

REPRESENTATIONS OF TWO-PARAMETER QUANTUM GROUPS AND SCHUR-WEYL DUALITY

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ABSTRACT. We determine the finite-dimensional simple modules for two-parameter quantum groups corresponding to the general linear and special linear Lie algebras \mathfrak{gl}_n and \mathfrak{sl}_n and present a complete reducibility result. These quantum groups have a natural n -dimensional module V . We prove an analogue of Schur-Weyl duality in this setting: the centralizer algebra of the quantum group action on the k -fold tensor power of V is a quotient of a Hecke algebra for all n and is isomorphic to the Hecke algebra in case $n \geq k$.

INTRODUCTION

Two-parameter general linear and special linear quantum groups were introduced by Takeuchi [T] in 1990. Our interest in these quantum groups arose from our investigations [BW1] of down-up algebras and their embeddings into certain Hopf algebras. These Hopf algebras depend on two parameters r and s , and the Drinfel'd double of such a Hopf algebra is essentially the two-parameter quantum group $U_{r,s}(\mathfrak{sl}_3)$ of Takeuchi (defined below). More generally, as shown in [BW2], $U_{r,s}(\mathfrak{sl}_n)$ is a Drinfel'd double of a Borel-type subalgebra, and there is an R -matrix which comes from the double construction and which reduces to the standard R -matrix for the one-parameter quantum group $U_q(\mathfrak{sl}_n)$ (a quotient of $U_{q,q^{-1}}(\mathfrak{sl}_n)$). In the analogous quantum function algebra setting, allowing two parameters unifies the Drinfel'd-Jimbo quantum groups ($r = q, s = q^{-1}$) [D, Ji1] with the Dipper-Donkin quantum groups ($r = 1, s = q^{-1}$) [DD].

In this work we study the representations of the two-parameter quantum groups $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ and $U = U_{r,s}(\mathfrak{sl}_n)$, defined in Section 1. Our Hopf algebra \tilde{U} is isomorphic as an algebra to Takeuchi's quantum group $U_{r,s^{-1}}$ (see [T]), but as a Hopf algebra, it has the opposite coproduct. Our main goal is to prove a two-parameter analogue of Schur-Weyl duality, for which we need a result on complete reducibility of modules. To this end, in Sections 2 and 3 we adapt the methods of [Ja] and [L] to classify the finite-dimensional simple \tilde{U} -modules when rs^{-1} is

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not a root of unity. We use a quantum Casimir operator to prove that all finite-dimensional \tilde{U} -modules on which \tilde{U}^0 (the subalgebra generated by the grouplike elements) acts semisimply are completely reducible. These results hold equally well for U . The hypothesis on \tilde{U}^0 is necessary: In Remark 3.9, we present examples of finite-dimensional modules that are not completely reducible. These examples exist because the two-parameter quantum groups contain more grouplike elements than their one-parameter counterparts.

The construction of the R -matrix, which provides an isomorphism $R_{M',M} : M' \otimes M \rightarrow M \otimes M'$ for any two \tilde{U} -modules M, M' in category \mathcal{O} (defined in Section 3), is summarized in Section 4. On the tensor power $V^{\otimes k}$ of the natural module V (which is described in Section 1), the transformations $R_i = \text{Id}^{\otimes(i-1)} \otimes R_{V,V} \otimes \text{Id}^{\otimes(k-i-1)}$ ($1 \leq i < k$) commute with the action of \tilde{U} , and so they generate a subalgebra of $\text{End}_{\tilde{U}}(V^{\otimes k})$. This yields a map from the two-parameter Hecke algebra $H_k(r, s)$ to $\text{End}_{\tilde{U}}(V^{\otimes k})$. In the final section we prove a two-parameter analogue of Schur-Weyl duality: The transformations R_i generate the centralizer algebra $\text{End}_{\tilde{U}}(V^{\otimes k})$, and in case $n \geq k$, this centralizer algebra is isomorphic to $H_k(r, s)$. The proof of this result is elementary, relying only on basic facts about representations and explicit computations and is new in the one-parameter case as well (compare [DPS, Du, Ji2, KT, LR]). The proof in the $n \geq k$ case is similar to one for classical (nonquantum) Schur-Weyl duality given by De Concini and Procesi [DP]. It is a consequence of our result, Lemma 6.2 below, that $V^{\otimes k}$ is a cyclic \tilde{U} -module in this case.

Throughout we will work over an algebraically closed field \mathbb{K} .

§1. PRELIMINARIES

First we recall the definitions of the two-parameter quantum groups from [BW2], and some basics about their representations. Let $\epsilon_1, \dots, \epsilon_n$ denote an orthonormal basis of a Euclidean space E with an inner product $\langle \cdot, \cdot \rangle$. Let $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \dots, n-1\}$ and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$. Then Φ is a finite root system of type A_{n-1} with Π a base of simple roots.

Fix nonzero elements r, s in \mathbb{K} with $r \neq s$.

Let $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ be the unital associative algebra over \mathbb{K} generated by elements e_j, f_j , ($1 \leq j < n$), and $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$), which satisfy the following relations.

$$(R1) \quad \text{The } a_i^{\pm 1}, b_j^{\pm 1} \text{ all commute with one another and } a_i a_i^{-1} = b_j b_j^{-1} = 1,$$

$$(R2) \quad a_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} e_j a_i \quad \text{and} \quad a_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} f_j a_i,$$

$$(R3) \quad b_i e_j = s^{\langle \epsilon_i, \alpha_j \rangle} e_j b_i \quad \text{and} \quad b_i f_j = s^{-\langle \epsilon_i, \alpha_j \rangle} f_j b_i,$$

$$(R4) \quad [e_i, f_j] = \frac{\delta_{i,j}}{r-s} (a_i b_{i+1} - a_{i+1} b_i),$$

$$(R5) \quad [e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } |i-j| > 1,$$

$$(R6) \quad e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + r s e_{i+1} e_i^2 = 0,$$

$$e_i e_{i+1}^2 - (r+s) e_{i+1} e_i e_{i+1} + r s e_{i+1}^2 e_i = 0,$$

$$(R7) \quad f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0,$$

$$f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_{i+1}^2 f_i = 0.$$

We will be interested in the subalgebra $U = U_{r,s}(\mathfrak{sl}_n)$ of $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ generated by the elements e_j, f_j, ω_j , and ω'_j ($1 \leq j < n$), where

$$(1.1) \quad \omega_j = a_j b_{j+1} \quad \text{and} \quad \omega'_j = a_{j+1} b_j.$$

These elements satisfy (R5)-(R7) along with the following relations:

$$(R1') \quad \text{The } \omega_i^{\pm 1}, \omega_j^{\pm 1} \text{ all commute with one another and } \omega_i \omega_i^{-1} = \omega'_j (\omega'_j)^{-1} = 1,$$

$$(R2') \quad \omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i \quad \text{and} \quad \omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i,$$

$$(R3') \quad \omega'_i e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega'_i \quad \text{and} \quad \omega'_i f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega'_i,$$

$$(R4') \quad [e_i, f_j] = \frac{\delta_{i,j}}{r-s} (\omega_i - \omega'_i).$$

When $r = q$ and $s = q^{-1}$, the algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by the elements $b_i - a_i^{-1}$, $1 \leq i \leq n$, is just the quantum general linear group $U_q(\mathfrak{gl}_n)$, and $U_{r,s}(\mathfrak{sl}_n)$ modulo the ideal generated by the elements $\omega'_j - \omega_j^{-1}$, $1 \leq j < n$, is $U_q(\mathfrak{sl}_n)$.

The algebras \tilde{U} and U are Hopf algebras, where the $a_i^{\pm 1}, b_i^{\pm 1}$ are group-like elements, and the remaining Hopf structure is given by

$$(1.2) \quad \begin{aligned} \Delta(e_i) &= e_i \otimes 1 + \omega_i \otimes e_i, & \Delta(f_i) &= 1 \otimes f_i + f_i \otimes \omega'_i, \\ \varepsilon(e_i) &= \varepsilon(f_i) = 0, & S(e_i) &= -\omega_i^{-1} e_i, & S(f_i) &= -f_i (\omega'_i)^{-1}. \end{aligned}$$

Let $\Lambda = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, the weight lattice of \mathfrak{gl}_n , and $Q = \mathbb{Z}\Phi$ the root lattice. We assume Λ is equipped with the partial order in which $\nu \leq \lambda$ if and only if $\lambda - \nu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$. Corresponding to $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra \tilde{U}^0 of \tilde{U} generated by the elements $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$) to \mathbb{K} given by

$$(1.3) \quad \hat{\lambda}(a_i) = r^{\langle \epsilon_i, \lambda \rangle} \quad \text{and} \quad \hat{\lambda}(b_i) = s^{\langle \epsilon_i, \lambda \rangle}.$$

The restriction $\hat{\lambda} : U^0 \rightarrow \mathbb{K}$ of $\hat{\lambda}$ to the subalgebra U^0 of U generated by $\omega_j^{\pm 1}, (\omega'_j)^{\pm 1}$ ($1 \leq j < n$) satisfies

$$(1.4) \quad \hat{\lambda}(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle} \quad \text{and} \quad \hat{\lambda}(\omega'_j) = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle}.$$

Similarly for $U = U_{r,s}(\mathfrak{sl}_n)$, we let $\Lambda_{\mathfrak{sl}} = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_{n-1}$, the weight lattice of \mathfrak{sl}_n , where ϖ_i is the fundamental weight

$$\varpi_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{n} \sum_{j=1}^n \epsilon_j.$$

If we fix n th roots $r^{1/n}$ and $s^{1/n}$ of r and s , respectively, then we may define algebra homomorphisms $\hat{\lambda} : U^0 \rightarrow \mathbb{K}$ by (1.4) for any $\lambda \in \Lambda_{\mathfrak{sl}}$.

Let M be a module for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ of dimension $d < \infty$. As \mathbb{K} is algebraically closed, we have

$$M = \bigoplus_{\chi} M_{\chi},$$

where each $\chi : \tilde{U}^0 \rightarrow \mathbb{K}$ is an algebra homomorphism, and M_{χ} is the generalized eigenspace given by

$$(1.5) \quad M_{\chi} = \{m \in M \mid (a_i - \chi(a_i)1)^d m = 0 = (b_i - \chi(b_i)1)^d m, \text{ for all } i\}.$$

When $M_{\chi} \neq 0$ we say that χ is a *weight* and M_{χ} is the corresponding *weight space*. (If M decomposes into genuine eigenspaces relative to \tilde{U}^0 (resp. U^0), then we say that \tilde{U}^0 (resp. U^0) *acts semisimply on M* .)

From relations (R2) and (R3) we deduce that

$$(1.6) \quad \begin{aligned} e_j M_{\chi} &\subseteq M_{\chi \cdot \widehat{\alpha}_j} \\ f_j M_{\chi} &\subseteq M_{\chi \cdot (-\widehat{\alpha}_j)}, \end{aligned}$$

where $\widehat{\alpha}_j$ is as in (1.3), and $\chi \cdot \psi$ is the homomorphism with values $(\chi \cdot \psi)(a_i) = \chi(a_i)\psi(a_i)$ and $(\chi \cdot \psi)(b_i) = \chi(b_i)\psi(b_i)$. In fact, if $(a_i - \chi(a_i)1)^k m = 0$, then applying relation (R2) yields $(a_i - \chi(a_i)r^{(\epsilon_i, \alpha_j)}1)^k e_j m = 0$, and similarly for b_i and for f_j . Therefore, the sum of eigenspaces is a submodule of M , and if M is simple, this sum must be M itself. Thus in (1.5), we may replace the power d by 1 whenever M is simple, and \tilde{U}^0 must act semisimply in this case. We also can see from (1.6) that for each simple M there is a homomorphism χ so that all the weights of M are of the form $\chi \cdot \hat{\zeta}$, where $\zeta \in Q$.

It is shown in [BW2, Prop. 3.5] that if $\hat{\zeta} = \hat{\eta}$, then $\zeta = \eta$ ($\zeta, \eta \in \Lambda$) provided rs^{-1} is not a root of unity. As a result, we have the following proposition.

Proposition 1.7. [BW2, Cor. 3.14] *Let M be a finite-dimensional module for $U_{r,s}(\mathfrak{sl}_n)$ or for $U_{r,s}(\mathfrak{gl}_n)$. If rs^{-1} is not a root of unity, then the elements e_i, f_i ($1 \leq i < n$) act nilpotently on M .*

When rs^{-1} is not a root of unity, a finite-dimensional simple module M is a *highest weight* module by Proposition 1.7 and (1.6). Thus there is some weight ψ and a nonzero vector $v_0 \in M_{\psi}$ such that $e_j v_0 = 0$ for all $j = 1, \dots, n-1$, and $M = \tilde{U} \cdot v_0$. It follows from the defining relations that \tilde{U} has a triangular decomposition: $\tilde{U} = U^- \tilde{U}^0 U^+$, where U^+ (resp., U^-) is the subalgebra generated by the elements e_i (resp., f_i). Applying this decomposition to v_0 , we see that $M = \bigoplus_{\zeta \in Q^+} M_{\psi \cdot (-\widehat{\zeta})}$, where $Q^+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$.

When all the weights of a module M are of the form $\hat{\lambda}$, where $\lambda \in \Lambda$, then for brevity we say that M has weights in Λ . Rather than writing $M_{\hat{\lambda}}$ for the weight

space, we simplify the notation by writing M_λ . Note then (1.6) can be rewritten as $e_j M_\lambda \subseteq M_{\lambda+\alpha_j}$ and $f_j M_\lambda \subseteq M_{\lambda-\alpha_j}$. Any simple \tilde{U} -module having one weight in Λ has all its weights in Λ .

Next we give an example of a simple \tilde{U} -module with weights in Λ , which is the analogue of the natural representation for \mathfrak{gl}_n .

The natural representation for $U_{r,s}(\mathfrak{gl}_n)$ and $U_{r,s}(\mathfrak{sl}_n)$.

Consider an n -dimensional vector space V over \mathbb{K} with basis $\{v_j \mid 1 \leq j \leq n\}$. We define an action of the generators of $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ by specifying their matrices relative to this basis:

$$\begin{aligned} e_j &= E_{j,j+1}, & f_j &= E_{j+1,j}, & (1 \leq j < n) \\ a_i &= rE_{i,i} + \sum_{k \neq i} E_{k,k}, & & & (1 \leq i \leq n) \\ b_i &= sE_{i,i} + \sum_{k \neq i} E_{k,k}, & & & (1 \leq i \leq n). \end{aligned}$$

It follows that $\omega_j = a_j b_{j+1} = rE_{j,j} + sE_{j+1,j+1} + \sum_{k \neq j,j+1} E_{k,k}$ and $\omega'_j = a_{j+1} b_j = sE_{j,j} + rE_{j+1,j+1} + \sum_{k \neq j,j+1} E_{k,k}$. It may be verified that this extends to an action of \tilde{U} (hence of $U = U_{r,s}(\mathfrak{sl}_n)$); that is, relations (R1)–(R7) hold.

It follows from the fact that $a_i v_j = r^{\langle \epsilon_i, \epsilon_j \rangle} v_j$ and $b_i v_j = s^{\langle \epsilon_i, \epsilon_j \rangle} v_j$ for all i, j that v_j corresponds to the weight $\epsilon_j = \epsilon_1 - (\alpha_1 + \cdots + \alpha_{j-1})$. Thus, $V = \bigoplus_{j=1}^n V_{\epsilon_j}$ is the natural analogue of the n -dimensional representation of \mathfrak{gl}_n and \mathfrak{sl}_n , and it is a simple module for both \tilde{U} and U . When $r = q$ and $s = q^{-1}$, b_i acts as a_i^{-1} on V , and so V is a module for the quotient $U_q(\mathfrak{gl}_n)$ of $U_{q,q^{-1}}(\mathfrak{gl}_n)$ by the ideal generated by $b_i - a_i^{-1}$ ($1 \leq i \leq n$). This is the natural module for the one-parameter quantum group $U_q(\mathfrak{gl}_n)$, and an analogous statement is true for $U_q(\mathfrak{sl}_n)$.

§2. CLASSIFICATION OF FINITE-DIMENSIONAL SIMPLE MODULES

Often results will be stated only for \tilde{U} -modules, but generally everything holds as well for U -modules. We will indicate where there are differences in the theory.

Let $\tilde{U}^{\geq 0}$ denote the subalgebra of \tilde{U} generated by a_i, b_i ($1 \leq i \leq n$) and e_i ($1 \leq i < n$). Let ψ be any algebra homomorphism from \tilde{U}^0 to \mathbb{K} and V^ψ be the one-dimensional $\tilde{U}^{\geq 0}$ -module on which e_i acts as multiplication by 0 ($1 \leq i < n$), and \tilde{U}^0 acts via ψ . We define the *Verma module* $M(\psi)$ with highest weight ψ to be the \tilde{U} -module induced from V^ψ , that is

$$M(\psi) = \tilde{U} \otimes_{\tilde{U}^{\geq 0}} V^\psi.$$

Let $v_\psi = 1 \otimes v \in M(\psi)$, where v is any nonzero vector of V^ψ . Then $e_i \cdot v_\psi = 0$ ($1 \leq i < n$) and $a \cdot v_\psi = \psi(a)v_\psi$ for any $a \in \tilde{U}^0$ by construction.

Notice that \tilde{U}^0 acts semisimply on $M(\psi)$ by relations (R2) and (R3). If N is a \tilde{U} -submodule of $M(\psi)$, then N is also a \tilde{U}^0 -submodule of the \tilde{U}^0 -module $M(\psi)$,

and so \tilde{U}^0 acts semisimply on N as well. If N is a *proper* submodule, it must be that $N \subset \sum_{\mu \in \tilde{Q}^+ \setminus \{0\}} M(\psi)_{\psi, (\tilde{-}\mu)}$ by (1.6), as $M(\psi)_\psi = \mathbb{K}v_\psi$ generates $M(\psi)$. Therefore $M(\psi)$ has a unique maximal submodule, namely the sum of all proper submodules, and a unique simple quotient, $L(\psi)$. In fact, all finite-dimensional simple \tilde{U} -modules are of this form, as the following theorem demonstrates.

Theorem 2.1. *Let $\psi : \tilde{U}^0 \rightarrow \mathbb{K}$ be an algebra homomorphism. Let M be a \tilde{U} -module, on which \tilde{U}^0 acts semisimply and which contains a nonzero element $m \in M_\psi$ such that $e_i m = 0$ for all i ($1 \leq i < n$). Then there is a unique homomorphism of \tilde{U} -modules $F : M(\psi) \rightarrow M$ with $F(v_\psi) = m$. In particular, if rs^{-1} is not a root of unity and M is a finite-dimensional simple \tilde{U} -module, then $M \cong L(\psi)$ for some weight ψ .*

Proof. By the hypothesis on m , $\mathbb{K}m$ is a one-dimensional $\tilde{U}^{\geq 0}$ -submodule of M , considered as a $\tilde{U}^{\geq 0}$ -module by restriction. In fact, mapping v_ψ to m yields a $\tilde{U}^{\geq 0}$ -homomorphism from V^ψ to $\mathbb{K}m$. By the definition of $M(\psi)$, we have $\text{Hom}_{\tilde{U}}(M(\psi), M) \cong \text{Hom}_{\tilde{U}^{\geq 0}}(V^\psi, M)$, so there is a unique \tilde{U} -module homomorphism $F : M(\psi) \rightarrow M$ with $F(v_\psi) = m$, namely $F(u \otimes v) = u.m$ for all $u \in \tilde{U}$.

For the final assertion, note that \tilde{U}^0 acts semisimply on any finite-dimensional simple module M , and by (1.6) and Proposition 1.7, there is some nonzero vector $m \in M_\psi$ such that $e_i m = 0$ ($1 \leq i < n$). By the first part, M is a quotient of $M(\psi)$, and so $M \cong L(\psi)$, as $L(\psi)$ is the unique simple quotient of $M(\psi)$. \square

As a special case, we will consider the modules $L(\lambda) = L(\hat{\lambda})$ where $\lambda \in \Lambda$. Let $\Lambda^+ \subset \Lambda$ be the subset of *dominant* weights, that is

$$\Lambda^+ = \{\lambda \in \Lambda \mid \langle \alpha_i, \lambda \rangle \geq 0 \text{ for } 1 \leq i < n\}.$$

Similarly, the set of dominant weights for \mathfrak{sl}_n is

$$\Lambda_{\mathfrak{sl}}^+ = \{\lambda \in \Lambda_{\mathfrak{sl}} \mid \langle \alpha_i, \lambda \rangle \geq 0 \text{ for } 1 \leq i < n\} = \left\{ \sum_{i=1}^{n-1} \ell_i \varpi_i \mid \ell_i \in \mathbb{Z}_{\geq 0} \right\}.$$

We will show that if $L(\lambda)$ is finite-dimensional, then λ is dominant. This requires an identity for commuting e_i past powers of f_i . For $k \geq 1$, let

$$(2.2) \quad [k] = \frac{r^k - s^k}{r - s}.$$

Then the following lemma may be proven by induction.

Lemma 2.3. *If $k \geq 1$, then*

$$\begin{aligned} e_i f_i^k &= f_i^k e_i + [k] f_i^{k-1} \frac{r^{1-k} \omega_i - s^{1-k} \omega'_i}{r - s} \\ e_i^k f_i &= f_i e_i^k + [k] e_i^{k-1} \frac{s^{1-k} \omega_i - r^{1-k} \omega'_i}{r - s}. \end{aligned}$$

Lemma 2.4. *Assume rs^{-1} is not a root of unity. Let M be a nonzero finite-dimensional module for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ on which \tilde{U}^0 acts semisimply, and let $\lambda \in \Lambda$. Suppose there is some nonzero vector $v \in M_\lambda$ with $e_i.v = 0$ for all i ($1 \leq i < n$). Then $\lambda \in \Lambda^+$. A similar statement is true for $U = U_{r,s}(\mathfrak{sl}_n)$ with Λ replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda_{\mathfrak{sl}}^+$.*

Proof. Proposition 1.7 implies that for any given value of i there is some $k \geq 0$ such that $f_i^{k+1}.v = 0$ and $f_i^k.v \neq 0$. Applying e_i to $f_i^{k+1}.v = 0$ and using Lemma 2.3 and the fact that $e_i.v = 0$, we have

$$0 = [k+1]f_i^k \frac{r^{-k}\omega_i - s^{-k}\omega'_i}{r-s}.v = \frac{[k+1]}{r-s} (r^{-k}\hat{\lambda}(\omega_i) - s^{-k}\hat{\lambda}(\omega'_i))f_i^k.v.$$

Now $[k+1]/(r-s) \neq 0$ as rs^{-1} is not a root of unity. Therefore, since $f_i^k.v \neq 0$, we have $r^{-k}\hat{\lambda}(\omega_i) = s^{-k}\hat{\lambda}(\omega'_i)$. Equivalently,

$$r^{-k}r^{\langle \epsilon_i, \lambda \rangle} s^{\langle \epsilon_{i+1}, \lambda \rangle} = s^{-k}r^{\langle \epsilon_{i+1}, \lambda \rangle} s^{\langle \epsilon_i, \lambda \rangle}, \quad \text{or} \quad r^{-k+\langle \alpha_i, \lambda \rangle} = s^{-k+\langle \alpha_i, \lambda \rangle}.$$

Again, because rs^{-1} is not a root of unity, this forces $\langle \alpha_i, \lambda \rangle = k \geq 0$, so $\lambda \in \Lambda^+$. \square

Corollary 2.5. *When rs^{-1} is not a root of unity, any finite-dimensional simple \tilde{U} -module with weights in Λ is isomorphic to $L(\lambda)$ for some $\lambda \in \Lambda^+$. An analogous result holds for U with Λ replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda_{\mathfrak{sl}}^+$.*

Next we will show that all modules $L(\lambda)$ with $\lambda \in \Lambda^+$ are indeed finite-dimensional, and that all other finite-dimensional simple \tilde{U} -modules are shifts of these by one-dimensional modules. In doing this, it helps to consider first the special case of simple $U_{r,s}(\mathfrak{sl}_2)$ -modules.

Highest weight modules for $U = U_{r,s}(\mathfrak{sl}_2)$.

For simplicity we drop the subscripts and just write e, f, ω, ω' for the generators of $U = U_{r,s}(\mathfrak{sl}_2)$. Any homomorphism $\phi : U^0 \rightarrow \mathbb{K}$ is determined by its values on ω and ω' . By abuse of notation, we adopt the shorthand $\phi = \phi(\omega)$ and $\phi' = \phi(\omega')$.

Corresponding to each such ϕ , there is a Verma module $M(\phi) = U \otimes_{U_{\geq 0}} \mathbb{K}v$ with basis $v_j = f^j \otimes v$ ($0 \leq j < \infty$) such that the U -action is given by:

$$(2.6) \quad \begin{aligned} f.v_j &= v_{j+1} \\ e.v_j &= [j] \frac{\phi r^{-j+1} - \phi' s^{-j+1}}{r-s} v_{j-1} \quad (v_{-1} := 0) \\ \omega.v_j &= \phi r^{-j\langle \epsilon_1, \alpha_1 \rangle} s^{-j\langle \epsilon_2, \alpha_1 \rangle} v_j = \phi r^{-j} s^j v_j \\ \omega'.v_j &= \phi' r^{-j\langle \epsilon_2, \alpha_1 \rangle} s^{-j\langle \epsilon_1, \alpha_1 \rangle} v_j = \phi' r^j s^{-j} v_j. \end{aligned}$$

Note that $M(\phi)$ is a simple U -module if and only if $[j] \frac{\phi r^{-j+1} - \phi' s^{-j+1}}{r-s} \neq 0$ for any $j \geq 1$.

Suppose $[\ell + 1] \frac{\phi r^{-\ell} - \phi' s^{-\ell}}{r - s} = 0$ for some $\ell \geq 0$. Then either $r^{\ell+1} = s^{\ell+1}$, which implies rs^{-1} is a root of unity, or $\phi' = \phi r^{-\ell} s^\ell$. Assuming that rs^{-1} is not a root of unity and $\phi' = \phi r^{-\ell} s^\ell$, we see that the elements v_i , $i \geq \ell + 1$, span a maximal submodule. The quotient is the $(\ell + 1)$ -dimensional simple module $L(\phi)$, which we can suppose is spanned by v_0, v_1, \dots, v_ℓ and has U -action given by

$$(2.7) \quad \begin{aligned} f.v_j &= v_{j+1}, & (v_{\ell+1} &= 0) \\ e.v_j &= \phi r^{-\ell} [j][\ell + 1 - j] v_{j-1} & (v_{-1} &= 0) \\ \omega.v_j &= \phi r^{-j} s^j v_j \\ \omega'.v_j &= \phi r^{-\ell+j} s^{\ell-j} v_j. \end{aligned}$$

When $M(\phi)$ is not simple and rs^{-1} is not a root of unity, $j = \ell + 1$ is the unique value such that $[j] \frac{\phi r^{-j+1} - \phi' s^{-j+1}}{r - s} = 0$. In this case, $M(\phi)$ has a unique proper submodule, namely the maximal submodule generated by $v_{\ell+1}$ as above.

We now have the following classification of simple modules for $U_{r,s}(\mathfrak{sl}_2)$.

Proposition 2.8.

- (i) Assume $U = U_{r,s}(\mathfrak{sl}_2)$, where rs^{-1} is not a root of unity. Let $\phi : U^0 \rightarrow \mathbb{K}$ be an algebra homomorphism such that $\phi(\omega') = \phi(\omega)r^{-\ell}s^\ell$ for some $\ell \geq 0$. Then there is an $(\ell + 1)$ -dimensional simple U -module $L(\phi)$ spanned by vectors v_0, v_1, \dots, v_ℓ and having U -action given by (2.7). Any $(\ell + 1)$ -dimensional simple U -module is isomorphic to some such $L(\phi)$.
- (ii) If $\nu = \nu_1 \epsilon_1 + \nu_2 \epsilon_2 \in \Lambda^+$, then $\nu_1 - \nu_2 = \ell$ for some $\ell \in \mathbb{Z}_{\geq 0}$, and $\nu(\omega') = r^{\nu_2} s^{\nu_1} = r^{\nu_1 - \ell} s^{\nu_2 + \ell} = \nu(\omega)r^{-\ell}s^\ell$ in this case. Thus, the module $L(\nu)$ is $(\ell + 1)$ -dimensional and has U -action given by (2.7) with $\phi = r^{\nu_1} s^{\nu_2} = r^{\nu_1} s^{\nu_1 - \ell}$.

Finite-dimensionality of $L(\lambda)$ for $\lambda \in \Lambda^+$.

We show below that the simple modules $L(\lambda)$ for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ with $\lambda \in \Lambda^+$ are finite-dimensional. For this it suffices to prove that $M(\lambda)$ has a \tilde{U} -submodule of finite codimension, as $L(\lambda)$ is the quotient of $M(\lambda)$ by its unique maximal submodule.

As λ is dominant, $k_i = \langle \alpha_i, \lambda \rangle$ for $i = 1, \dots, n - 1$, are nonnegative integers. Define a \tilde{U} -submodule $M'(\lambda)$ of $M(\lambda)$ by

$$(2.9) \quad M'(\lambda) = \sum_{i=1}^{n-1} \tilde{U} f_i^{k_i+1} .v_\lambda.$$

Our goal is to prove that the module $L'(\lambda) := M(\lambda)/M'(\lambda)$ is nonzero and finite-dimensional.

By Lemma 2.3 we have $e_i f_i^{k_i+1} .v_\lambda = 0$. If $j \neq i$, $e_j f_i^{k_i+1} .v_\lambda = f_i^{k_i+1} e_j .v_\lambda = 0$ by the defining relations. Consequently, by Theorem 2.1, $\tilde{U} f_i^{k_i+1} .v_\lambda$ is a homomorphic image of $M(\lambda - (k_i + 1)\alpha_i)$, and so all its weights are less than or equal to $\lambda - (k_i + 1)\alpha_i$. This implies that $v_\lambda \notin M'(\lambda)$, hence $L'(\lambda) \neq 0$.

Lemma 2.10. *The elements e_j, f_j ($1 \leq j < n$) act locally nilpotently on $L'(\lambda)$.*

Proof. As the Verma module $M(\lambda)$ is spanned over \mathbb{K} by all elements $x_1 \cdots x_t.v_\lambda$ where $x_1, \dots, x_t \in \{f_1, \dots, f_{n-1}\}$, $t \in \mathbb{Z}_{\geq 0}$, it is enough to argue by induction on t that a sufficiently high power of e_j (resp., f_j) takes such an element to $M'(\lambda)$. If $t = 0$, then $e_j.v_\lambda = 0 \in M'(\lambda)$, and $f_j^{k_j+1}.v_\lambda \in M'(\lambda)$ by construction. Now assume that there are positive integers N_j such that

$$e_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda) \quad \text{and} \quad f_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda).$$

Suppose that $x_1 = f_i$. If $j \neq i$, then $e_j^{N_j} x_1 \cdots x_t.v_\lambda = f_i e_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda)$. Otherwise by Lemma 2.3,

$$e_i^{N_i+1} x_1 \cdots x_t.v_\lambda = f_i e_i^{N_i+1} x_2 \cdots x_t.v_\lambda + [N_i + 1] e_i^{N_i} \frac{s^{-N_i \omega_i} - r^{-N_i \omega_i'}}{r - s} x_2 \cdots x_t.v_\lambda.$$

Applying relation (R2') and the induction hypothesis, we see that these terms are both in $M'(\lambda)$.

Now $f_i^{N_i-1} x_1 \cdots x_t.v_\lambda = f_i^{N_i} x_2 \cdots x_t.v_\lambda \in M'(\lambda)$, and if $|i - j| > 1$, we also have $f_j^{N_j} x_1 \cdots x_t.v_\lambda = f_i f_j^{N_j} x_2 \cdots x_t.v_\lambda \in M'(\lambda)$. Finally, we need to show that if $|i - j| = 1$, then $f_j^{N_j+1} x_1 \cdots x_t.v_\lambda \in M'(\lambda)$. This will follow from the induction hypothesis once we know that $f_j^{N_j+1} f_i \in \mathbb{K} f_j f_i f_j^{N_j} + \mathbb{K} f_i f_j^{N_j+1}$.

We argue by induction on $m \geq 1$ that

$$f_j^{m+1} f_i \in \mathbb{K} f_j f_i f_j^m + \mathbb{K} f_i f_j^{m+1}.$$

Indeed if $m = 1$, this follows from relation (R7); but if $m > 1$, then by induction and (R7),

$$f_j^{m+1} f_i \in f_j(\mathbb{K} f_j f_i f_j^{m-1} + \mathbb{K} f_i f_j^m) \subseteq \mathbb{K} f_j f_i f_j^m + \mathbb{K} f_i f_j^{m+1}. \quad \square$$

Lemma 2.11. *Assume rs^{-1} is not a root of unity, and let V be a module for $U = U_{r,s}(\mathfrak{sl}_2)$ on which U^0 acts semisimply. Suppose $V = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} V_{\lambda - j\alpha}$ for some weight $\lambda \in \Lambda$; each weight space of V is finite-dimensional; and e and f act locally nilpotently on V . Then V is finite-dimensional, and the weights of V are preserved under the simple reflection taking α to $-\alpha$.*

Proof. Let $\mu = \mu_1 \epsilon_1 + \mu_2 \epsilon_2$ be a weight of V , and $v \in V_\mu \setminus \{0\}$. As e acts locally nilpotently on V , there is a nonnegative integer k such that $e^{k+1}.v = 0$ and $e^k.v \neq 0$. By Theorem 2.1, $Ue^k.v$ is a homomorphic image of $M(\mu + k\alpha)$. But since f acts locally nilpotently on $Ue^k.v$, this image cannot be isomorphic to $M(\mu + k\alpha)$. Thus because $M(\mu + k\alpha)$ has a unique proper submodule, $Ue^k.v \cong L(\mu + k\alpha)$, and so it is finite-dimensional. Corollary 2.5 implies that $\mu + k\alpha$ is dominant. As there are only finitely many dominant weights less than or equal to the given weight λ (under the partial order $\nu \leq \lambda$ if and only if $\lambda - \nu \in \mathbb{Z}_{\geq 0}\alpha$), and each weight space is finite-dimensional, it must be that V itself is finite-dimensional.

In particular, V has a composition series with factors isomorphic to $L(\nu)$ for some $\nu \in \Lambda^+$. Any weight μ of V is a weight of some such $L(\nu)$ with $\nu = \nu_1\epsilon_1 + \nu_2\epsilon_2 \in \Lambda^+$. By (ii) of Proposition 2.8, $L(\nu)$ has weights $\nu, \nu - \alpha, \dots, \nu - \ell\alpha$ where $\ell = \nu_1 - \nu_2$. Thus, $\mu = \nu - j\alpha$ for some $j \in \{0, 1, \dots, \ell\}$. But then $\mu - \langle \mu, \alpha \rangle \alpha = \nu - (\ell - j)\alpha$ is a weight of $L(\nu)$ since $\ell - j \in \{0, 1, \dots, \ell\}$, hence it is a weight of V . Thus, the weights of V are preserved under the simple reflection taking α to $-\alpha$. \square

Lemma 2.12. *Assume rs^{-1} is not a root of unity, and let $\lambda \in \Lambda^+$. Then the module $L(\lambda)$ for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ is finite-dimensional. A similar statement holds for $U = U_{r,s}(\mathfrak{sl}_n)$, where Λ^+ is replaced by $\Lambda_{\mathfrak{sl}}^+$.*

Proof. This follows once we show that $L'(\lambda) = M(\lambda)/M'(\lambda)$, where $M'(\lambda)$ is as in (2.9), is finite-dimensional. We will prove that the set of weights of $L'(\lambda)$ is preserved under the action of the symmetric group S_n (the Weyl group of \mathfrak{gl}_n) on Λ which is generated by the simple reflections $s_i : \mu \rightarrow \mu - \langle \mu, \alpha_i \rangle \alpha_i$ ($1 \leq i < n$). Each S_n -orbit contains a dominant weight, and there are only finitely many dominant weights less than or equal to λ . As the weights in $M(\lambda)$ are all less than or equal to λ , and the weight spaces are finite-dimensional, the same is true of $L'(\lambda)$. Therefore $L'(\lambda)$ is finite-dimensional.

To see that s_i preserves the set of weights of $L'(\lambda)$, let $\mu = \mu_1\epsilon_1 + \dots + \mu_n\epsilon_n$ be a weight of $L'(\lambda)$. Consider $L'(\lambda)$ as a module for the copy U_i of $U_{r,s}(\mathfrak{sl}_2)$ generated by $e_i, f_i, \omega_i, \omega'_i$, and let $L'_i(\mu)$ be the U_i -submodule of $L'(\lambda)$ generated by $L'(\lambda)_\mu$. As all weights of $L'(\lambda)$ are less than or equal to λ , we have

$$L'_i(\mu) = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L'_i(\mu)_{\lambda' - j\alpha_i}$$

for some weight $\lambda' \leq \lambda$. By Lemmas 2.10 and 2.11, the simple reflection s_i preserves the weights of $L'_i(\mu)$, so in particular, $s_i(\mu)$ is also a weight of $L'(\lambda)$. \square

Remark 2.13. It will follow from Lemma 3.7 in the next section that $L(\lambda) \cong L'(\lambda)$, since $L(\lambda)$ is the unique simple quotient of $M(\lambda)$, $L'(\lambda)$ is a finite-dimensional quotient of $M(\lambda)$, and by that lemma, every finite-dimensional quotient is simple. Thus, we have

Corollary 2.14. *When rs^{-1} is not a root of unity, the finite-dimensional simple \tilde{U} -modules having weights in Λ are precisely the modules $L(\lambda)$ where $\lambda \in \Lambda^+$. Moreover, $L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$. Similar statements hold for $U = U_{r,s}(\mathfrak{sl}_n)$, where Λ is replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda_{\mathfrak{sl}}^+$.*

Proof. The first statement is a consequence of Corollary 2.5 and Lemma 2.12 (see also Remark 2.13). Assume there is an isomorphism of \tilde{U} -modules from $L(\lambda)$ to $L(\mu)$. The highest weight vector of $L(\lambda)$ must be sent to a weight vector of $L(\mu)$, so $\lambda \leq \mu$. As a similar argument shows that $\mu \leq \lambda$, we have $\lambda = \mu$. \square

Shifts by one-dimensional modules.

Suppose now that we have a one-dimensional module L for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$. Then by Theorem 2.1, $L = L(\chi)$ for some algebra homomorphism $\chi : \tilde{U}^0 \rightarrow \mathbb{K}$, with the elements e_i, f_i ($1 \leq i < n$) acting as multiplication by 0. Relation (R4) yields

$$(2.15) \quad \chi(\omega_i) = \chi(a_i b_{i+1}) = \chi(a_{i+1} b_i) = \chi(\omega'_i) \quad (1 \leq i < n).$$

Conversely, if an algebra homomorphism χ satisfies this equation, then $L(\chi)$ is one-dimensional by relation (R4). We will write $L_\chi = L(\chi)$ to emphasize that the module is one-dimensional.

Proposition 2.16. *Assume rs^{-1} is not a root of unity and $L(\psi)$ is the finite-dimensional simple module for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ with highest weight ψ . Then there exists a homomorphism $\chi : \tilde{U}^0 \rightarrow \mathbb{K}$ such that (2.15) holds and an element $\lambda \in \Lambda^+$ so that $\psi = \chi \cdot \hat{\lambda}$. Thus, the weights of $L(\psi)$ belong to $\chi \cdot \hat{\Lambda}$. A similar statement holds for $U = U_{r,s}(\mathfrak{sl}_n)$ with Λ replaced by $\Lambda_{\mathfrak{sl}}$ and Λ^+ by $\Lambda_{\mathfrak{sl}}^+$.*

Proof. When $L(\psi)$ is viewed as a module for the copy U_i of $U_{r,s}(\mathfrak{sl}_2)$ generated by $e_i, f_i, \omega_i, \omega'_i$, it has a composition series whose factors are simple U_i -modules as described by Proposition 2.8. As the highest weight vector of $L(\psi)$ gives a highest weight vector of some composition factor, there is a weight ϕ_i of U_i and a nonnegative integer ℓ_i so that $\psi(\omega_i) = \phi_i(\omega_i)$ and $\psi(\omega'_i) = \phi_i(\omega'_i) = \phi_i(\omega_i) r^{-\ell_i} s^{\ell_i} = \psi(\omega_i) r^{-\ell_i} s^{\ell_i}$.

For the case $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$, set $\ell_n = 0$ and define $\lambda_i = \ell_i + \dots + \ell_n$ for $i = 1, \dots, n$. Let $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$, which belongs to Λ^+ . Now define $\chi : \tilde{U}^0 \rightarrow \mathbb{K}$ by the formulas

$$\begin{aligned} \chi(a_i) &= \psi(a_i) r^{-\langle \epsilon_i, \lambda \rangle} = \psi(a_i) r^{-(\ell_i + \dots + \ell_n)} \\ \chi(b_i) &= \psi(b_i) s^{-\langle \epsilon_i, \lambda \rangle} = \psi(b_i) s^{-(\ell_i + \dots + \ell_n)}. \end{aligned}$$

Then it follows that $\chi(\omega'_i) = \chi(\omega_i)$ for $i = 1, \dots, n-1$, and $\psi = \chi \cdot \hat{\lambda}$ as desired.

For the case $U = U_{r,s}(\mathfrak{sl}_n)$, fix an n th root $(rs)^{1/n}$ of rs in \mathbb{K} . As above, there exist nonnegative integers ℓ_i ($1 \leq i < n$) so that $\psi(\omega'_i) = \psi(\omega_i) r^{-\ell_i} s^{\ell_i}$. Let $\lambda = \sum_{i=1}^{n-1} \ell_i \varpi_i \in \Lambda_{\mathfrak{sl}}^+$. Define $\chi : U^0 \rightarrow \mathbb{K}$ by

$$\begin{aligned} \chi(\omega_i) &= \psi(\omega_i) r^{-(\ell_i + \dots + \ell_{n-1})} s^{-(\ell_{i+1} + \dots + \ell_{n-1})} (rs)^{c/n} \\ \chi(\omega'_i) &= \psi(\omega'_i) r^{-(\ell_{i+1} + \dots + \ell_{n-1})} s^{-(\ell_i + \dots + \ell_{n-1})} (rs)^{c/n} \end{aligned}$$

where $c = \sum_{j=1}^{n-1} j \ell_j$. Then (2.15) holds, and $\psi = \chi \cdot \hat{\lambda}$. \square

Remark 2.17. If M is any finite-dimensional module, then $M = \bigoplus_{i=1}^m \bigoplus_{\lambda \in \Lambda} M_{\psi_i \cdot \hat{\lambda}}$ for some weights ψ_i such that $\psi_i \cdot \hat{\Lambda}$ ($1 \leq i \leq m$) are distinct cosets in $\text{Hom}(\tilde{U}^0, \mathbb{K}) / \hat{\Lambda}$ (viewed as a \mathbb{Z} -module under the action $k \cdot \psi = \psi^k$). Then $M_i := \bigoplus_{\lambda \in \Lambda} M_{\psi_i \cdot \hat{\lambda}}$ is a submodule, and $M = \bigoplus_{i=1}^m M_i$. Therefore, if M is an indecomposable \tilde{U} -module, $M = \bigoplus_{\lambda \in \Lambda} M_{\psi \cdot \hat{\lambda}}$ for some $\psi \in \text{Hom}(\tilde{U}^0, \mathbb{K})$. A simple submodule S of M has weights in $\psi \cdot \hat{\Lambda}$. By replacing ψ with the homomorphism χ for S given by Proposition 2.16, we may assume that for any indecomposable module M , there is a χ satisfying (2.15) so that $M = \bigoplus_{\lambda \in \Lambda} M_{\chi \cdot \hat{\lambda}}$.

Lemma 2.18. *Let $\chi : \widetilde{U}^0 \rightarrow \mathbb{K}$ be an algebra homomorphism with $\chi(\omega_i) = \chi(\omega'_i)$ ($1 \leq i < n$). Let M be a finite-dimensional \widetilde{U} -module whose weights are all in $\chi \cdot \hat{\Lambda}$. If \widetilde{U}^0 acts semisimply on M , then*

$$M \cong L_\chi \otimes N$$

for some \widetilde{U} -module N whose weights are all in Λ . A similar statement holds for $U = U_{r,s}(\mathfrak{sl}_n)$ with Λ replaced by $\Lambda_{\mathfrak{sl}}$.

Proof. Let $\chi^{-1} : \widetilde{U}^0 \rightarrow \mathbb{K}$ be the algebra homomorphism defined by $\chi^{-1}(a_i) = \chi(a_i^{-1}) = (\chi(a_i))^{-1}$ and $\chi^{-1}(b_i) = \chi(b_i^{-1}) = (\chi(b_i))^{-1}$ for $1 \leq i \leq n$. Note that $L_\chi \otimes L_{\chi^{-1}}$ is isomorphic to the trivial module L_ε corresponding to the counit. Let

$$N = L_{\chi^{-1}} \otimes M.$$

Then $M \cong L_\chi \otimes N$ as L_ε is a multiplicative identity (up to isomorphism) for \widetilde{U} -modules. The weights of N are all in $\chi^{-1} \cdot \chi \cdot \hat{\Lambda} = \hat{\Lambda}$. \square

We now have a classification of finite-dimensional simple modules for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ and for $U = U_{r,s}(\mathfrak{sl}_n)$.

Theorem 2.19. *Assume rs^{-1} is not a root of unity. The finite-dimensional simple \widetilde{U} -modules are precisely the modules*

$$L_\chi \otimes L(\lambda),$$

where $\chi : \widetilde{U}^0 \rightarrow \mathbb{K}$ is an algebra homomorphism with $\chi(\omega_i) = \chi(\omega'_i)$ ($1 \leq i < n$), and $\lambda \in \Lambda^+$. An analogous statement holds for U with Λ^+ replaced by $\Lambda_{\mathfrak{sl}}^+$.

Proof. Let M be a finite-dimensional simple \widetilde{U} -module. By Theorem 2.1, Proposition 2.16, and Lemma 2.18, $M \cong L_\chi \otimes N$ for some χ satisfying (2.15) and some simple module N with weights in Λ . By Corollary 2.5, $N \cong L(\lambda)$ for some $\lambda \in \Lambda^+$. Conversely, any \widetilde{U} -module of this form is finite-dimensional by Lemma 2.12 and simple by its construction. \square

Remark 2.20. If $r = q$ and $s = q^{-1}$ for some $q \in \mathbb{K}$, the classification of finite-dimensional simple $U_q(\mathfrak{sl}_n)$ -modules is a consequence of Theorem 2.19 applied to $U_{q,q^{-1}}(\mathfrak{sl}_n)$: The simple $U_q(\mathfrak{sl}_n)$ -modules are precisely those simple $U_{q,q^{-1}}(\mathfrak{sl}_n)$ -modules on which ω'_i acts as ω_i^{-1} , so that

$$\chi(\omega_i) = \chi(\omega'_i) = \chi(\omega_i^{-1}).$$

This implies $\chi(\omega_i) = \pm 1$ ($1 \leq i < n$). Each choice of algebra homomorphism $\chi : U^0 \rightarrow \mathbb{K}$ with $\chi(\omega_i) = \chi(\omega'_i) = \pm 1$ yields a one-dimensional $U_{q,q^{-1}}(\mathfrak{sl}_n)$ -module L_χ , and so the simple $U_q(\mathfrak{sl}_n)$ -modules are the $L_\chi \otimes L(\lambda)$ with $\lambda \in \Lambda_{\mathfrak{sl}}^+$ and χ as above. We have

$$\hat{\lambda}(\omega_i) = q^{\langle \epsilon_i, \lambda \rangle} q^{-\langle \epsilon_{i+1}, \lambda \rangle} = q^{\langle \alpha_i, \lambda \rangle}.$$

Thus, we recover the results of [Ja, 5.2 and 5.10].

Remark 2.21. We can interpret Proposition 2.8 in light of Theorem 2.19: Let $L(\phi)$ be the simple $U_{r,s}(\mathfrak{sl}_2)$ -module described in the proposition. Let $\lambda = (\ell/2)\alpha \in \Lambda_{\mathfrak{sl}}^+$ and define $\chi : U^0 \rightarrow \mathbb{K}$ by $\chi(\omega) = \phi(\omega)r^{-\ell/2}s^{\ell/2}$, $\chi(\omega') = \phi(\omega')r^{\ell/2}s^{-\ell/2} = \phi(\omega)r^{-\ell/2}s^{\ell/2} = \chi(\omega)$. Then $\phi = \chi \cdot \hat{\lambda}$ and $L(\phi) \cong L_\chi \otimes L(\lambda)$.

§3. COMPLETE REDUCIBILITY

In this section we will establish complete reducibility of all finite-dimensional modules for \tilde{U} (resp., U) on which \tilde{U}^0 (resp., U^0) acts semisimply. Statements made for \tilde{U} hold equally well for U , and we will point out where there are differences. It is helpful to work in a more general context.

Let \mathcal{O} denote the category of modules M for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ which satisfy the conditions:

- (O1) \tilde{U}^0 acts semisimply on M , and the set $\text{wt}(M)$ of weights of M belongs to Λ :
 $M = \bigoplus_{\lambda \in \text{wt}(M)} M_\lambda$, where $M_\lambda = \{m \in M \mid a_i \cdot m = r^{\langle \epsilon_i, \lambda \rangle}, \quad b_i \cdot m = s^{\langle \epsilon_i, \lambda \rangle}$
for all i };
- (O2) $\dim_{\mathbb{K}} M_\lambda < \infty$ for all $\lambda \in \text{wt}(M)$;
- (O3) $\text{wt}(M) \subseteq \bigcup_{\mu \in F} (\mu - Q^+)$ for some finite set $F \subset \Lambda$.

The morphisms in \mathcal{O} are \tilde{U} -module homomorphisms. In defining category \mathcal{O} for $U = U_{r,s}(\mathfrak{sl}_n)$, we replace Λ by the weight lattice $\Lambda_{\mathfrak{sl}}$ of \mathfrak{sl}_n .

All finite-dimensional \tilde{U} -modules which satisfy (O1) belong to category \mathcal{O} , as do all highest weight modules with weights in Λ such as the Verma modules $M(\lambda)$.

We recall the definition of the quantum Casimir operator [BW2, Sec. 4]. It is a consequence of (R2) and (R3) that the subalgebra U^+ of \tilde{U} (or of $U = U_{r,s}(\mathfrak{sl}_n)$) generated by 1 and e_i ($1 \leq i < n$) has the decomposition $U^+ = \bigoplus_{\zeta \in Q^+} U_\zeta^+$ where

$$U_\zeta^+ = \{z \in U^+ \mid a_i z = r^{\langle \epsilon_i, \zeta \rangle} z a_i, \quad b_i z = s^{\langle \epsilon_i, \zeta \rangle} z b_i \quad (1 \leq i < n)\}.$$

The weight space U_ζ^+ is spanned by all the monomials $e_{i_1} \cdots e_{i_\ell}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_\ell} = \zeta$. Similarly, the subalgebra U^- generated by 1 and the f_i has the decomposition $U^- = \bigoplus_{\zeta \in Q^+} U_{-\zeta}^-$. The spaces U_ζ^+ and $U_{-\zeta}^-$ are nondegenerately paired by the Hopf pairing specified by

$$(3.1) \quad \begin{aligned} (f_i, e_j) &= \frac{\delta_{i,j}}{s-r} \\ (\omega'_i, \omega_j) &= r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_j+1, \alpha_i \rangle} \\ (b_n, a_n) &= 1, \quad (b_n, \omega_j) = s^{-\langle \epsilon_n, \alpha_j \rangle}, \quad (\omega'_i, a_n) = r^{\langle \epsilon_n, \alpha_i \rangle}. \end{aligned}$$

(See [BW2, Sec. 2].) The Hopf algebras \tilde{U} and U are Drinfel'd doubles of certain Hopf subalgebras with respect to this pairing [BW2, Thm. 2.7]. Let $d_\zeta = \dim_{\mathbb{K}} U_\zeta^+$. Assume $\{u_k^\zeta\}_{k=1}^{d_\zeta}$ is a basis for U_ζ^+ , and $\{v_k^\zeta\}_{k=1}^{d_\zeta}$ is the dual basis for $U_{-\zeta}^-$ with respect to the pairing.

Now let

$$(3.2) \quad \Omega = \sum_{\zeta \in Q^+} \sum_{k=1}^{d_\zeta} S(v_k^\zeta) u_k^\zeta,$$

where S denotes the antipode. All but finitely many terms in this sum will act as multiplication by 0 on any weight space M_λ of $M \in \mathcal{O}$. Therefore Ω is a well-defined operator on such a module M .

The second part of the Casimir operator involves a function $g : \Lambda \rightarrow \mathbb{K}^\#$ defined as follows. If ρ denotes half the sum of the positive roots, then $2\rho = \sum_{j=1}^n (n+1-2j)\epsilon_j \in \Lambda$. For $\lambda \in \Lambda$, set

$$(3.3) \quad g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho, \lambda \rangle}.$$

When M is a \tilde{U} -module in \mathcal{O} , we define the linear operator $\Xi : M \rightarrow M$ by

$$\Xi(m) = g(\lambda)m$$

for all $m \in M_\lambda$, $\lambda \in \Lambda$. Then Ξ is well-defined, as $\hat{\lambda} = \hat{\mu}$ if and only if $\lambda = \mu$ ($\lambda, \mu \in \Lambda$) [BW2, Prop. 3.5]. (When $U = U_{r,s}(\mathfrak{sl}_n)$, it is necessary to first fix roots $r^{1/2n}$ and $s^{1/2n}$ of r and s in \mathbb{K} .) Then Ξ , as given above, is well-defined. We have the following result from [BW2].

Proposition 3.4. [BW2, Thm. 4.20] *The operator $\Omega\Xi : M \rightarrow M$ commutes with the action of \tilde{U} on any \tilde{U} -module $M \in \mathcal{O}$.*

We require the next lemma in order to prove complete reducibility.

Lemma 3.5. *Assume rs^{-1} is not a root of unity, and let $\lambda, \mu \in \Lambda^+$. If $\lambda \geq \mu$ and $g(\lambda) = g(\mu)$, then $\lambda = \mu$.*

Proof. Because $\lambda \geq \mu$, we may suppose $\lambda = \mu + \beta$ where $\beta = \sum_{i=1}^{n-1} k_i \alpha_i$ and $k_i \in \mathbb{Z}_{\geq 0}$. By assumption we have

$$(rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho, \lambda \rangle} = g(\lambda) = g(\mu) = (rs^{-1})^{\frac{1}{2}\langle \mu+2\rho, \mu \rangle},$$

and as rs^{-1} is not a root of unity, it must be that $\langle \lambda + 2\rho, \lambda \rangle = \langle \mu + 2\rho, \mu \rangle$, or equivalently, $2\langle \mu + \rho, \beta \rangle + \langle \beta, \beta \rangle = 0$. Since $\mu \in \Lambda^+$, $\mu = \mu_1 \epsilon_1 + \mu_2 \epsilon_2 + \cdots + \mu_n \epsilon_n$ where $\mu_i \in \mathbb{Z}$ for all i and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. A calculation yields

$$0 = \langle 2\mu + 2\rho, \beta \rangle + \langle \beta, \beta \rangle = \sum_{i=1}^{n-1} 2k_i (\mu_i - \mu_{i+1} + 1) + \sum_{i=1}^n (k_i - k_{i-1})^2, \quad (k_0 = 0 = k_n).$$

The only way this can happen is if $k_i = 0$ for all i and $\lambda = \mu$. \square

Lemma 3.6. *Assume rs^{-1} is not a root of unity.*

- (i) $\Omega\Xi$ acts as multiplication by the scalar $g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho, \lambda \rangle}$ on the Verma module $M(\lambda)$ with $\lambda \in \Lambda$, hence on any submodule or quotient of $M(\lambda)$.
- (ii) The eigenvalues of the operator $\Omega\Xi : M \rightarrow M$ are integral powers of $(rs^{-1})^{\frac{1}{2}}$ on any finite-dimensional $M \in \mathcal{O}$. (For $U = U_{r,s}(\mathfrak{sl}_n)$, the eigenvalues are integral powers of $(rs^{-1})^{1/2n^2}$.)

Proof. By its construction, $\Omega\Xi$ acts by multiplication by $g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle\lambda+2\rho,\lambda\rangle}$ on the maximal vector v_λ of $M(\lambda)$. But since $M(\lambda) = \tilde{U}.v_\lambda$ and $\Omega\Xi$ commutes with \tilde{U} on modules in \mathcal{O} , $\Omega\Xi$ acts as multiplication by $(rs^{-1})^{\frac{1}{2}\langle\lambda+2\rho,\lambda\rangle}$ on all of $M(\lambda)$.

If $M \in \mathcal{O}$ is finite-dimensional, it has a composition series. Each factor is a finite-dimensional simple \tilde{U} -module with weights in Λ , and in particular, is a quotient of $M(\lambda)$ for some $\lambda \in \Lambda$. On such a factor, $\Omega\Xi$ acts as multiplication by $g(\lambda)$. Therefore the action of $\Omega\Xi$ on M may be expressed by an upper triangular matrix with each diagonal entry equal to $g(\lambda)$ for some $\lambda \in \Lambda$. \square

Lemma 3.7. *Assume rs^{-1} is not a root of unity. Let $\lambda \in \Lambda$ and M be a nonzero finite-dimensional quotient of the Verma module $M(\lambda)$. Then M is simple.*

Proof. First observe that by Lemma 2.4, $\lambda \in \Lambda^+$. Assume M' is a proper submodule of M . As M is generated by its one-dimensional subspace M_λ , we must have $M'_\lambda = 0$. Let $\mu \in \Lambda$ be maximal such that $M'_\mu \neq 0$, and note that $\mu < \lambda$. Let m' be a nonzero vector of M'_μ . It follows from the maximality of μ that $e_i.m' = 0$ for all i ($1 \leq i < n$). Letting $M'' = \tilde{U}.m'$, a nonzero finite-dimensional quotient of $M(\mu)$, we see that $\mu \in \Lambda^+$ as well. By Lemma 3.6 (i), $\Omega\Xi$ acts as multiplication by $g(\lambda)$ on M , and by $g(\mu)$ on M'' . This forces $g(\lambda) = g(\mu)$, which contradicts Lemma 3.5 as $\mu < \lambda$. \square

Finally we state the needed complete reducibility result, whose proof parallels that of [L, Thm. 6.2.2].

Theorem 3.8. *Assume rs^{-1} is not a root of unity. Let M be a nonzero finite-dimensional \tilde{U} -module on which \tilde{U}^0 acts semisimply. Then M is completely reducible.*

Proof. Suppose first that M has weights in Λ . Then M is a direct sum of generalized eigenspaces for $\Omega\Xi$, which by Proposition 3.4, is a direct sum decomposition of M as a \tilde{U} -module. Therefore, we may assume M is itself a generalized eigenspace of $\Omega\Xi$, so that $(\Omega\Xi - (rs^{-1})^c)^d(M) = 0$ for some $c \in \frac{1}{2}\mathbb{Z}$, $d = \dim_{\mathbb{K}} M$, by Lemma 3.6 (ii).

Let $P = \{m \in M \mid e_i.m = 0 \text{ (} 1 \leq i < n \text{)}\}$, and note that $P = \bigoplus_{\lambda \in \Lambda} P_\lambda$, $P_\lambda = P \cap M_\lambda$. If $m \in P_\lambda - \{0\}$, the \tilde{U} -submodule $\tilde{U}.m$ of M is a nonzero quotient of $M(\lambda)$ by Theorem 2.1. By Lemma 3.7, each such $\tilde{U}.m$ is a simple \tilde{U} -module, and so the \tilde{U} -submodule M' of M generated by P is a sum of simple \tilde{U} -modules. That is, M' is completely reducible. Let $M'' = M/M'$.

Assuming $M'' \neq 0$, there is a weight $\mu \in \Lambda$ maximal such that $M''_\mu \neq 0$. It follows from the maximality of μ that $e_i.m'' = 0$ for $m'' \in M''_\mu - \{0\}$ and for all i ($1 \leq i < n$). By Lemma 2.4, $\mu \in \Lambda^+$, and by Theorem 2.1 and Lemma 3.6, $\Omega\Xi$ acts as multiplication by $g(\mu)$ on the U -module $U.m''$ generated by m'' . This implies $g(\mu) = (rs^{-1})^c$.

Let $m \in M_\mu$ be a representative for $m'' \in (M/M')_\mu$, and $M_1 = \tilde{U}.m$. Then there is a weight $\eta \in \Lambda$ maximal such that $M_1 \cap M_\eta \neq 0$. Let $m_1 \in M_1 \cap M_\eta - \{0\}$,

so that $e_i.m_1 = 0$ for all i ($1 \leq i < n$). Again applying Theorem 2.1 and Lemmas 2.4 and 3.6, we have $\eta \in \Lambda^+$ and $\Omega\Xi(m_1) = g(\eta)m_1$. Therefore $g(\eta) = (rs^{-1})^c$.

We now have $g(\mu) = g(\eta)$, where $\eta, \mu \in \Lambda^+$, and $\eta \geq \mu$ by construction. By Lemma 3.5, $\eta = \mu$, so M_1 is the one-dimensional space spanned by m , and $e_i.m = 0$ ($1 \leq i < n$), that is $m \in P$. This implies $m'' = 0$, a contradiction to the assumption that $M'' \neq 0$. Therefore $M'' = 0$, and $M = M'$ is completely reducible.

Finally, when M does not have weights in Λ , we may assume that M is indecomposable. By Remark 2.17, M has all its weights in $\chi \cdot \hat{\Lambda}$ for some χ satisfying (2.15). By Lemma 2.18, $M \cong L_\chi \otimes N$ for some \tilde{U} -module N whose weights are all in Λ . Note that \tilde{U}^0 acts semisimply on N as well ($N = L_{\chi^{-1}} \otimes M$), and so N is completely reducible by the above argument. This implies that M itself is completely reducible. \square

Remark 3.9. It is necessary to include the hypothesis that \tilde{U}^0 acts semisimply in Theorem 3.8, as the next examples illustrate. (Recall that \tilde{U}^0 does indeed act semisimply on any simple \tilde{U} -module, as remarked in the text following (1.6).) Let $V = \mathbb{K}^m$ for $m \geq 2$ and $\xi, \xi' \in \mathbb{K} \setminus \{0\}$. We define a \tilde{U} -module structure on V by requiring that e_i, f_i act as multiplication by 0 and a_i, b_i act via the $m \times m$ Jordan blocks with diagonal entries ξ, ξ' , respectively. Then relations (R1)-(R7) of \tilde{U} hold on V . The scalars ξ, ξ' may be chosen so that V has weights in Λ , for example choose an integer c , let $\lambda = c(\epsilon_1 + \dots + \epsilon_n)$, and set $\xi = r^c = \hat{\lambda}(a_i)$, $\xi' = s^c = \hat{\lambda}(b_i)$. Clearly V is not completely reducible. A related example for $U_{r,s}(\mathfrak{sl}_n)$ is given by sending each ω_i, ω'_i to the Jordan block with 1s on the diagonal, thus corresponding to the weight 0.

§4. THE R -MATRIX

In this section we recall the definition of the R -matrix from [BW2, Sec. 4] and use it to prove a more general result on the commutativity of the tensor product of finite-dimensional modules than was given there (compare [BW2, Thm. 4.11] with Theorem 4.2 below). Let M, M' be \tilde{U} -modules in category \mathcal{O} . We define an isomorphism of \tilde{U} -modules $R_{M',M} : M' \otimes M \rightarrow M \otimes M'$ as follows. If $\lambda = \sum_{i=1}^n \lambda_i \alpha_i \in \Lambda$, where $\alpha_n = \epsilon_n$, set

$$\omega_\lambda = \omega_1^{\lambda_1} \dots \omega_{n-1}^{\lambda_{n-1}} a_n^{\lambda_n} \quad \text{and} \quad \omega'_\lambda = (\omega'_1)^{\lambda_1} \dots (\omega'_{n-1})^{\lambda_{n-1}} b_n^{\lambda_n}.$$

Also let

$$\Theta = \sum_{\zeta \in Q^+} \sum_{k=1}^{d_\zeta} v_k^\zeta \otimes u_k^\zeta,$$

where the notation is as in the paragraph following (3.1). Define

$$R_{M',M} = \Theta \circ \tilde{f} \circ P$$

where $P(m' \otimes m) = m \otimes m'$, $\tilde{f}(m \otimes m') = (\omega'_\mu, \omega_\lambda)^{-1}(m \otimes m')$ when $m \in M_\lambda$ and $m' \in M'_\mu$, and the Hopf pairing $(\ , \)$ is defined in (3.1). (There is an equivalent

definition of \tilde{f} that works in the case $U = U_{r,s}(\mathfrak{sl}_n)$, given in [BW2] after (4.3).) Then $R_{M',M}$ is an isomorphism of \tilde{U} -modules that satisfies the quantum Yang-Baxter equation and the hexagon identities [BW2, Thms. 4.11, 5.4, and 5.7].

We will show that the tensor product of *any* two finite-dimensional \tilde{U} -modules in \mathcal{O} is commutative (up to module isomorphism), starting first with the special case that one of the modules is a one-dimensional module $L_\chi = L(\chi)$, as defined in Section 2.

Lemma 4.1. *Let M be a \tilde{U} -module in category \mathcal{O} , and let L_χ be a one-dimensional \tilde{U} -module. Then*

$$L_\chi \otimes M \cong M \otimes L_\chi.$$

Proof. Fix a basis element v of L_χ . Define a linear function $F : L_\chi \otimes M \rightarrow M \otimes L_\chi$ as follows. If $m \in M_\lambda$ and $\lambda = -\sum_{i=1}^n c_i \alpha_i$, then

$$F(v \otimes m) = \chi_1^{c_1} \cdots \chi_n^{c_n} m \otimes v,$$

where $\chi_i = \chi(\omega_i) = \chi(\omega'_i)$ ($1 \leq i < n$) and $\chi_n = \chi(a_n)$. Clearly F is bijective, and we check that F is a \tilde{U} -homomorphism:

$$e_i.F(v \otimes m) = \chi_1^{c_1} \cdots \chi_n^{c_n} (e_i \otimes 1 + \omega_i \otimes e_i)(m \otimes v) = \chi_1^{c_1} \cdots \chi_n^{c_n} e_i.m \otimes v$$

On the other hand, as $e_i.m \in M_{\lambda+\alpha_i}$, we have

$$F(e_i.(v \otimes m)) = \chi_i F(v \otimes e_i m) = \chi_i (\chi_1^{c_1} \cdots \chi_i^{c_i-1} \cdots \chi_n^{c_n}) e_i.m \otimes v = e_i.F(v \otimes m).$$

Similarly, F commutes with f_i . As the action by a_i, b_i preserves the weight spaces, F commutes with a_i, b_i ($1 \leq i \leq n$) as well. Therefore F is an isomorphism of \tilde{U} -modules. \square

Theorem 4.2. *Let M and M' be finite-dimensional modules for \tilde{U} (resp., U) with \tilde{U}^0 (resp., U^0) acting semisimply. Then*

$$M \otimes M' \cong M' \otimes M.$$

Proof. As the tensor product distributes over direct sums, we may assume that M and M' are indecomposable. Therefore the weights of M are all in $\chi \cdot \hat{\Lambda}$ for some algebra homomorphism $\chi : \tilde{U}^0 \rightarrow \mathbb{K}$ with $\chi(\omega_i) = \chi(\omega'_i)$. (See Remark 2.17.) By Lemma 2.18, $M \cong L_\chi \otimes N$ for some module N with weights in Λ . Similarly, $M' \cong L_{\chi'} \otimes N'$ for some χ' . By Lemma 4.1 and [BW2, Thm. 4.11],

$$\begin{aligned} M \otimes M' &\cong (L_\chi \otimes N) \otimes (L_{\chi'} \otimes N') \cong (L_\chi \otimes L_{\chi'}) \otimes (N \otimes N') \\ &\cong (L_{\chi'} \otimes L_\chi) \otimes (N' \otimes N) \\ &\cong (L_{\chi'} \otimes N') \otimes (L_\chi \otimes N) \cong M' \otimes M. \quad \square \end{aligned}$$

§5. TENSOR POWERS OF THE NATURAL MODULE

In this section we consider tensor powers $V^{\otimes k} = V \otimes V \otimes \cdots \otimes V$ (k factors) of the natural module V for \tilde{U} (defined in Section 1). Set $R = R_{V,V}$, and for $1 \leq i < k$, let R_i be the \tilde{U} -module isomorphism on $V^{\otimes k}$ defined by

$$R_i(z_1 \otimes z_2 \otimes \cdots \otimes z_k) = z_1 \otimes \cdots \otimes z_{i-1} \otimes R(z_i \otimes z_{i+1}) \otimes z_{i+2} \otimes \cdots \otimes z_k.$$

Then it is a consequence of the quantum Yang-Baxter equation that the braid relations hold:

$$(5.1) \quad \begin{aligned} R_i \circ R_{i+1} \circ R_i &= R_{i+1} \circ R_i \circ R_{i+1} & \text{for } 1 \leq i < k \\ R_i \circ R_j &= R_j \circ R_i & \text{for } |i - j| \geq 2. \end{aligned}$$

We would like to argue that

$$(5.2) \quad R_i^2 = (1 - rs^{-1})R_i + rs^{-1}\text{Id}$$

for all $i = 1, \dots, k-1$. For this it suffices to work with the 2-fold tensor product $V \otimes V$.

Proposition 5.3. *Whenever $s \neq -r$, the \tilde{U} -module $V \otimes V$ decomposes into two simple \tilde{U} -submodules, $S_{r,s}^2(V)$ (the (r, s) -symmetric tensors) and $\Lambda_{r,s}^2(V)$ (the (r, s) -antisymmetric tensors). These modules are defined as follows:*

- (i) $S_{r,s}^2(V)$ is the span of $\{v_i \otimes v_i \mid 1 \leq i \leq n\} \cup \{v_i \otimes v_j + sv_j \otimes v_i \mid 1 \leq i < j \leq n\}$.
- (ii) $\Lambda_{r,s}^2(V)$ is the span of $\{v_i \otimes v_j - rv_j \otimes v_i \mid 1 \leq i < j \leq n\}$.

Proof. A computation shows that $S_{r,s}^2(V)$ and $\Lambda_{r,s}^2(V)$ are \tilde{U} -submodules of $V \otimes V$. Note that each weight space of $S_{r,s}^2(V)$ is one-dimensional and is spanned by one of the weight vectors listed in (i). Therefore any submodule of $S_{r,s}^2(V)$ must contain one of these vectors. It may be checked that any of these vectors generates all of $S_{r,s}^2(V)$ in case $s \neq -r$. In particular, $v_1 \otimes v_1$ is a highest weight vector, and given any other vector in (i), there is an element of U taking it to $v_1 \otimes v_1$. Therefore $S_{r,s}^2(V)$ is simple. A similar argument proves that $\Lambda_{r,s}^2(V)$ is simple, with highest weight vector $v_1 \otimes v_2 - rv_2 \otimes v_1$. \square

Remark 5.4. The $s = -r$ case is “nongeneric,” and in this exceptional case, $V \otimes V$ need not be completely reducible. For example, when $n = 2$ what happens is that $v_1 \otimes v_2 - rv_2 \otimes v_1$ spans a one-dimensional module (as it does for $n = 2$ generic) that is not complemented in $V \otimes V$. Modulo that submodule, $v_1 \otimes v_1$ spans a one-dimensional module. Modulo the resulting two-dimensional module, $v_1 \otimes v_2 + rv_2 \otimes v_1$ and $v_2 \otimes v_2$ span a two-dimensional module.

Proposition 5.5. *The minimum polynomial of $R = R_{V,V}$ on $V \otimes V$ is $(t-1)(t+rs^{-1})$ if $s \neq -r$.*

Proof. It follows from the definition of R that $R(v_1 \otimes v_1) = v_1 \otimes v_1$ and $R(v_1 \otimes v_2 - rv_2 \otimes v_1) = -rs^{-1}(v_1 \otimes v_2 - rv_2 \otimes v_1)$. By Proposition 5.3, $S_{r,s}^2(V)$ and $\Lambda_{r,s}^2(V)$ are simple, and in fact, $v_1 \otimes v_1$ and $v_1 \otimes v_2 - rv_2 \otimes v_1$ are the highest weight vectors. In particular, each is a cyclic module generated by its highest weight vector. As $Ra(v_1 \otimes v_1) = aR(v_1 \otimes v_1) = a(v_1 \otimes v_1)$ for all $a \in \tilde{U}$, this implies that $S_{r,s}^2(V)$ is in the eigenspace of R corresponding to eigenvalue 1. Analogously, $\Lambda_{r,s}^2(V)$ corresponds to the eigenvalue $-rs^{-1}$, and since $V \otimes V$ is the direct sum of those submodules, we have the desired result. \square

From Proposition 5.5 it follows that R acts as

$$(5.6) \quad r \sum_{i < j} E_{j,i} \otimes E_{i,j} + s^{-1} \sum_{i < j} E_{i,j} \otimes E_{j,i} + (1 - rs^{-1}) \sum_{i < j} E_{j,j} \otimes E_{i,i} + \sum_i E_{i,i} \otimes E_{i,i}$$

on $V \otimes V$. Indeed, (5.6) is a linear operator that acts on $S_{r,s}^2(V)$ as multiplication by 1, and on $\Lambda_{r,s}^2(V)$ as multiplication by $-rs^{-1}$. By Proposition 5.5, R has the same properties, and so R is equal to this sum on $V \otimes V$.

§6. QUANTUM SCHUR-WEYL DUALITY

Assume $r, s \in \mathbb{K}$. Let $H_k(r, s)$ be the unital associative algebra over \mathbb{K} with generators T_i , $1 \leq i < k$, subject to the relations:

- (H1) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $1 \leq i < k$
- (H2) $T_i T_j = T_j T_i$, $|i - j| \geq 2$
- (H3) $T_i^2 = (s - r)T_i + rs1$.

When $r \neq 0$, the elements $t_i = r^{-1}T_i$ satisfy the braid relations (H1), (H2), along with the relation

$$(H3') \quad t_i^2 = (q - 1)t_i + q1,$$

where $q = r^{-1}s$. The Hecke algebra $H_k(q)$ (of type A_{k-1}) is generated by elements t_i , $1 \leq i < k$, which satisfy (H1), (H2), (H3'). It has dimension $k!$ and is semisimple whenever q is not a root of unity. At $q = 1$, the Hecke algebra $H_k(q)$ is isomorphic to $\mathbb{K}S_k$, the group algebra of the symmetric group S_k , where we may identify t_i with the transposition $(i \ i + 1)$.

The two-parameter Hecke algebra $H_k(r, s)$ defined above is isomorphic to $H_k(r^{-1}s)$ whenever $r \neq 0$. Thus, it is semisimple whenever $r^{-1}s$ is not a root of unity. For any $\sigma \in S_k$, we may define $T_\sigma = T_{i_1} \cdots T_{i_\ell}$ where $\sigma = t_{i_1} \cdots t_{i_\ell}$ is a reduced expression for σ as a product of transpositions. It follows from (H1) and (H2) that T_σ is independent of the reduced expression and these elements give a basis.

The results of Section 5 show that the \tilde{U} -module $V^{\otimes k}$ affords a representation of the Hecke algebra $H_k(r, s)$:

$$(6.1) \quad \begin{aligned} H_k(r, s) &\rightarrow \text{End}_{\tilde{U}}(V^{\otimes k}) \\ T_i &\mapsto sR_i \quad (1 \leq i < k). \end{aligned}$$

When $k = 2$ and $s \neq -r$, $V^{\otimes 2} = S_{r,s}^2(V) \oplus \wedge_{r,s}^2(V)$ is a decomposition of $V^{\otimes 2}$ into simple \tilde{U} -modules by Proposition 5.3. The maps $p_1 = (sR_1 + r)/(s + r)$ and $p_2 = (s - sR_1)/(s + r)$, ($R_1 = R_{V,V}$), are the corresponding projections onto the simple summands. Thus, the map in (6.1) is an isomorphism for $k = 2$. More generally, we will show next that it is surjective whenever rs^{-1} is not a root of unity, and it is an isomorphism when $n \geq k$. This is the two-parameter version of the well-known result of Jimbo [Ji] that $H_k(q) \cong \text{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes k})$ and is the analogue of classical Schur-Weyl duality, $\mathbb{K}S_k \cong \text{End}_{\mathfrak{gl}_n}(V^{\otimes k})$ for $n \geq k$. We will apply the following lemma. The case $n < k$ is dealt with separately, and it uses the isomorphism $H_k(r, s) \cong \text{End}_{\tilde{U}}(V^{\otimes k})$ in the $n = k$ case.

Lemma 6.2. *If $n \geq k$ and V is the natural representation of \tilde{U} , then $V^{\otimes k}$ is a cyclic \tilde{U} -module generated by $v_1 \otimes \cdots \otimes v_k$.*

Proof. Let $\underline{v} = v_1 \otimes \cdots \otimes v_k$. We begin by showing that if $\sigma \in S_k$, then $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \tilde{U}.\underline{v}$.

Suppose we have an arbitrary permutation $x_1 \otimes \cdots \otimes x_k$ ($x_i \in \{v_1, \dots, v_k\}$ for all i) of the factors of \underline{v} . For some $\ell < m$, assume that $x_\ell = v_j$ and $x_m = v_{j+1}$. Then because of the formulas

$$(6.3) \quad \begin{aligned} \Delta^{k-1}(e_j) &= \sum_{i=1}^k \underbrace{\omega_j \otimes \cdots \otimes \omega_j}_{i-1} \otimes e_j \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-i} \\ \Delta^{k-1}(f_j) &= \sum_{i=1}^k \underbrace{1 \otimes \cdots \otimes 1}_{k-i} \otimes f_j \otimes \underbrace{\omega'_j \otimes \cdots \otimes \omega'_j}_{i-1}, \end{aligned}$$

there are nonzero scalars c and c' such that

$$(ce_j f_j + c').(x_1 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_m \otimes \cdots \otimes x_\ell \otimes \cdots \otimes x_k.$$

Similarly, there are nonzero scalars d and d' such that

$$(de_j f_j + d').(x_1 \otimes \cdots \otimes x_m \otimes \cdots \otimes x_\ell \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_k.$$

As the transpositions $(j \ j+1)$ generate S_k , $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \tilde{U}.\underline{v}$ for all $\sigma \in S_k$.

Next we will use induction on k to establish the following. For any k elements $i_1, \dots, i_k \in \{1, \dots, n\}$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_k$, there is a $u \in \tilde{U}$ such that $u.\underline{v} = v_{i_1} \otimes \cdots \otimes v_{i_k}$ and u does not contain any terms with factors of $e_m, e_{m+1}, \dots, e_{n-1}, f_{m+1}, f_{m+2}, \dots, f_{n-2}$, or f_{n-1} where $m = \max\{i_k, k\}$. If $k = 1$, we may apply $f_{m-1} \cdots f_1$ to $\underline{v} = v_1$ to obtain v_m for any $m \in \{1, \dots, n\}$. If

$k > 1$, let ℓ be such that $i_\ell < i_k$, $i_{\ell+1} = i_{\ell+2} = \cdots = i_k$. (If no such ℓ exists, that is if $i_1 = \cdots = i_k$, then set $\ell = 0$ and apply u' from (6.5) below to $v_1 \otimes \cdots \otimes v_k$ to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$.) By induction, there is an element $u \in \widetilde{U}$ such that

$$(6.4) \quad u.(v_1 \otimes \cdots \otimes v_\ell) = v_{i_1} \otimes \cdots \otimes v_{i_\ell},$$

where u has no terms with factors of $e_{m'}, e_{m'+1}, \dots, e_{n-1}, f_{m'+1}, \dots, f_{n-1}$ ($m' = \max\{i_\ell, \ell\}$).

Suppose initially that $i_\ell \leq \ell$. Then $m' = \ell$, and so $u.(v_1 \otimes \cdots \otimes v_k)$ is a nonzero scalar multiple of $(v_{i_1} \otimes \cdots \otimes v_{i_\ell}) \otimes (v_{\ell+1} \otimes \cdots \otimes v_k)$. Now apply

$$(6.5) \quad u' = \begin{cases} (f_{i_k-1} f_{i_k-2} \cdots f_{\ell+1}) \cdots (f_{i_k-1} f_{i_k-2}) (f_{i_k-1}) (e_{i_k} e_{i_k+1} \cdots e_{k-1}) \cdots (e_{i_k} e_{i_k+1}) (e_{i_k}) & \text{if } i_k < k \\ (f_{i_k-1} f_{i_k-2} \cdots f_{\ell+1}) \cdots (f_{i_k-1} f_{i_k-2} \cdots f_{k-1}) (f_{i_k-1} f_{i_k-2} \cdots f_k) & \text{if } i_k \geq k \end{cases}$$

to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$, as desired. (Note that we did not use any factors of $e_m, e_{m+1}, \dots, e_{n-1}, f_{m+1}, \dots, f_{n-1}$ for $m = \max\{i_k, k\}$.)

If on the other hand, $i_\ell > \ell$ (so that $m' = i_\ell$ and $i_k > \ell + 1$), first apply u' from (6.5) to $v_1 \otimes \cdots \otimes v_k$ to obtain a nonzero scalar multiple of

$$(v_1 \otimes \cdots \otimes v_\ell) \otimes (v_{i_k} \otimes \cdots \otimes v_{i_k}),$$

and then apply u from (6.4) to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$, as desired.

Finally, if $i_1, \dots, i_k \in \{1, \dots, n\}$ are *any* k elements (not necessarily in nondecreasing numerical order), let $\sigma \in S_k$ be a permutation such that

$$i_{\sigma(1)} \leq i_{\sigma(2)} \leq \cdots \leq i_{\sigma(k)}.$$

By the first paragraph of the proof, there is an element of \widetilde{U} taking \underline{v} to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$. Now we may apply u from (6.4) and u' from (6.5) in the appropriate order (as above) to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$. \square

This leads to the two-parameter analogue of Schur-Weyl duality.

Theorem 6.6. *Assume rs^{-1} is not a root of unity. Then:*

- (i) $H_k(r, s)$ maps surjectively onto $\text{End}_{\widetilde{U}}(V^{\otimes k})$.
- (ii) If $n \geq k$, then $H_k(r, s)$ is isomorphic to $\text{End}_{\widetilde{U}}(V^{\otimes k})$.

Proof. We establish part (ii) first. Assume $F \in \text{End}_{\widetilde{U}}(V^{\otimes k})$ and $\underline{v} = v_1 \otimes \cdots \otimes v_k$. As F commutes with the action of \widetilde{U} , $F(\underline{v})$ must have the same weight as \underline{v} , that is,

$\epsilon_1 + \cdots + \epsilon_k$. The only vectors of $V^{\otimes k}$ with this weight are the linear combinations of the permutations of $v_1 \otimes \cdots \otimes v_k$, so that

$$(6.7) \quad F(\underline{v}) = \sum_{\sigma \in S_k} c_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

for some scalars $c_\sigma \in \mathbb{K}$. We will show that there is an element R^σ in the image of $H_k(r, s)$ in $\text{End}_{\tilde{U}}(V^{\otimes k})$ such that $R^\sigma(\underline{v}) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. (Previously we constructed an element $u \in \tilde{U}$ with this property.)

We begin with the transposition $\tau = t_j = (j \ j + 1)$. For any tensor product $v_{i_1} \otimes \cdots \otimes v_{i_k}$ of distinct basis vectors, we have by (5.6) that

$$v_{i_{\tau(1)}} \otimes \cdots \otimes v_{i_{\tau(k)}} = \begin{cases} r^{-1}R_j(v_{i_1} \otimes \cdots \otimes v_{i_k}) & \text{if } i_j < i_{j+1} \\ (sR_j + (r-s)\text{Id})(v_{i_1} \otimes \cdots \otimes v_{i_k}) & \text{if } i_j > i_{j+1}. \end{cases}$$

Therefore, if $\sigma = t_{j_1} \cdots t_{j_m}$, a product of such transpositions, we can set $R^{t_{j_\ell}} := r^{-1}R_{j_\ell}$ or $R^{t_{j_\ell}} := sR_{j_\ell} + (r-s)\text{Id}$, depending on the numerical order of the appropriate indices i_{j_ℓ} and $i_{j_\ell+1}$ in $R^{t_{j_\ell-1}} \circ \cdots \circ R^{t_{j_1}} \underline{v}$. Then defining $R^\sigma = R^{t_{j_m}} \circ \cdots \circ R^{t_{j_1}} \in \text{End}_{\tilde{U}}(V^{\otimes k})$, we have the desired map such that $R^\sigma(\underline{v}) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Now let $F_0 = F - \sum_{\sigma \in S_k} c_\sigma R^\sigma \in \text{End}_{\tilde{U}}(V^{\otimes k})$ (with the c_σ coming from (6.7)). By Lemma 6.2, $F_0(V^{\otimes k}) = F_0(\tilde{U}\underline{v}) = \tilde{U}.F_0(\underline{v}) = 0$. Therefore $F = \sum_{\sigma \in S_k} c_\sigma R^\sigma$ is in the image of $H_k(r, s)$. Consequently, the map $H_k(r, s) \rightarrow \text{End}_{\tilde{U}}(V^{\otimes k})$ in (6.1) is a surjection, and $\text{End}_{\tilde{U}}(V^{\otimes k})$ is the \mathbb{K} -linear span of $\{R^\sigma \mid \sigma \in S_k\}$. Now suppose that $\sum_{\sigma \in S_k} c_\sigma R^\sigma = 0$ for some scalars $c_\sigma \in \mathbb{K}$. Then in particular, $0 = \sum_{\sigma \in S_k} c_\sigma R^\sigma(\underline{v}) = \sum_{\sigma \in S_k} c_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$, which implies $c_\sigma = 0$ for all $\sigma \in S_k$. Therefore $\{R^\sigma \mid \sigma \in S_k\}$ is a basis for the vector space $\text{End}_{\tilde{U}}(V^{\otimes k})$ and $\dim_{\mathbb{K}} \text{End}_{\tilde{U}}(V^{\otimes k}) = k! = \dim_{\mathbb{K}} H_k(r, s)$. It follows that $H_k(r, s)$ is isomorphic to $\text{End}_{\tilde{U}}(V^{\otimes k})$ for $n \geq k$, as asserted.

Next we turn to the proof of (i) and assume here that $n < k$. For $i = n, k$, let $\tilde{U}_i = U_{r,s}(\mathfrak{gl}_i)$, let Λ_i be the weight lattice of \mathfrak{gl}_i , and let V_i be the natural \tilde{U}_i -module. By (ii), we may identify $H_k(r, s)$ with $\text{End}_{\tilde{U}_k}(V_k^{\otimes k})$. We will show that $H_k(r, s)$ maps surjectively onto $\text{End}_{\tilde{U}_n}(V_n^{\otimes k})$.

Consider $V_k^{\otimes k}$ as a \tilde{U}_n -module via the inclusion of \tilde{U}_n into \tilde{U}_k , and regard $V_n^{\otimes k}$ as a \tilde{U}_n -submodule of $V_k^{\otimes k}$ in the obvious way. Now $V_n^{\otimes k}$ is a finite-dimensional \tilde{U}_n -module on which \tilde{U}_n^0 acts semisimply, so by Theorem 3.8, it is completely reducible. Therefore,

$$(6.8) \quad V_n^{\otimes k} = L_1 \oplus \cdots \oplus L_t$$

for simple \tilde{U}_n -modules L_i . It suffices to show that the projections onto the simple summands L_i can be obtained from $H_k(r, s)$.

Consider

$$(6.9) \quad \tilde{U}_k.V_n^{\otimes k} = \tilde{U}_k.L_1 + \cdots + \tilde{U}_k.L_t,$$

the \tilde{U}_k -submodule of $V_k^{\otimes k}$ generated by $V_n^{\otimes k}$. By Corollary 2.5, each L_i is isomorphic to some $L(\lambda_i)$, $\lambda_i \in \Lambda_n^+$, and in particular is generated by a highest weight vector m_i with $e_j.m_i = 0$ for all j ($1 \leq j < n$). We claim that $e_j.m_i = 0$ as well when $n \leq j < k$. This follows from the expression for $\Delta^{k-1}(e_j)$ in (6.3) and the action of e_j on the natural module V_k for \tilde{U}_k given by $e_j.v_i = \delta_{i,j+1}v_j$, because m_i must be some linear combination of vectors $v_{i_1} \otimes \cdots \otimes v_{i_k}$ with $i_1, \dots, i_k \in \{1, \dots, n\}$. Therefore m_i is also a highest weight vector for the finite-dimensional \tilde{U}_k -module $\tilde{U}_k.L_i$. By Theorem 2.1 and Lemma 3.7, $\tilde{U}_k.L_i = \tilde{U}_k.m_i$ is a simple \tilde{U}_k -module. Therefore (6.9) must be a direct sum:

$$\tilde{U}_k.V_n^{\otimes k} = \tilde{U}_k.L_1 \oplus \cdots \oplus \tilde{U}_k.L_t.$$

Because $V_k^{\otimes k}$ is a completely reducible \tilde{U}_k -module, there is some complementary \tilde{U}_k -submodule W such that

$$(6.10) \quad V_k^{\otimes k} = \tilde{U}_k.L_1 \oplus \cdots \oplus \tilde{U}_k.L_t \oplus W.$$

Let $\pi_i \in H_k(r, s)$ be the projection of $V_k^{\otimes k}$ onto $\tilde{U}_k.L_i$. Then, π_i commutes with the \tilde{U}_k -action, and acts as the identity map on $\tilde{U}_k.L_i$ and as 0 on the other summands in (6.10). Since $L_j \subseteq \tilde{U}_k.L_j$ for all j , the map π_i restricted to $V_n^{\otimes k}$ commutes with the \tilde{U}_n -action and is the projection onto L_i . Thus, $H_k(r, s) \rightarrow \text{End}_{\tilde{U}_n}(V_n^{\otimes k})$ is onto. \square

Finally, we note that because $V^{\otimes k}$ is a semisimple \tilde{U} -module, it follows from the double commutant theorem (see for example, [GW, Thm. 3.3.7]) that $\text{End}_{H_k(r,s)}(V^{\otimes k})$ is isomorphic to the image of \tilde{U} in $\text{End}_{\mathbb{K}}(V^{\otimes k})$.

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