b)$ acts trivially on $\mathrm{H}^*_P(P,\mathbb{F}) \otimes \mathbb{F}Z(P)$, and $(b')^Q \cong \mathbb{F}$ as $(b')^Q$ is the center of the matrix algebra b'. □

As P acts trivially on $\mathrm{H}^*_P(P,\mathbb{F}) \otimes \mathbb{F}Z(P)$, and $\mathbb{F}Z(P)$ is a local ring, we have the following corollary.

Corollary 4.9. $\mathrm{HH}^*_P(B)$ *is isomorphic to* $\mathrm{H}^*_P(P,\mathbb{F})^{N_G(b)}$*, modulo radicals.*

In case P is elementary abelian, this result has the following consequence for the cohomology ring of the block B.

Corollary 4.10. *Let G be a finite group with elementary abelian Sylow p-subgroup P, and B a block of G with defect group P. Then* $\mathrm{LH}^*_P(B)$ *is isomorphic to* $\mathrm{HH}^*(B)$*, modulo radicals.*

Proof. As $\mathrm{cor}_Q^P : \mathrm{H}^*(Q,\mathbb{F}) \to \mathrm{H}^*(P,\mathbb{F})$ is 0 in case $Q < P$, Corollary 4.9 implies that $\mathrm{HH}^*_P(B)$ is isomorphic to $\mathrm{H}^*(P,\mathbb{F})^{N_G(b)}$, modulo radicals. By Remark 2.3, this is precisely $\mathrm{LH}^*(B)$. □

We conclude with an example suggesting that Corollary 4.10 may be strengthened in some cases to yield an isomorphism, modulo radicals, between $\mathrm{HH}^*(B)$ and $\mathrm{LH}^*(B)$.

Example 4.11. Let $p > 2$, $t < p$, and $G = S_{t+2p}$, the symmetric group on $t+2p$ letters. See [12] for the block theory of G. A Sylow p-subgroup P of G is elementary abelian of rank 2. The normalizer of P is

$$N = N_G(P) \cong ((C_p \rtimes C_{p-1}) \wr C_2) \times S_t,$$

where C_i denotes a cyclic group of order i. Let B be a block of G with defect group P. By [6, Corollary 3.2], B is derived equivalent to its Brauer correspondent b in $N_G(P)$. Further, Remark 2.3 applies as P is abelian, and so

$$\mathrm{HH}^*(B) \cong \mathrm{HH}^*(b) \quad \text{and} \quad \mathrm{LH}^*(B) \cong \mathrm{LH}^*(b).$$

Now b is Morita equivalent to the principal block b' of

$$N' = (C_p \rtimes C_{p-1}) \wr C_2$$

(see the introduction of [6]). Therefore $\mathrm{HH}^*(b) \cong \mathrm{HH}^*(b')$. Another application of Remark 2.3 shows that

$$\mathrm{LH}^*(b) \cong \mathrm{H}^*(N, \mathbb{F}) \cong \mathrm{H}^*(N', \mathbb{F}) \cong \mathrm{LH}^*(b'),$$

as $C_N(P) = P = C_{N'}(P)$ implies $b_p = \mathbb{F}P = b'_P$ and $N = N' \times S_t$ with S_t a p'-group.

Finally, N' satisfies the hypotheses of Theorem 3.1, so that $\mathrm{HH}^*(b')$ and $\mathrm{LH}^*(b')$ are isomorphic modulo their radicals. Combining all of the above isomorphisms, we now have that $\mathrm{LH}^*(B)$ is isomorphic to $\mathrm{HH}^*(B)$, modulo their radicals.

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